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q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions

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Convexity;
Quasi-convexity

Abstract In this paper, we prove the correct q-Hermite–Hadamard inequality, some new q-Hermite–Hadamard inequalities, and generalized q-Hermite–Hadamard inequality. By using the left hand part of the correct q-Hermite–Hadamard inequality, we have a new equality. Finally using the new equality, we give some q-midpoint type integral inequalities through q-differentiable convex and q-differentiable quasi-convex functions. Many results given in this paper provide extensions of others given in previous works.
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1. Introduction

The study of calculus without limits is known as quantum calculus or q-calculus. The famous mathematician Euler initiated the study q-calculus in the eighteenth century by introducing the parameter q in Newton's work of infinite series. In early twentieth century, Jackson (1910) has started a symmetric study of q-calculus and introduced q-definite integrals. The subject of quantum calculus has numerous applications in various areas of mathematics and physics such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions, quantum theory, mechanics and in theory of relativity. This subject has received outstanding attention by many researchers and hence it is considered as an in-corporative subject between mathematics and physics. Interested readers are referred to Ernst (2012), Gauchman (2004), and Kac and Cheung (2001) for some current advances in the theory of quantum calculus and theory of inequalities in quantum calculus.

In recent articles, Tariboon and Ntouyas (2013, 2014) studied the concept of q-derivatives and q-integrals over the intervals of the form $[a, b] \subset \mathbb{R}$ and settled a number of quantum analogs of some well-known results such as Holder inequality, Hermite–Hadamard inequality and Ostrowski inequality, Cauchy–Bunyakovsky–Schwarz, Gruss, Gruss–Cebysev and other integral inequalities using classical convexity. Also, Noor et al. (2015), Noor et al. (2015),

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Sudsutad et al. (2015), and Zhuang et al., 2016, have contributed to the ongoing research and have developed some integral inequalities which provide quantum estimates for the right part of the quantum analog of Hermite–Hadamard inequality through q-differentiable convex and q-differentiable quasi-convex functions.

Let real function f be defined on some non-empty interval I of real line $\mathbb{R}$. The function f said to be convex on I, if the inequality

$$f(ta+(1-t)b) \leqslant tf(a)+(1-t)f(b)$$

holds for all $a, b \in I$ and $t \in [0,1]$. The function f said to be quasi-convex on I, if the inequality

$$f(ta+(1-t)b) \leqslant \sup\{f(a), f(b)\}$$

holds for all $a, b \in I$ and $t \in [0,1]$.

Kırmacı (2004) obtained inequalities for differentiable convex mappings which are connected with midpoint type inequality, Alomari et al. (2009) obtained inequalities for differentiable quasi-convex mappings which are connected with midpoint type inequality. They used the following lemma to prove their theorems.

Lemma 1 Kırmacı (2004). *Let $f: I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a<b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) = b-a\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)f'(ta+(1-t)b)dt\right] \tag{1.1}$$

2. Preliminaries and definitions of q-calculus

Throughout this paper, let $a<b$ and $0<q<1$ be a constant. The following definitions and theorems for q-derivative and q-integral of a function f on $[a,b]$ are given in Tariboon and Ntouyas (2013, 2014).

Definition 2. For a continuous function $f: [a,b] \to \mathbb{R}$ then q-derivative of f at $x \in [a,b]$ is characterized by the expression

$$_aD_q f(x) = \frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \tag{2.1}$$

Since $f: [a,b] \to \mathbb{R}$ is a continuous function, thus we have $_aD_q f(a) = \lim_{x\to a} {_aD_q f(x)}$. The function f is said to be q-differentiable on $[a,b]$ if $_aD_q f(t)$ exists for all $x \in [a,b]$. If $a=0$ in (2.1), then $_0D_q f(x) = D_q f(x)$, where $D_q f(x)$ is familiar q-derivative of f at $x \in [a,b]$ defined by the expression (see Kac and Cheung, 2001)

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad x \neq 0. \tag{2.2}$$

Definition 3. Let $f: [a,b] \to \mathbb{R}$ be a continuous function. Then the q-definite integral on $[a,b]$ is delineated as

$$\int_a^x f(t)\,{_ad_q}t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x+(1-q^n)a) \tag{2.3}$$

for $x \in [a,b]$.

If $a=0$ in (2.3), then $\int_0^x f(t)\,{_0d_q}t = \int_0^x f(t)d_qt$, where $\int_0^x f(t)d_qt$ is familiar q-definite integral on $[0,x]$ defined by the expression (see Kac and Cheung, 2001)

$$\int_0^x f(t)\,{_0d_q}t = \int_0^x f(t)d_qt = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x). \tag{2.4}$$

If $c \in (a,x)$, then the q-definite integral on $[c,x]$ is expressed as

$$\int_c^x f(t)\,{_ad_q}t = \int_a^x f(t)\,{_ad_q}t - \int_a^c f(t)\,{_ad_q}t. \tag{2.5}$$

Theorem 4 Tariboon and Ntouyas (2014, Theorem 3.2). *Let $f: [a,b] \to \mathbb{R}$ be a convex continuous function on $[a,b]$ and $0<q<1$. Then we have*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)\,{_ad_q}t \leqslant \frac{qf(a)+f(b)}{1+q}. \tag{2.6}$$

Kunt and İşcan (2016) give the following example to prove that the left hand side of (2.6) is not correct:

Example 5. Let $[a,b]=[0,1]$. Then the function $f(t)=1-t$ is a convex continuous function on $[0,1]$. Therefore the function f satisfies Theorem 4 assumptions. Then, from the inequality (2.6) the following inequality must be hold for all $q \in (0,1)$

$$f\left(\frac{0+1}{2}\right) \leqslant \frac{1}{1-0}\int_0^1 f(t)\,{_0d_q}t$$

$$1-\frac{1}{2} \leqslant (1-q)\sum_{n=0}^{\infty} q^n(1-q^n)$$

$$\frac{1}{2} \leqslant (1-q)\left(\frac{1}{1-q} - \frac{1}{1-q^2}\right)$$

Then we have

$$\frac{1}{2} \leqslant \frac{q}{1+q}. \tag{2.7}$$

If we choose $q=\frac{1}{2}$ in (2.7) we have the following contradiction

$$\frac{1}{2} \leqslant \frac{1}{3}.$$

It means that the left hand side of (2.6) is not correct.

In the next section we give the correct q-Hermite–Hadamard inequality, some q-Hermite–Hadamard inequalities, and generalized q-Hermite–Hadamard inequality.

3. q-Hermite–Hadamard inequalities

In this section we prove q-Hermite–Hadamard inequality and varieties of q-Hermite–Hadamard inequalities.

Theorem 6 (*q-Hermite–Hadamard inequality*). *Let* $f:[a,b]\to\mathbb{R}$ *be a convex differentiable function on* (a,b) *and* $0<q<1$. *Then we have*

$$f\left(\frac{qa+b}{1+q}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\leqslant\frac{qf(a)+f(b)}{1+q}. \tag{3.1}$$

Proof. Since f is differentiable function on (a,b), there is a tangent line for the function f at the point $\frac{qa+b}{1+q}\in(a,b)$. This tangent line can be expressed as a function $h(x)=f\left(\frac{qa+b}{1+q}\right)+f'\left(\frac{qa+b}{1+q}\right)\left(x-\frac{qa+b}{1+q}\right)$. Since f is a convex function on $[a,b]$, than we have the following inequality

$$h(x)=f\left(\frac{qa+b}{1+q}\right)+f'\left(\frac{qa+b}{1+q}\right)\left(x-\frac{qa+b}{1+q}\right)\leqslant f(x) \tag{3.2}$$

for all $x\in[a,b]$ (see Fig. 1). q-Integrating the inequality (3.2) on $[a,b]$, we have

$$\begin{aligned}\int_a^b h(x)\,{}_ad_qx&=\int_a^b\left[f\left(\frac{qa+b}{1+q}\right)+f'\left(\frac{qa+b}{1+q}\right)\left(x-\frac{qa+b}{1+q}\right)\right]{}_ad_qx\\&=(b-a)f\left(\frac{qa+b}{1+q}\right)+f'\left(\frac{qa+b}{1+q}\right)\left(\int_a^b x\,{}_ad_qx-(b-a)\frac{qa+b}{1+q}\right)\\&=(b-a)f\left(\frac{qa+b}{1+q}\right)+f'\left(\frac{qa+b}{1+q}\right)\left((1-q)(b-a)\sum_{n=0}^{\infty}q^n((1-q^n)a+q^nb)-(b-a)\frac{qa+b}{1+q}\right)\\&=(b-a)f\left(\frac{qa+b}{1+q}\right)+f'\left(\frac{qa+b}{1+q}\right)\left((1-q)(b-a)\left[\left(\frac{1}{1-q}-\frac{1}{1-q^2}\right)a+\frac{1}{1-q^2}b\right]-(b-a)\frac{qa+b}{1+q}\right)\\&=(b-a)f\left(\frac{qa+b}{1+q}\right)+f'\left(\frac{qa+b}{1+q}\right)\left((b-a)\frac{qa+b}{1+q}-(b-a)\frac{qa+b}{1+q}\right)\\&=(b-a)f\left(\frac{qa+b}{1+q}\right)\leqslant\int_a^b f(x)\,{}_ad_qx.\end{aligned} \tag{3.3}$$

On the other hand, line connecting the points $(a,f(a))$ and $(b,f(b))$ can be expressed as a function $k(x)=f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$. Since f is a convex function on $[a,b]$, than we have the following inequality

$$f(x)\leqslant k(x)=f(a)+\frac{f(b)-f(a)}{b-a}(x-a) \tag{3.4}$$

for all $x\in[a,b]$ (see Fig. 1). q-Integrating the inequality (3.4) on $[a,b]$, we have

$$\begin{aligned}\int_a^b k(x)\,{}_ad_qx&=\int_a^b\left(f(a)+\frac{f(b)-f(a)}{b-a}(x-a)\right){}_ad_qx\\&=(b-a)f(a)+\frac{f(b)-f(a)}{b-a}\int_a^b(x-a)\,{}_ad_qx\\&=(b-a)f(a)+\frac{f(b)-f(a)}{b-a}\left(\int_a^b x\,{}_ad_qx-a(b-a)\right)\\&=(b-a)f(a)+\frac{f(b)-f(a)}{b-a}\left((1-q)(b-a)\sum_{n=0}^{\infty}q^n((1-q^n)a+q^nb)-a(b-a)\right)\\&=(b-a)f(a)+\frac{f(b)-f(a)}{b-a}\left((1-q)(b-a)\left[\left(\frac{1}{1-q}-\frac{1}{1-q^2}\right)a+\frac{1}{1-q^2}b\right]-a(b-a)\right)\\&=(b-a)f(a)+(f(b)-f(a))\left(\frac{qa+b}{1+q}-a\right)\\&=(b-a)f(a)+(b-a)\frac{f(b)-f(a)}{1+q}\\&=(b-a)\frac{qf(a)+f(b)}{1+q}\geqslant\int_a^b f(x)\,{}_ad_qx.\end{aligned} \tag{3.5}$$

A combination of (3.3) and (3.5) gives (3.1). Thus the proof is accomplished. □

Remark 7. In Theorem 6, if we take $q\to1^-$, we recapture the well known Hermite–Hadamard inequality for convex function.

Theorem 8. *Let* $f:[a,b]\to\mathbb{R}$ *be a convex differentiable function on* (a,b) *and* $0<q<1$. *Then we have*

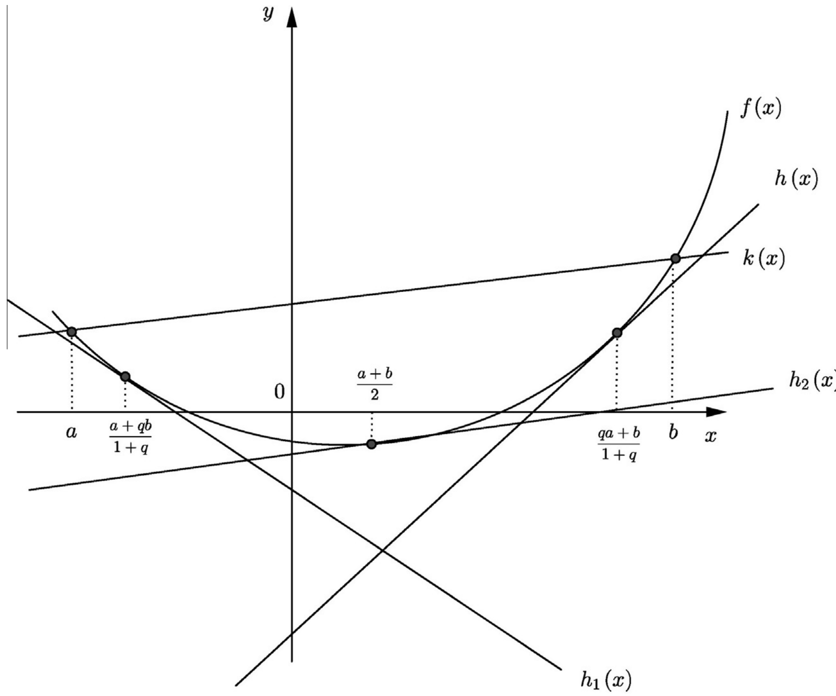

Fig. 1 Tangent and chord line for a convex function.

$$f\left(\frac{a+qb}{1+q}\right)+\frac{(1-q)(b-a)}{1+q}f'\left(\frac{a+qb}{1+q}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx \leqslant \frac{qf(a)+f(b)}{1+q}. \tag{3.6}$$

Proof. Since f is differentiable function on (a,b), there is a tangent line for the function f at the point $\frac{a+qb}{1+q}\in(a,b)$. This tangent line can be expressed as a function $h_1(x)=f\left(\frac{a+qb}{1+q}\right)+f'\left(\frac{a+qb}{1+q}\right)\left(x-\frac{a+qb}{1+q}\right)$. Since f is a convex function on $[a,b]$, than we have the following inequality

$$h_1(x)=f\left(\frac{a+qb}{1+q}\right)+f'\left(\frac{a+qb}{1+q}\right)\left(x-\frac{a+qb}{1+q}\right)\leqslant f(x) \tag{3.7}$$

for all $x\in[a,b]$ (see Fig. 1). q-Integrating the inequality (3.7) on $[a,b]$, we have

$$\begin{aligned}\int_a^b h_1(x)\,{}_ad_qx &= \int_a^b\left[f\left(\frac{a+qb}{1+q}\right)+f'\left(\frac{a+qb}{1+q}\right)\left(x-\frac{a+qb}{1+q}\right)\right]{}_ad_qx\\
&=(b-a)f\left(\frac{a+qb}{1+q}\right)+f'\left(\frac{a+qb}{1+q}\right)\left(\int_a^b x\,{}_ad_qx-(b-a)\frac{a+qb}{1+q}\right)\\
&=(b-a)f\left(\frac{a+qb}{1+q}\right)+f'\left(\frac{a+qb}{1+q}\right)\left((1-q)(b-a)\sum_{n=0}^{\infty}q^n((1-q^n)a+q^nb)-(b-a)\frac{a+qb}{1+q}\right)\\
&=(b-a)f\left(\frac{a+qb}{1+q}\right)+f'\left(\frac{a+qb}{1+q}\right)\left((1-q)(b-a)\left[\left(\frac{1}{1-q}-\frac{1}{1-q^2}\right)a+\frac{1}{1-q^2}b\right]-(b-a)\frac{a+qb}{1+q}\right)\\
&=(b-a)f\left(\frac{a+qb}{1+q}\right)+f'\left(\frac{a+qb}{1+q}\right)\left((b-a)\frac{qa+b}{1+q}-(b-a)\frac{a+qb}{1+q}\right)\\
&=(b-a)f\left(\frac{a+qb}{1+q}\right)+\frac{(1-q)(b-a)^2}{1+q}f'\left(\frac{a+qb}{1+q}\right)\leqslant\int_a^b f(x)\,{}_ad_qx.\end{aligned} \tag{3.8}$$

A combination of (3.5) and (3.8) gives (3.6). Thus the proof is accomplished. □

Theorem 9. *Let $f:[a,b]\to\mathbb{R}$ be a convex differentiable function on (a,b) and $0<q<1$. Then we have*

$$f\left(\frac{a+b}{2}\right)+\frac{(1-q)(b-a)}{2(1+q)}f'\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx \leqslant \frac{qf(a)+f(b)}{1+q}. \tag{3.9}$$

Proof. Since f is differentiable function on (a,b), there is a tangent line for the function f at the point $\frac{a+b}{2}\in(a,b)$. This tangent line can be expressed as a function $h_2(x)=f\left(\frac{a+b}{2}\right)+f'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)$. Since f is a convex function on $[a,b]$, we have the following inequality

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$$h_2(x) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) \leqslant f(x). \tag{3.10}$$

for all $x \in [a,b]$ (see Fig. 1). q-Integrating the inequality (3.10) on $[a,b]$, we have

$$\begin{aligned}
\int_a^b h_2(x)\, {}_ad_qx &= \int_a^b \left[f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right)\right] {}_ad_qx \\
&= (b-a)f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(\int_a^b x\, {}_ad_qx - (b-a)\frac{a+b}{2}\right) \\
&= (b-a)f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left((1-q)(b-a)\sum_{n=0}^{\infty} q^n((1-q^n)a + q^n b) - (b-a)\frac{a+b}{2}\right) \\
&= (b-a)f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left((1-q)(b-a)\left[\left(\frac{1}{1-q} - \frac{1}{1-q^2}\right)a + \frac{1}{1-q^2}b\right] - (b-a)\frac{a+b}{2}\right) \\
&= (b-a)f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left((b-a)\frac{qa+b}{1+q} - (b-a)\frac{a+b}{2}\right) \\
&= (b-a)f\left(\frac{a+b}{2}\right) + \frac{(1-q)(b-a)^2}{2(1+q)} f'\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(x)\, {}_ad_qx.
\end{aligned} \tag{3.11}$$

A combination of (3.5) and (3.11) gives (3.9). Thus the proof is accomplished. □

Theorem 10. [Generalized q-Hermite–Hadamard inequality] *Let $f : [a,b] \to \mathbb{R}$ be a convex differentiable function on (a,b) and $0 < q < 1$. Then we have*

$$\max\{I_1, I_2, I_3\} \leqslant \frac{1}{b-a}\int_a^b f(x)\, {}_ad_qx \leqslant \frac{qf(a) + f(b)}{1+q}. \tag{3.12}$$

where

$$\begin{aligned}
I_1 &= f\left(\frac{qa+b}{1+q}\right), \\
I_2 &= f\left(\frac{a+qb}{1+q}\right) + \frac{(1-q)(b-a)}{1+q} f'\left(\frac{a+qb}{1+q}\right), \\
I_3 &= f\left(\frac{a+b}{2}\right) + \frac{(1-q)(b-a)}{2(1+q)} f'\left(\frac{a+b}{2}\right).
\end{aligned}$$

Proof. A combination of (3.1), (3.6), and (3.9) gives (3.12). Thus the proof is accomplished. □

4. Midpoint type inequalities via q-calculus

In this section we proved an equality for the q-analog of midpoint type inequality. By using this equality we have q-midpoint type integral inequalities through q-differentiable convex and q-differentiable quasi-convex functions. We will use the following Lemma to prove our main results.

Lemma 11. *Let $f : [a,b] \to \mathbb{R}$ be a q-differentiable function on (a,b). If ${}_aD_qf$ is continuous and integrable on $[a,b]$, then the following identity holds:*

$$\begin{aligned}
&f\left(\frac{qa+b}{1+q}\right) - \frac{1}{(b-a)}\int_a^b f(x)\, {}_ad_qx \\
&= q(b-a)\left[\int_0^{\frac{1}{1+q}} t\, {}_aD_qf(tb + (1-t)a)\, {}_0d_qt \right. \\
&\quad \left. + \int_{\frac{1}{1+q}}^1 \left(t - \frac{1}{q}\right) {}_aD_qf(tb + (1-t)a)\, {}_0d_qt\right]
\end{aligned} \tag{4.1}$$

Proof. Using (2.1), we have

$$\begin{aligned}
{}_aD_qf(tb + (1-t)a) &= \frac{f(tb + (1-t)a) - f(q[tb + (1-t)a] + (1-q)a)}{(1-q)[tb + (1-t)a - a]} \\
&= \frac{f(tb + (1-t)a) - f(qtb + (1-qt)a)}{t(1-q)(b-a)}.
\end{aligned} \tag{4.2}$$

Calculating following integrals by using (2.3) and (4.2), we have

$$\begin{aligned}
&q(b-a)\left[\int_0^{\frac{1}{1+q}} t\, {}_aD_qf(tb + (1-t)a)\, {}_0d_qt + \int_{\frac{1}{1+q}}^1 \left(t - \frac{1}{q}\right) {}_aD_qf(tb + (1-t)a)\, {}_0d_qt\right] \\
&= q(b-a)\left[\int_0^{\frac{1}{1+q}} t\, {}_aD_qf(tb + (1-t)a)\, {}_0d_qt + \int_{\frac{1}{1+q}}^1 \left(t - \frac{1}{q}\right) {}_aD_qf(tb + (1-t)a)\, {}_0d_qt - \frac{1}{q}\int_0^{\frac{1}{1+q}} {}_aD_qf(tb + (1-t)a)\, {}_0d_qt \right. \\
&\quad \left. + \frac{1}{q}\int_0^{\frac{1}{1+q}} {}_aD_qf(tb + (1-t)a)\, {}_0d_qt\right]
\end{aligned}$$

$$= q(b-a)\left[\int_0^1 t\,{}_aD_qf(tb+(1-t)a)\,{}_0d_qt - \frac{1}{q}\int_0^1 {}_aD_qf(tb+(1-t)a)\,{}_0d_qt + \frac{1}{q}\int_0^{\frac{1}{1+q}} {}_aD_qf(tb+(1-t)a)\,{}_0d_qt\right]$$

$$= q(b-a)\begin{bmatrix} \int_0^1 t\frac{f(tb+(1-t)a)-f(qtb+(1-qt)a)}{t(1-q)(b-a)}\,{}_0d_qt \\ -\frac{1}{q}\int_0^1 \frac{f(tb+(1-t)a)-f(qtb+(1-qt)a)}{t(1-q)(b-a)}\,{}_0d_qt \\ +\frac{1}{q}\int_0^{\frac{1}{1+q}} \frac{f(tb+(1-t)a)-f(qtb+(1-qt)a)}{t(1-q)(b-a)}\,{}_0d_qt \end{bmatrix} = \begin{bmatrix} \frac{q}{1-q}\int_0^1 f(tb+(1-t)a)-f(qtb+(1-qt)a)\,{}_0d_qt \\ -\frac{1}{1-q}\int_0^1 \frac{f(tb+(1-t)a)}{t}-\frac{f(qtb+(1-qt)a)}{t}\,{}_0d_qt \\ +\frac{1}{1-q}\int_0^{\frac{1}{1+q}} \frac{f(tb+(1-t)a)}{t}-\frac{f(qtb+(1-qt)a)}{t}\,{}_0d_qt \end{bmatrix}$$

$$= \left[q\left(\sum_{n=0}^{\infty} q^n f(q^nb+(1-q^n)a) - \sum_{n=0}^{\infty} q^n f\left(q^{n+1}b+\left(1-q^{n+1}\right)a\right)\right) - \left(\sum_{n=0}^{\infty} f(q^nb+(1-q^n)a) - \sum_{n=0}^{\infty} f\left(q^{n+1}b+\left(1-q^{n+1}\right)a\right)\right)\right.$$
$$\left. + \left(\sum_{n=0}^{\infty} f\left(\frac{q^n}{1+q}b+\left(1-\frac{q^n}{1+q}\right)a\right) - \sum_{n=0}^{\infty} f\left(\frac{q^{n+1}}{1+q}b+\left(1-\frac{q^{n+1}}{1+q}\right)a\right)\right)\right]$$

$$= q\left(\frac{1}{q}f(b) - \left(\frac{1}{q}-1\right)\sum_{n=0}^{\infty} q^n f(q^nb+(1-q^n)a)\right) - (f(b)-f(a)) + \left(f\left(\frac{qa+b}{1+q}\right)-f(a)\right) = f\left(\frac{qa+b}{1+q}\right) - \frac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx.$$

Thus the proof is accomplished. □

Remark 12. In Lemma 11, if we take $q \to 1^-$, we recapture Lemma 1.

We can now prove some quantum estimates of q-midpoint type integral inequalities by using convexity and quasi-convexity of the absolute values of the q-derivatives.

Theorem 13. *Let $f : [a,b] \to \mathbb{R}$ be a q-differentiable function on (a,b), ${}_aD_qf$ be continuous and integrable on $[a,b]$ and $0<q<1$. If $\left|{}_aD_qf\right|$ is convex on $[a,b]$, then the following q-midpoint type inequality holds:*

$$\left|f\left(\frac{qa+b}{1+q}\right) - \frac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx\right| \leqslant q(b-a)\left[\left|{}_aD_qf(b)\right|\frac{3}{(1+q)^3(1+q+q^2)} + \left|{}_aD_qf(a)\right|\frac{-1+2q+2q^2}{(1+q)^3(1+q+q^2)}\right] \tag{4.3}$$

Proof. Taking absolute value on both sides of (4.1) and using the fact that $\left|{}_aD_qf\right|$ is convex on $[a,b]$, then we have

$$\begin{aligned} &\left|f\left(\frac{qa+b}{1+q}\right) - \frac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx\right| \\ &\leqslant q(b-a)\left[\int_0^{\frac{1}{1+q}} t\left|{}_aD_qf(tb+(1-t)a)\right|{}_0d_qt + \int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right)\left|{}_aD_qf(tb+(1-t)a)\right|{}_0d_qt\right] \\ &\leqslant q(b-a)\begin{bmatrix} \int_0^{\frac{1}{1+q}} t\left[t\left|{}_aD_qf(b)\right| + (1-t)\left|{}_aD_qf(a)\right|\right]{}_0d_qt \\ +\int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right)\left[t\left|{}_aD_qf(b)\right| + (1-t)\left|{}_aD_qf(a)\right|\right]{}_0d_qt \end{bmatrix} \\ &\leqslant q(b-a)\begin{bmatrix} \left|{}_aD_qf(b)\right|\int_0^{\frac{1}{1+q}} t^2\,{}_0d_qt + \left|{}_aD_qf(a)\right|\int_0^{\frac{1}{1+q}} t(1-t)\,{}_0d_qt \\ \left|{}_aD_qf(b)\right|\int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right)t\,{}_0d_qt + \left|{}_aD_qf(a)\right|\int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right)(1-t)\,{}_0d_qt \end{bmatrix} \\ &\leqslant q(b-a)\begin{bmatrix} \left|{}_aD_qf(b)\right|\left[\int_0^{\frac{1}{1+q}} t^2\,{}_0d_qt + \int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right)t\,{}_0d_qt\right] \\ \left|{}_aD_qf(a)\right|\left[\int_0^{\frac{1}{1+q}} t(1-t)\,{}_0d_qt + \int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right)(1-t)\,{}_0d_qt\right] \end{bmatrix} \end{aligned} \tag{4.4}$$

We evaluate the appearing definite q-integrals as follows

$$\begin{aligned} \int_0^{\frac{1}{1+q}} t^2\,{}_0d_qt &= (1-q)\frac{1}{1+q}\sum_{n=0}^{\infty} q^n\left(\frac{q^n}{1+q}\right)^2 \\ &= (1-q)\frac{1}{(1+q)^3}\frac{1}{1-q^3} = \frac{1}{(1+q)^3(1+q+q^2)}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \int_0^{\frac{1}{1+q}} t(1-t)\,{}_0d_qt &= \int_0^{\frac{1}{1+q}} t\,{}_0d_qt - \int_0^{\frac{1}{1+q}} t^2\,{}_0d_qt \\ &= (1-q)\frac{1}{1+q}\sum_{n=0}^{\infty} q^n\left(\frac{q^n}{1+q}\right) - \frac{1}{(1+q)^3(1+q+q^2)} \\ &= \frac{1}{(1+q)^3} - \frac{1}{(1+q)^3(1+q+q^2)} \\ &= \frac{q}{(1+q)^2(1+q+q^2)}, \end{aligned} \tag{4.6}$$

$$\int_{\frac{1}{1+q}}^{1}\left(\frac{1}{q}-t\right)t\,{}_0d_qt=\int_0^1\left(\frac{1}{q}-t\right)t\,{}_0d_qt-\int_0^{\frac{1}{1+q}}\left(\frac{1}{q}-t\right)t\,{}_0d_qt$$
$$=\frac{1}{q}\int_0^1 t\,{}_0d_qt-\int_0^1 t^2\,{}_0d_qt-\frac{1}{q}\int_0^{\frac{1}{1+q}}t\,{}_0d_qt+\int_0^{\frac{1}{1+q}}t^2\,{}_0d_qt$$
$$=(1-q)\left[\frac{1}{q}\sum_{n=0}^{\infty}q^{2n}-\sum_{n=0}^{\infty}q^{3n}-\frac{1}{q(1+q)^2}\sum_{n=0}^{\infty}q^{2n}+\frac{1}{(1+q)^3}\sum_{n=0}^{\infty}q^{3n}\right]$$
$$=\left[\frac{1}{q(1+q)}-\frac{1}{1+q+q^2}-\frac{1}{q(1+q)^3}+\frac{1}{(1+q)^3(1+q+q^2)}\right]$$
$$=\frac{2}{(1+q)^3(1+q+q^2)},\tag{4.7}$$

$$\int_{\frac{1}{1+q}}^{1}\left(\frac{1}{q}-t\right)(1-t)\,{}_0d_qt=\int_{\frac{1}{1+q}}^{1}\left(\frac{1}{q}-t\right){}_0d_qt-\int_{\frac{1}{1+q}}^{1}\left(\frac{1}{q}-t\right)t\,{}_0d_qt$$
$$=\int_0^1\left(\frac{1}{q}-t\right){}_0d_qt-\int_0^{\frac{1}{1+q}}\left(\frac{1}{q}-t\right){}_0d_qt-\frac{2}{(1+q)^3(1+q+q^2)}$$
$$=\frac{-1+q+q^2}{(1+q)^3(1+q+q^2)}.\tag{4.8}$$

Making use of (4.4)–(4.8), gives us the desired result (4.3). Thus the proof is accomplished. □

Corollary 14. *In Theorem 13, if we take $q\to 1^-$, we have the following midpoint type inequality for convex functions:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int_a^b f(x)dx\right|\leqslant\frac{(b-a)[|f'(a)|+|f'(b)|]}{8}.\tag{4.9}$$

Remark 15. In (4.9), we recapture the inequality Kırmacı, 2004, Theorem 2.2.

Theorem 16. *Let $f:[a,b]\to\mathbb{R}$ be a q-differentiable function on (a,b), ${}_aD_qf$ be continuous and integrable on $[a,b]$ and $0<q<1$. If $\left|{}_aD_qf\right|^r$ is convex on $[a,b]$ for $r\geqslant 1$, then the following q-midpoint type inequality holds:*

$$\left|f\left(\frac{qa+b}{1+q}\right)-\frac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx\right|\leqslant q(b-a)\frac{1}{(1+q)^{3-\frac{3}{r}}}\left[\begin{array}{l}\left(\left|{}_aD_qf(b)\right|^r\frac{1}{(1+q)^3(1+q+q^2)}+\left|{}_aD_qf(a)\right|^r\frac{q}{(1+q)^2(1+q+q^2)}\right)^{\frac{1}{r}}\\+\left(\left|{}_aD_qf(b)\right|^r\frac{2}{(1+q)^3(1+q+q^2)}+\left|{}_aD_qf(a)\right|^r\frac{-1+q+q^2}{(1+q)^3(1+q+q^2)}\right)^{\frac{1}{r}}\end{array}\right].\tag{4.10}$$

Proof. Taking absolute value on both sides of (4.1), applying the power mean inequality and using the fact that $\left|{}_aD_qf\right|^r$ is convex on $[a,b]$ for $r\geqslant 1$, we get that

$$\begin{aligned}&\left|f\left(\tfrac{qa+b}{1+q}\right)-\tfrac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx\right|\\
&\leqslant q(b-a)\left[\int_0^{\frac{1}{1+q}}t\left|{}_aD_qf(tb+(1-t)a)\right|{}_0d_qt+\int_{\frac{1}{1+q}}^1\left(\tfrac{1}{q}-t\right)\left|{}_aD_qf(tb+(1-t)a)\right|{}_0d_qt\right]\\
&\leqslant q(b-a)\left[\begin{array}{l}\left(\int_0^{\frac{1}{1+q}}t\,{}_0d_qt\right)^{1-\frac{1}{r}}\left(\int_0^{\frac{1}{1+q}}t\left|{}_aD_qf(tb+(1-t)a)\right|^r{}_0d_qt\right)^{\frac{1}{r}}\\+\left(\int_{\frac{1}{1+q}}^1\left(\tfrac{1}{q}-t\right){}_0d_qt\right)^{1-\frac{1}{r}}\left(\int_{\frac{1}{1+q}}^1\left(\tfrac{1}{q}-t\right)\left|{}_aD_qf(tb+(1-t)a)\right|^r{}_0d_qt\right)^{\frac{1}{r}}\end{array}\right]\\
&\leqslant q(b-a)\tfrac{1}{(1+q)^{3-\frac{3}{r}}}\left[\begin{array}{l}\left(\int_0^{\frac{1}{1+q}}t\left[t\left|{}_aD_qf(b)\right|^r+(1-t)\left|{}_aD_qf(a)\right|^r\right]{}_0d_qt\right)^{\frac{1}{r}}\\+\left(\int_{\frac{1}{1+q}}^1\left(\tfrac{1}{q}-t\right)\left[t\left|{}_aD_qf(b)\right|^r+(1-t)\left|{}_aD_qf(a)\right|^r\right]{}_0d_qt\right)^{\frac{1}{r}}\end{array}\right]\\
&\leqslant q(b-a)\tfrac{1}{(1+q)^{3-\frac{3}{r}}}\left[\begin{array}{l}\left(\begin{array}{l}\left|{}_aD_qf(b)\right|^r\int_0^{\frac{1}{1+q}}t^2\,{}_0d_qt\\+\left|{}_aD_qf(a)\right|^r\int_0^{\frac{1}{1+q}}t(1-t)\,{}_0d_qt\end{array}\right)^{\frac{1}{r}}\\+\left(\begin{array}{l}\left|{}_aD_qf(b)\right|^r\int_{\frac{1}{1+q}}^1\left(\tfrac{1}{q}-t\right)t\,{}_0d_qt\\+\left|{}_aD_qf(a)\right|^r\int_{\frac{1}{1+q}}^1\left(\tfrac{1}{q}-t\right)(1-t)\,{}_0d_qt\end{array}\right)^{\frac{1}{r}}\end{array}\right].\end{aligned}\tag{4.11}$$

Making use of (4.5)–(4.8) in (4.11), gives us the desired result (4.10). Thus the proof is accomplished. □

Corollary 17. *In Theorem 16, if we take $q \to 1^-$, we have the following midpoint type inequality for convex functions:*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)}\int_a^b f(x)dx\right| \leqslant (b-a)\frac{1}{2^{3-\frac{3}{r}}}\left[\begin{array}{c}\left(|f'(b)|^r\frac{1}{24} + |f'(a)|^r\frac{1}{12}\right)^{\frac{1}{r}} \\ +\left(|f'(b)|^r\frac{1}{12} + |f'(a)|^r\frac{1}{24}\right)^{\frac{1}{r}}\end{array}\right]. \tag{4.12}$$

Theorem 18. *Let $f:[a,b]\to\mathbb{R}$ be a q-differentiable function on (a,b), ${}_aD_qf$ be continuous and integrable on $[a,b]$ and $0<q<1$. If $\left|{}_aD_qf\right|^r$ is convex on $[a,b]$ for $r>1$, the following q-midpoint type inequality holds:*

$$\left|f\left(\frac{qa+b}{1+q}\right) - \frac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx\right| \leqslant q(b-a)\left[\begin{array}{c}\left(\frac{1}{(1+q)^{s+1}}\frac{1-q}{1-q^{s+1}}\right)^{\frac{1}{s}}\left(\frac{\left|{}_aD_qf(b)\right|^r}{(1+q)^3} + \frac{(2q+q^2)\left|{}_aD_qf(a)\right|^r}{(1+q)^3}\right)^{\frac{1}{r}} \\ +\left(\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right)^s{}_0d_qt\right)^{\frac{1}{s}}\left(\frac{(2q+q^2)\left|{}_aD_qf(b)\right|^r}{(1+q)^3} + \frac{(-q+q^2+q^3)\left|{}_aD_qf(a)\right|^r}{(1+q)^3}\right)^{\frac{1}{r}}\end{array}\right] \tag{4.13}$$

where $r^{-1}+s^{-1}=1$.

Proof. Taking absolute value on both sides of (4.1), applying the Hölder inequality and using the fact that $\left|{}_aD_qf\right|^r$ is convex on $[a,b]$ for $r>1$, we get that

$$\begin{aligned}
&\left|f\left(\frac{qa+b}{1+q}\right) - \frac{1}{(b-a)}\int_a^b f(x){}_ad_qx\right| \\
&\leqslant q(b-a)\left[\int_0^{\frac{1}{1+q}} t\left|{}_aD_qf(tb+(1-t)a)\right|{}_0d_qt + \int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right)\left|{}_aD_qf(tb+(1-t)a)\right|{}_0d_qt\right] \\
&\leqslant q(b-a)\left[\begin{array}{c}\left(\int_0^{\frac{1}{1+q}} t^s{}_0d_qt\right)^{\frac{1}{s}}\left(\int_0^{\frac{1}{1+q}}\left|{}_aD_qf(tb+(1-t)a)\right|^r{}_0d_qt\right)^{\frac{1}{r}} \\ +\left(\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right)^s{}_0d_qt\right)^{\frac{1}{s}}\left(\int_{\frac{1}{1+q}}^1\left|{}_aD_qf(tb+(1-t)a)\right|^r{}_0d_qt\right)^{\frac{1}{r}}\end{array}\right] \\
&\leqslant q(b-a)\left[\begin{array}{c}\left(\int_0^{\frac{1}{1+q}} t^s{}_0d_qt\right)^{\frac{1}{s}}\left(\int_0^{\frac{1}{1+q}}\left[t\left|{}_aD_qf(b)\right|^r + (1-t)\left|{}_aD_qf(a)\right|^r\right]{}_0d_qt\right)^{\frac{1}{r}} \\ +\left(\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right)^s{}_0d_qt\right)^{\frac{1}{s}}\left(\int_{\frac{1}{1+q}}^1\left[t\left|{}_aD_qf(b)\right|^r + (1-t)\left|{}_aD_qf(a)\right|^r\right]{}_0d_qt\right)^{\frac{1}{r}}\end{array}\right] \\
&\leqslant q(b-a)\left[\begin{array}{c}\left(\int_0^{\frac{1}{1+q}} t^s{}_0d_qt\right)^{\frac{1}{s}}\left(\begin{array}{c}\left|{}_aD_qf(b)\right|^r\int_0^{\frac{1}{1+q}} t\,{}_0d_qt \\ +\left|{}_aD_qf(a)\right|^r\int_0^{\frac{1}{1+q}}(1-t)\,{}_0d_qt\end{array}\right)^{\frac{1}{r}} \\ +\left(\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right)^s{}_0d_qt\right)^{\frac{1}{s}}\left(\begin{array}{c}\left|{}_aD_qf(b)\right|^r\int_{\frac{1}{1+q}}^1 t\,{}_0d_qt \\ +\left|{}_aD_qf(a)\right|^r\int_{\frac{1}{1+q}}^1(1-t){}_0d_qt\end{array}\right)^{\frac{1}{r}}\end{array}\right] \\
&\leqslant q(b-a)\left[\begin{array}{c}\left(\frac{1}{(1+q)^{s+1}}\frac{1-q}{1-q^{s+1}}\right)^{\frac{1}{s}}\left(\frac{\left|{}_aD_qf(b)\right|^r}{(1+q)^3} + \frac{(2q+q^2)\left|{}_aD_qf(a)\right|^r}{(1+q)^3}\right)^{\frac{1}{r}} \\ +\left(\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right)^s{}_0d_qt\right)^{\frac{1}{s}}\left(\frac{(2q+q^2)\left|{}_aD_qf(b)\right|^r}{(1+q)^3} + \frac{(-q+q^2+q^3)\left|{}_aD_qf(a)\right|^r}{(1+q)^3}\right)^{\frac{1}{r}}\end{array}\right]
\end{aligned}$$

Thus the proof is accomplished. □

Corollary 19. *In Theorem 18, If we take $q\to 1^-$, we have the following midpoint type inequality for convex functions:*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)}{16}\left(\frac{4}{s+1}\right)^{\frac{1}{s}}\left[\begin{array}{c}\left(|f'(b)|^r + 3|f'(a)|^r\right)^{\frac{1}{r}} \\ +\left(3|f'(b)|^r + |f'(a)|^r\right)^{\frac{1}{r}}\end{array}\right]. \tag{4.14}$$

Remark 20. In (4.14), we recapture the inequality Kırmacı, 2004, Theorem 2.3.

Theorem 21. *Let $f:[a,b]\to\mathbb{R}$ be a q-differentiable function on (a,b), ${}_aD_qf$ be continuous and integrable on $[a,b]$ and $0<q<1$. If $\left|{}_aD_qf\right|^r$ is convex on $[a,b]$ for $r>1$, the following q-midpoint type inequality holds:*

$$\left|f\left(\frac{qa+b}{1+q}\right) - \frac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx\right| \leqslant q(b-a)\left(\frac{1}{1+q}\right)^{\frac{3}{s}} \times\left[\begin{array}{c}\left(\left|{}_aD_qf(b)\right|^r\frac{1}{(1+q)^3(1+q+q^2)} + \left|{}_aD_qf(a)\right|^r\frac{q}{(1+q)^2(1+q+q^2)}\right)^{\frac{1}{r}} \\ +\left(\left|{}_aD_qf(b)\right|^r\frac{2}{(1+q)^3(1+q+q^2)} + \left|{}_aD_qf(a)\right|^r\frac{-1+q+q^2}{(1+q)^3(1+q+q^2)}\right)^{\frac{1}{r}}\end{array}\right] \tag{4.15}$$

where $r^{-1}+s^{-1}=1$.

Proof. Taking absolute value on both sides of (4.1), applying the Hölder inequality and using the fact that $\left|{}_aD_qf\right|^r$ is convex on $[a,b]$ for $r>1$, we get that

$$\begin{aligned}
&\left|f\left(\tfrac{qa+b}{1+q}\right)-\tfrac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx\right| \\
&\leqslant q(b-a)\left[\begin{array}{c}\int_0^{\frac{1}{1+q}} t^{\frac{1}{s}}t^{\frac{1}{r}}\left|{}_aD_qf(tb+(1-t)a)\right|{}_0d_qt \\ +\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right)^{\frac{1}{s}}\left(\frac{1}{q}-t\right)^{\frac{1}{r}}\left|{}_aD_qf(tb+(1-t)a)\right|\end{array}{}_0d_qt\right] \\
&\leqslant q(b-a)\left[\begin{array}{c}\left(\int_0^{\frac{1}{1+q}} t\,{}_0d_qt\right)^{\frac{1}{s}}\left(\int_0^{\frac{1}{1+q}} t\left|{}_aD_qf(tb+(1-t)a)\right|^r{}_0d_qt\right)^{\frac{1}{r}} \\ +\left(\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right){}_0d_qt\right)^{\frac{1}{s}}\left(\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right)\left|{}_aD_qf(tb+(1-t)a)\right|^r{}_0d_qt\right)^{\frac{1}{r}}\end{array}\right] \\
&\leqslant q(b-a)\left(\tfrac{1}{1+q}\right)^{\frac{3}{s}}\left[\begin{array}{c}\left(\int_0^{\frac{1}{1+q}} t\left|{}_aD_qf(tb+(1-t)a)\right|^r{}_0d_qt\right)^{\frac{1}{r}} \\ +\left(\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right)\left|{}_aD_qf(tb+(1-t)a)\right|^r{}_0d_qt\right)^{\frac{1}{r}}\end{array}\right] \\
&\leqslant q(b-a)\left(\tfrac{1}{1+q}\right)^{\frac{3}{s}}\left[\begin{array}{c}\left|{}_aD_qf(b)\right|^r\int_0^{\frac{1}{1+q}} t^2\,{}_0d_qt+\left|{}_aD_qf(a)\right|^r\int_0^{\frac{1}{1+q}} t(1-t){}_0d_qt \\ \left|{}_aD_qf(b)\right|^r\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right)t\,{}_0d_qt+\left|{}_aD_qf(a)\right|^r\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right)(1-t){}_0d_qt\end{array}\right].
\end{aligned} \tag{4.16}$$

Making use of (4.5), (4.8) in (4.16), gives us the desired result (4.15). Thus the proof is accomplished. □

Corollary 22. *In Theorem 21, if we take $q\to 1^-$, we have the following midpoint type inequality for convex functions:*

$$\begin{aligned}
&\left|f\left(\tfrac{a+b}{2}\right)-\tfrac{1}{(b-a)}\int_a^b f(x)dx\right| \\
&\leqslant (b-a)\left(\tfrac{1}{2}\right)^{\frac{3}{s}}\left[\begin{array}{c}\left(|f'(b)|^r\frac{1}{24}+|f'(a)|^r\frac{1}{12}\right)^{\frac{1}{r}} \\ +\left(|f'(b)|^r\frac{1}{12}+|f'(a)|^r\frac{1}{24}\right)^{\frac{1}{r}}\end{array}\right]
\end{aligned} \tag{4.17}$$

Some results related to quasi-convexity are presented in the following theorems.

Theorem 23. *Let $f:[a,b]\to\mathbb{R}$ be a q-differentiable function on (a,b), ${}_aD_qf$ be continuous and integrable on $[a,b]$ and $0<q<1$. If $\left|{}_aD_qf\right|^r$ is quasi-convex on $[a,b]$ for $r\geqslant 1$, the following q-midpoint type inequality holds:*

$$\begin{aligned}
&\left|f\left(\frac{qa+b}{1+q}\right)-\frac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx\right| \\
&\leqslant (b-a)\frac{2q}{(1+q)^3}\sup\left\{\left|{}_aD_qf(a)\right|,\ \left|{}_aD_qf(b)\right|\right\}.
\end{aligned} \tag{4.18}$$

Proof. Taking absolute value on both sides of (4.1), applying the power mean inequality and using the fact that $\left|{}_aD_qf\right|^r$ is quasi-convex on $[a,b]$ for $r\geqslant 1$, we get that

$$\begin{aligned}
&\left|f\left(\tfrac{qa+b}{1+q}\right)-\tfrac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx\right| \\
&\leqslant q(b-a)\left[\int_0^{\frac{1}{1+q}} t\left|{}_aD_qf(tb+(1-t)a)\right|{}_0d_qt+\int_{\frac{1}{1+q}}^1\left(\tfrac{1}{q}-t\right)\left|{}_aD_qf(tb+(1-t)a)\right|{}_0d_qt\right] \\
&\leqslant q(b-a)\left[\begin{array}{c}\left(\int_0^{\frac{1}{1+q}} t\,{}_0d_qt\right)^{1-\frac{1}{r}}\left(\int_0^{\frac{1}{1+q}} t\left|{}_aD_qf(tb+(1-t)a)\right|^r{}_0d_qt\right)^{\frac{1}{r}} \\ +\left(\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right){}_0d_qt\right)^{1-\frac{1}{r}}\left(\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right)\left|{}_aD_qf(tb+(1-t)a)\right|^r{}_0d_qt\right)^{\frac{1}{r}}\end{array}\right] \\
&\leqslant q(b-a)\left[\begin{array}{c}\left(\int_0^{\frac{1}{1+q}} t\,{}_0d_qt\right)^{1-\frac{1}{r}}\left(\int_0^{\frac{1}{1+q}} t\left[\sup\left\{\left|{}_aD_qf(a)\right|^r,\left|{}_aD_qf(b)\right|^r\right\}\right]{}_0d_qt\right)^{\frac{1}{r}} \\ +\left(\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right){}_0d_qt\right)^{1-\frac{1}{r}}\left(\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right)\left[\sup\left\{\left|{}_aD_qf(a)\right|^r,\left|{}_aD_qf(b)\right|^r\right\}\right]{}_0d_qt\right)^{\frac{1}{r}}\end{array}\right] \\
&\leqslant q(b-a)\left[\sup\left\{\left|{}_aD_qf(a)\right|^r,\left|{}_aD_qf(b)\right|^r\right\}\right]^{\frac{1}{r}}\left[\int_0^{\frac{1}{1+q}} t\,{}_0d_qt+\int_{\frac{1}{1+q}}^1\left(\tfrac{1}{q}-t\right){}_0d_qt\right] \\
&\leqslant (b-a)\tfrac{2q}{(1+q)^3}\sup\left\{\left|{}_aD_qf(a)\right|,\left|{}_aD_qf(b)\right|\right\}.
\end{aligned}$$

Hence the inequality (4.18) is established. Thus the proof is accomplished. □

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Corollary 24. *In Theorem 23, if we take $q \to 1^-$, we have the following midpoint type inequality for quasi-convex functions:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)}{4}\sup\{|f'(a)|,|f'(b)|\} \tag{4.19}$$

Theorem 25. *Let $f:[a,b]\to\mathbb{R}$ be a q-differentiable function on (a,b), ${}_aD_qf$ be continuous and integrable on $[a,b]$ and $0<q<1$. If $|{}_aD_qf|^r$ is quasi-convex on $[a,b]$ for $r>1$, the following q-midpoint type inequality holds:*

$$\left|f\left(\frac{qa+b}{1+q}\right)-\frac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx\right| \leqslant q(b-a)\sup\{|{}_aD_qf(a)|,|{}_aD_qf(b)|\} \times\left[\left(\frac{1}{(1+q)^{s+1}}\frac{1-q}{1-q^{s+1}}\right)^{\frac{1}{s}}\left(\frac{1}{1+q}\right)^{\frac{1}{r}} +\left(\int_{\frac{1}{1+q}}^1\left(\frac{1}{q}-t\right)^s{}_0d_qt\right)^{\frac{1}{s}}\left(\frac{q}{1+q}\right)^{\frac{1}{r}}\right]$$

where $r^{-1}+s^{-1}=1$.

Proof. Taking absolute value on both sides of (4.1), applying the Hölder inequality and using the fact that $|{}_aD_qf|^r$ is quasi-convex on $[a,b]$ for $r>1$, we get that

$$\begin{aligned}
&\left|f\left(\tfrac{qa+b}{1+q}\right)-\tfrac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx\right| \\
&\leqslant q(b-a)\left[\int_0^{\frac{1}{1+q}} t\,|{}_aD_qf(tb+(1-t)a)|\,{}_0d_qt+\int_{\frac{1}{1+q}}^1\left(\tfrac{1}{q}-t\right)|{}_aD_qf(tb+(1-t)a)|\,{}_0d_qt\right] \\
&\leqslant q(b-a)\left[\begin{array}{c}\left(\int_0^{\frac{1}{1+q}} t^s\,{}_0d_qt\right)^{\frac{1}{s}}\left(\int_0^{\frac{1}{1+q}}|{}_aD_qf(tb+(1-t)a)|^r\,{}_0d_qt\right)^{\frac{1}{r}} \\ +\left(\int_{\frac{1}{1+q}}^1\left(\tfrac{1}{q}-t\right)^s{}_0d_qt\right)^{\frac{1}{s}}\left(\int_{\frac{1}{1+q}}^1|{}_aD_qf(tb+(1-t)a)|^r\,{}_0d_qt\right)^{\frac{1}{r}}\end{array}\right] \\
&\leqslant q(b-a)\left[\begin{array}{c}\left(\tfrac{1}{(1+q)^{s+1}}\tfrac{1-q}{1-q^{s+1}}\right)^{\frac{1}{s}}\left(\int_0^{\frac{1}{1+q}}\left[\sup\{|{}_aD_qf(a)|^r,|{}_aD_qf(b)|^r\}\right]{}_0d_qt\right)^{\frac{1}{r}} \\ +\left(\int_{\frac{1}{1+q}}^1\left(\tfrac{1}{q}-t\right)^s{}_0d_qt\right)^{\frac{1}{s}}\left(\int_{\frac{1}{1+q}}^1\left[\sup\{|{}_aD_qf(a)|^r,|{}_aD_qf(b)|^r\}\right]{}_0d_qt\right)^{\frac{1}{r}}\end{array}\right] \\
&\leqslant q(b-a)\left[\sup\{|{}_aD_qf(a)|^r,|{}_aD_qf(b)|^r\}\right]^{\frac{1}{r}} \\
&\times\left[\left(\tfrac{1}{(1+q)^{s+1}}\tfrac{1-q}{1-q^{s+1}}\right)^{\frac{1}{s}}\left(\int_0^{\frac{1}{1+q}}{}_0d_qt\right)^{\frac{1}{r}}+\left(\int_{\frac{1}{1+q}}^1\left(\tfrac{1}{q}-t\right)^s{}_0d_qt\right)^{\frac{1}{s}}\left(\int_{\frac{1}{1+q}}^1{}_0d_qt\right)^{\frac{1}{r}}\right] \\
&\leqslant q(b-a)\sup\{|{}_aD_qf(a)|,|{}_aD_qf(b)|\} \\
&\times\left[\left(\tfrac{1}{(1+q)^{s+1}}\tfrac{1-q}{1-q^{s+1}}\right)^{\frac{1}{s}}\left(\tfrac{1}{1+q}\right)^{\frac{1}{r}}+\left(\int_{\frac{1}{1+q}}^1\left(\tfrac{1}{q}-t\right)^s{}_0d_qt\right)^{\frac{1}{s}}\left(\tfrac{q}{1+q}\right)^{\frac{1}{r}}\right]
\end{aligned}$$

Thus the proof is accomplished. □

Corollary 26. *In Theorem 25, if we take $q\to 1^-$, we have the following midpoint type inequality for quasi-convex functions:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)}{2}\left(\frac{1}{s+1}\right)^{\frac{1}{s}}\sup\{|f'(a)|,|f'(b)|\}. \tag{4.20}$$

Competing interests

The authors declare that they have no competing interests.

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