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# Average kinetic energy of heavy quark ($\mu_\pi^2$) inside heavy meson in $0^-$ state by Bethe–Salpeter method

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**Abstract**

The average kinetic energy of the heavy quark inside $B$ or $D$ meson is computed by means of the instantaneous Bethe–Salpeter method. We first solve the relativistic Salpeter equation and obtain the relativistic wave function and mass of $0^-$ state, then we use the relativistic wave function to calculate the average kinetic energy of the heavy quark inside heavy meson of $0^-$ state. We find that the relativistic corrections to the average kinetic energy of the heavy quark inside $B$ or $D$ meson are quite large and cannot be ignored. We estimate $\mu_\pi^2(=-\lambda_1)\approx 0.24(B^0, B^\pm)$, $0.20(D^0, D^\pm)$, $0.33(B_s)$, $0.26(D_s)$, $0.83(B_c)$ and $0.62(\eta_c)$ GeV$^2$.



## 1. Introduction

In recent years, the study of hadronic processes involving heavy quarks has attracted continuous interest both in experiment and in theory. The difficulty of full theory of QCD, which is dynamic theory describing the quark and gluon, lead us to the theoretical achievements of the heavy quark effective theory (HQET) [1]. The latter describes the dynamics of heavy hadrons, i.e., hadrons containing a heavy quark $Q$, when $m_Q \to \infty$. The theory is based upon an effective Lagrangian written in terms of effective fields, which is a systematic expansion in the inverse powers of the heavy quark mass $m_Q$. The $\mathcal{O}(\frac{1}{m_Q})$ Lagrangian reads as follows:

$$\mathcal{L} = \bar{h}_v i v\cdot D h_v + \frac{1}{2m_Q}\bar{h}_v\big[(iD_\perp)^2\big]h_v + \frac{g_s}{2m_Q}\bar{h}_v\frac{\sigma_{\mu\nu}G^{\mu\nu}}{2}h_v + \mathcal{O}\left(\frac{1}{m_Q^2}\right), \tag{1}$$

where the velocity-dependent field $h_v$ is the heavy quark field, and $v_\mu$ is the heavy quark four-velocity within the heavy hadron. Then the total momentum is written as $p_Q = m_Q v + q$, where the residual momentum $q$ is the difference between the total momentum and the mechanical momentum; $D^\mu = \partial^\mu - igA^\mu$ is the covariant derivative, and $D_\perp^\mu = D^\mu - v^\mu v\cdot D$ contains its components perpendicular to the hadron velocity. In the hadron's rest frame we have $(iD_\perp)^2 = \vec{D}^2$. The second operator appearing in Eq. (1) corresponds to the kinetic energy

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resulting from the residual motion of the heavy quark, and the third one in Eq. (1) the Pauli chromomagnetic interaction operator which describes the interaction of the heavy quark spin with the chromomagnetic gluon field. Their matrix elements can be parameterized as follows [2]:

$$\mu_\pi^2(H_Q) = \frac{\langle H_Q|\bar{h}_v(\vec{D})^2 h_v|H_Q\rangle}{2M_H}, \tag{2}$$

$$\mu_G^2(H_Q) = \frac{\langle H_Q|\bar{h}_v \frac{g}{2}\sigma_{\mu\nu}G^{\mu\nu}h_v|H_Q\rangle}{2M_H}, \tag{3}$$

where $H_Q$ denotes generically a hadron containing the heavy quark $Q$ with the usual normalization $\langle H_Q|\bar{h}_v h_v \times |H_Q\rangle = 2M_H$.

These two quantities are interesting for several reasons. In the HQET, heavy hadron mass is expected to scale with the heavy quark mass $m_Q$ as:

$$M_H = m_Q + \bar{\Lambda} + \frac{\mu_\pi^2 - \mu_G^2}{2m_Q} + \cdots, \tag{4}$$

where $\bar{\Lambda}$ represents the difference between the mass of the hadron and that of the heavy quark in the $m_Q \to \infty$ limit. In this limit, it can be related to the trace anomaly of QCD [3]:

$$\bar{\Lambda} = \frac{1}{2M_H}\langle H_Q|\frac{\beta(\alpha_s)}{4\alpha_s}G^{\mu\nu}G_{\mu\nu}|H_Q\rangle,$$

where $\beta$ is the Gell-Mann-low function. Moreover, if the inclusive semileptonic width of a heavy hadron is calculated by an expansion in the powers of $1/m_Q$, the following results are found: the leading term of the expansion coincides with the free quark decay rate (spectator model); no corrections of order $1/m_Q$ appear in the rate, and the $1/m_Q^2$ corrections depend on $\mu_\pi^2$ and $\mu_G^2$ [17]. As a consequence, these parameters enter in the ratio of hadron lifetimes and in the lepton spectrum in inclusive transitions, which in principle are quantities directly comparable with experimental data. Many authors have given theoretical estimates of $\mu_\pi^2$ and $\mu_G^2$ using different methods, but different results are obtained for the estimation of $\mu_\pi^2$ (see Table 1). Even though there may be different definitions of these two quantities, our knowledge of them is still far from clear due to large discrepancies, and a more careful study is still needed.

In this Letter, we give a relativistically calculated version of $\mu_\pi^2$, i.e., we calculate the average kinetic energy of the heavy quark inside heavy meson in $0^-$ state by means of the Bethe–Salpeter method [18]. We solve the relativistic Salpeter equation [19] in Section 2, and give the mass and relativistic wave functions of heavy meson in $0^-$ state in Section 3. Finally, we use these relativistic wave functions to calculate the average kinetic energy of the heavy quark in Section 4. Discussions and conclusions are also in Section 4.

## 2. Instantaneous Bethe–Salpeter method

It has been known that the Bethe–Salpeter (BS) equation is one of the frameworks to describe bound state systems relativistically and has a very solid basis in quantum field theory. So it is very often used to describe bound state problems, and even in the current literature many authors would like to base the constituent quark model on the BS equation. For instance, in the constituent quark model the mesons, corresponding quark–antiquark bound states, are described by the BS equation as:

$$(\not{p}_Q - m_Q)\chi(q)(\not{p}_q + m_q) = i\int \frac{d^4k}{(2\pi)^4}V(p,k,q)\chi(k), \tag{5}$$

Table 1
Theoretical estimates of the parameter $\mu_\pi^2$ of $B_{u,d}$ (QCDSR: QCD sum rules, HQSR: heavy-quark sum rules, Exp.: experimental data on inclusive decays, QM: quark models)

| Reference | Method | $\mu_\pi^2$ [GeV$^2$] |
|---|---|---|
| Eletsky, Shuryak [4] | QCDSR | $0.18 \pm 0.06$ |
| Ball, Braun [5] | QCDSR | $0.52 \pm 0.12$ |
| Neubert [6] | QCDSR | $0.10 \pm 0.05$ |
| Giménez et al. [7] | Lattice | $-0.09 \pm 0.14$ |
| Kronfeld et al. [8] | Lattice | $0.45 \pm 0.12$ |
| Bigi et al. [3] | HQSR | $> 0.36$ |
| Gremm et al. [9] | Exp. | $0.19 \pm 0.10$ |
| Falk et al. [10] | Exp. | $0.1 \to 0.16$ |
| Chernyak [11] | Exp. | $0.14 \pm 0.03$ |
| Battaglia et al. [12] | Exp. | 0.17 |
| Hwang et al. [13] | QM | $0.4 \to 0.6$ |
| De Fazio [14] | QM | $0.66 \pm 0.13$ |
| Simula [15] | QM | $-0.089$ |
| Matsuki et al. [16] | QM | 0.238 |

where $\chi(q)$ is the BS wave function with the total momentum $p$ and relative momentum $q$, and $V(p,k,q)$ is the kernel between the quarks in the bound state. The momenta $p_Q$, $p_q$ are those of the constituent quarks 1 and 2: For a heavy meson with a heavy and a light valence quark, we can treat one of these two constituents as a heavy quark and the other as a light quark, e.g., we treat the quark as the heavy quark $p_1 = p_Q$ and the anti-quark as the light quark $p_2 = p_q$. The total momentum $p$ and the relative momentum $q$ are defined as:

$$p_Q = \alpha_1 p + q, \quad \alpha_1 = \frac{m_Q}{m_Q + m_q}, \qquad p_q = \alpha_2 p - q, \quad \alpha_2 = \frac{m_q}{m_Q + m_q}.$$

One can see that these definitions are just the same as in the HQET, where $\alpha_1 p$ is the mechanical momentum of the heavy quark which describes the heavy quark moving together with the meson, and the relative momentum $q$ is nothing but the residual momentum of the heavy quark inside meson. However, our method is not the HQET and we do not have the limit of $m_Q \to \infty$, so the light quark momentum have the meaning analogous to that of the heavy quark.

The BS wave function $\chi(q)$ should satisfy the following normalization condition:

$$\int \frac{d^4k\, d^4q}{(2\pi)^4} \mathrm{Tr}\left[\bar{\chi}(k) \frac{\partial}{\partial p_0}\left[S_1^{-1}(p_Q) S_2^{-1}(p_q) \delta^4(k-q) + V(p,k,q)\right]\chi(q)\right] = 2ip_0, \tag{6}$$

where $S_1(p_Q)$ and $S_2(p_q)$ are the propagators of the two constituents. In many applications, the kernel of the four-dimensional BS equation is "instantaneous", i.e., in the center of mass frame of the concerned bound state ($\vec{p} = 0$), the kernel $V(p,k,q)$ of the BS equation takes the simple form:

$$V(p,k,q) \Rightarrow V(k,q) = V\left(|\vec{k}|, |\vec{q}|, \cos\theta\right),$$

where $\theta$ is the angle between the vectors $\vec{k}$ and $\vec{q}$. Then the BS equation may be reduced to a three-dimensional one. Compared with the conditions to solve a three-dimensional equation, i.e., to evaluate its eigenvalues and eigenfunctions, the conditions to solve a four-dimensional one are much more complicated. Thus if the kernel of the BS equation for the considered problem is instantaneous, then we always would like to do the 'reduction' from four-dimensional to three-dimensional. Salpeter was the first to do this reduction, so the reduced BS equation with instantaneous kernel is also called the Salpeter equation. Here we briefly repeat his method and solve the full Salpeter equation. This equation is relativistic although it has an instantaneous kernel, so we will obtain the relativistic wave function of bound state.

Since in the HQET the heavy quark momentum is described by using the covariant derivative $D_\mu = \partial_\mu - igA_\mu$, and the kinetic energy of the residual motion of the heavy quark by using a covariant form $D_\perp$, it is convenient to write the BS equation in a covariant form. To do this, we divide the relative momentum $q$ into two parts, $q_\parallel$ and $q_\perp$, a parallel part and an orthogonal one to the total momentum of the bound state, respectively,

$$q^\mu = q_\parallel^\mu + q_\perp^\mu, \quad q_\parallel^\mu \equiv \left(p \cdot q/M_H^2\right)p^\mu, \quad q_\perp^\mu \equiv q^\mu - q_\parallel^\mu.$$

Correspondingly, we have two Lorentz invariant variables:

$$q_p = \frac{(p \cdot q)}{M_H}, \qquad q_T = \sqrt{q_p^2 - q^2} = \sqrt{-q_\perp^2}.$$

In the center of mass frame $\vec{p} = 0$, they turn out to be the usual component $q_0$ and $|\vec{q}|$, respectively. One can see that in the rest frame of bound state the orthogonal residual momentum of the heavy quark is just the orthogonal relative momentum, i.e., $i\vec{D} = \vec{q}$. Now the volume element of the relative momentum $k$ can be written in an invariant form:

$$d^4k = dk_p\, k_T^2\, dk_T\, ds\, d\phi, \tag{7}$$

where $\phi$ is the azimuthal angle, $s = (k_p q_p - k \cdot q)/(k_T q_T)$. The instantaneous interaction kernel can be rewritten as:

$$V\left(\left|\vec{k} - \vec{q}\right|\right) = V(k_\perp, q_\perp, s). \tag{8}$$

Let us introduce the notations $\varphi_p(q_\perp^\mu)$ and $\eta(q_\perp^\mu)$ for three-dimensional wave function as follows:

$$\varphi_p\left(q_\perp^\mu\right) \equiv i\int \frac{dq_p}{2\pi}\chi\left(q_\parallel^\mu, q_\perp^\mu\right), \qquad \eta\left(q_\perp^\mu\right) \equiv \int \frac{k_T^2\, dk_T\, ds}{(2\pi)^2} V(k_\perp, q_\perp, s)\varphi_p\left(k_\perp^\mu\right). \tag{9}$$

Then the BS equation can be rewritten as:

$$\chi(q_\parallel, q_\perp) = S_1(p_Q)\eta(q_\perp)S_2(p_q). \tag{10}$$

The propagators of the two constituents can be decomposed as:

$$S_i(p_i) = \frac{\Lambda_{ip}^+(q_\perp)}{J(i)q_p + \alpha_i M_H - \omega_{ip} + i\epsilon} + \frac{\Lambda_{ip}^-(q_\perp)}{J(i)q_p + \alpha_i M_H + \omega_{ip} - i\epsilon}, \tag{11}$$

with

$$\omega_{ip} = \sqrt{m_i^2 + q_T^2}, \qquad \Lambda_{ip}^\pm(q_\perp) = \frac{1}{2\omega_{ip}}\left[\frac{\not{p}}{M_H}\omega_{ip} \pm J(i)(m_i + \not{q}_\perp)\right], \tag{12}$$

where $i = 1, 2$ for heavy quark and light anti-quark, respectively, $\omega_{1p} = \omega_Q$, $\omega_{2p} = \omega_q$, and $J(i) = (-1)^{i+1}$. Here $\Lambda_{ip}^\pm(q_\perp)$ satisfy the relations:

$$\Lambda_{ip}^+(q_\perp) + \Lambda_{ip}^-(q_\perp) = \frac{\not{p}}{M_H}, \qquad \Lambda_{ip}^\pm(q_\perp)\frac{\not{p}}{M_H}\Lambda_{ip}^\pm(q_\perp) = \Lambda_{ip}^\pm(q_\perp), \qquad \Lambda_{ip}^\pm(q_\perp)\frac{\not{p}}{M_H}\Lambda_{ip}^\mp(q_\perp) = 0. \tag{13}$$

Due to these equations, $\Lambda^\pm$ may be considered as $p$-projection operators, while in the rest frame $\vec{p} = 0$ they turn to be the energy projection operator.

Introducing the notations $\varphi_p^{\pm\pm}(q_\perp)$ as:

$$\varphi_p^{\pm\pm}(q_\perp) \equiv \Lambda_{1p}^\pm(q_\perp)\frac{\not{p}}{M_H}\varphi_p(q_\perp)\frac{\not{p}}{M_H}\Lambda_{2p}^\pm(q_\perp), \tag{14}$$

and taking into account $\frac{\not p}{M_H}\frac{\not p}{M_H}=1$, we have

$$\varphi_p(q_\perp)=\varphi_p^{++}(q_\perp)+\varphi_p^{+-}(q_\perp)+\varphi_p^{-+}(q_\perp)+\varphi_p^{--}(q_\perp).$$

With contour integration over $q_p$ on both sides of Eq. (10), we obtain:

$$\varphi_p(q_\perp)=\frac{\Lambda_{1p}^{+}(q_\perp)\eta_p(q_\perp)\Lambda_{2p}^{+}(q_\perp)}{(M_H-\omega_Q-\omega_q)}-\frac{\Lambda_{1p}^{-}(q_\perp)\eta_p(q_\perp)\Lambda_{2p}^{-}(q_\perp)}{(M_H+\omega_Q+\omega_q)},$$

and we may decompose it further into four equations as follows:

$$(M_H-\omega_Q-\omega_q)\varphi_p^{++}(q_\perp)=\Lambda_{1p}^{+}(q_\perp)\eta_p(q_\perp)\Lambda_{2p}^{+}(q_\perp),$$
$$(M_H+\omega_Q+\omega_q)\varphi_p^{--}(q_\perp)=-\Lambda_{1p}^{-}(q_\perp)\eta_p(q_\perp)\Lambda_{2p}^{-}(q_\perp),\qquad \varphi_p^{+-}(q_\perp)=\varphi_p^{-+}(q_\perp)=0. \tag{15}$$

In Ref. [19], Salpeter considered the factor $(M_H-\omega_Q-\omega_q)$ being small, so he kept the first of Eq. (15) only. It is the ‘original’ instantaneous approximation proposed by Salpeter and followed by many authors in the literature. Whereas in this Letter we re-examine the BS equation with an instantaneous kernel, i.e., we try to deal with it exactly including the second of Eq. (15). The complete normalization condition (keeping all the four components appearing in Eq. (15)) for BS equation turns out to be:

$$\int\frac{q_T^2\,dq_T}{(2\pi)^2}\,\mathrm{tr}\left[\bar\varphi^{++}\frac{\not p}{M_H}\varphi^{++}\frac{\not p}{M_H}-\bar\varphi^{--}\frac{\not p}{M_H}\varphi^{--}\frac{\not p}{M_H}\right]=2p_0. \tag{16}$$

To solve the eigenvalue equation, one has to choose a definite kernel of the quark and anti-quark in the bound state. As usual we choose the Cornell potential, a linear scalar interaction (confinement one) plus a vector interaction (single gluon exchange):

$$I(r)=V_s(r)+V_0+\gamma_0\otimes\gamma^0V_v(r)=\lambda r+V_0-\gamma_0\otimes\gamma^0\frac{4}{3}\frac{\alpha_s}{r}, \tag{17}$$

where $\lambda$ is the string constant, $\alpha_s(r)$ is the running coupling constant. Usually, in order to fit the data of heavy quarkonia, a constant $V_0$ is often added to the scalar confining potential.

It is clear that there exists infrared divergence in the Coulomb-like potential. In order to avoid it, we introduce a factor $e^{-\alpha r}$:

$$V_s(r)=\frac{\lambda}{\alpha}\left(1-e^{-\alpha r}\right),\qquad V_v(r)=-\frac{4}{3}\frac{\alpha_s}{r}e^{-\alpha r}. \tag{18}$$

It is easy to show that when $\alpha r\ll 1$, the potential becomes identical with the original one. In the momentum space and the rest frame of the bound state, the potential reads:

$$I(\vec q)=V_s(\vec q)+\gamma_0\otimes\gamma^0V_v(\vec q),\quad V_s(\vec q)=-\left(\frac{\lambda}{\alpha}+V_0\right)\delta^3(\vec q)+\frac{\lambda}{\pi^2}\frac{1}{(\vec q^{\,2}+\alpha^2)^2},$$
$$V_v(\vec q)=-\frac{2}{3\pi^2}\frac{\alpha_s(\vec q)}{(\vec q^{\,2}+\alpha^2)}. \tag{19}$$

The coupling constant $\alpha_s(\vec q)$ is running:

$$\alpha_s(\vec q)=\frac{12\pi}{27}\frac{1}{\log\left(a+\frac{\vec q^{\,2}}{\Lambda_{\mathrm{QCD}}^2}\right)}.$$

Here the constants $\lambda$, $\alpha$, $a$, $V_0$ and $\Lambda_{\mathrm{QCD}}$ are the parameters that characterize the potential.

## 3. Heavy mesons in $0^-$ state

Following the method [20], the general form for the relativistic Salpeter wave function of the bound state $J^P = 0^-$ can be written as (in the center of mass system):

$$\varphi_{1S_0}(\vec{q}) = M_H\left[\gamma_0\varphi_1(\vec{q}) + \varphi_2(\vec{q}) + \frac{\not{q}_\perp}{M_H}\varphi_3(\vec{q}) + \frac{\gamma_0\not{q}_\perp}{M_H}\varphi_4(\vec{q})\right]\gamma_5, \tag{20}$$

where $q_\perp = (0, \vec{q})$, and $M_H$ is the mass of the corresponding meson. The equations

$$\varphi^{+-}_{1S_0}(\vec{q}) = \varphi^{-+}_{1S_0}(\vec{q}) = 0$$

give the constraints on the components of the wave function:

$$\varphi_3(\vec{q}) = \frac{\varphi_2(\vec{q})M_H(-\omega_Q + \omega_q)}{m_q\omega_Q + m_Q\omega_q}, \qquad \varphi_4(\vec{q}) = -\frac{\varphi_1(\vec{q})M_H(\omega_Q + \omega_q)}{m_q\omega_Q + m_Q\omega_q}.$$

Then we can rewrite the relativistic wave function of state $0^-$ as:

$$\varphi_{1S_0}(\vec{q}) = M_H\left[\gamma_0\varphi_1(\vec{q}) + \varphi_2(\vec{q}) - \not{q}_\perp\varphi_2(\vec{q})\frac{(\omega_Q - \omega_q)}{(m_q\omega_Q + m_Q\omega_q)} + \not{q}_\perp\gamma_0\varphi_1(\vec{q})\frac{(\omega_Q + \omega_q)}{(m_q\omega_Q + m_Q\omega_q)}\right]\gamma_5. \tag{21}$$

From this wave function we can obtain the wave functions corresponding to the positive and the negative projection, respectively:

$$\begin{aligned}\varphi^{++}_{1S_0}(\vec{q}) = \frac{M_H}{2}\Bigg[&\left(\varphi_1(\vec{q}) + \varphi_2(\vec{q})\frac{\omega_Q - \omega_q}{m_Q - m_q}\right)\left(\frac{m_Q - m_q}{\omega_Q - \omega_q} + \gamma_0 - \frac{\not{q}_\perp(m_Q - m_q)}{m_q\omega_Q + m_Q\omega_q}\right)\\ &+ \frac{\not{q}_\perp\gamma_0(\omega_Q + \omega_q)}{(m_q\omega_Q + m_Q\omega_q)}\left(\varphi_1(\vec{q}) + \varphi_2(\vec{q})\frac{m_Q + m_q}{\omega_Q + \omega_q}\right)\Bigg]\gamma_5,\end{aligned} \tag{22}$$

$$\begin{aligned}\varphi^{--}_{1S_0}(\vec{q}) = \frac{M_H}{2}\Bigg[&\left(-\varphi_1(\vec{q}) + \varphi_2(\vec{q})\frac{\omega_Q - \omega_q}{m_Q - m_q}\right)\left(\frac{m_Q - m_q}{\omega_Q - \omega_q} - \gamma_0 - \frac{\not{q}_\perp(m_Q - m_q)}{m_q\omega_Q + m_Q\omega_q}\right)\\ &+ \frac{\not{q}_\perp\gamma_0(\omega_Q + \omega_q)}{(m_q\omega_Q + m_Q\omega_q)}\left(\varphi_1(\vec{q}) - \varphi_2(\vec{q})\frac{m_Q + m_q}{\omega_Q + \omega_q}\right)\Bigg]\gamma_5.\end{aligned} \tag{23}$$

And there are two more equations from the reduced BS equation (15), which will give us coupled integral equations, and by solving them we obtain the numerical results for the mass and the wave function:

$$\begin{aligned}&(M_H - \omega_Q - \omega_2)\left[\varphi_1(\vec{q}) + \varphi_2(\vec{q})\frac{\omega_Q - \omega_q}{m_Q - m_q}\right]\\ &\quad = -\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{2\omega_Q\omega_q(E_Qm_q + E_qm_Q)}\\ &\qquad \times\big\{(E_Qm_q + E_qm_Q)(V_s - V_v)\big[\varphi_1(\vec{k})\big(\omega_Q\omega_q + m_Qm_q - \vec{q}^{\,2}\big) + \varphi_2(\vec{k})(m_q\omega_Q + m_Q\omega_q)\big]\\ &\qquad\quad - (V_s + V_v)\big[\varphi_1(\vec{k})(m_Q + m_q)(E_Q + E_q) + \varphi_2(\vec{k})(\omega_Q - \omega_q)(E_Q - E_q)\big]\vec{q}\cdot\vec{k}\big\},\end{aligned} \tag{24}$$

$$\begin{aligned}&(M_H + \omega_Q + \omega_q)\left[\varphi_1(\vec{q}) - \varphi_2(\vec{q})\frac{\omega_Q - \omega_q}{m_Q - m_q}\right]\\ &\quad = \int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{2\omega_Q\omega_q(E_Qm_q + E_qm_Q)}\\ &\qquad \times\big\{(E_Qm_q + E_qm_Q)(V_s - V_v)\big[\varphi_1(\vec{k})\big(\omega_Q\omega_q + m_Qm_q - \vec{q}^{\,2}\big) - \varphi_2(\vec{k})(m_q\omega_Q + m_Q\omega_q)\big]\\ &\qquad\quad - (V_s + V_v)\big[\varphi_1(\vec{k})(m_Q + m_q)(E_Q + E_q) - \varphi_2(\vec{k})(\omega_Q - \omega_q)(E_Q - E_q)\big]\vec{q}\cdot\vec{k}\big\},\end{aligned} \tag{25}$$

where $E_Q = \sqrt{m_Q^2 + k_T^2}$ and $E_q = \sqrt{m_q^2 + k_T^2}$. Finally the normalization condition is

$$\int \frac{d\vec{q}}{(2\pi)^3} 4\varphi_1(\vec{q})\varphi_2(\vec{q}) M_H^2 \left\{ \frac{\omega_Q - \omega_q}{m_Q - m_q} + \frac{m_Q - m_q}{\omega_Q - \omega_q} + \frac{2\vec{q}^{\,2}(\omega_Q m_Q + \omega_q m_q)}{(\omega_Q m_q + \omega_q m_Q)^2} \right\} = 2M_H. \tag{26}$$

## 4. Average kinetic energy of heavy quark inside heavy mesons in $0^-$ state

The average kinetic energy of the heavy quark inside heavy meson in $0^-$ state, in the BS method, is proportional to the average spatial momentum squared:

$$\mu_\pi^2 = \int \frac{d\vec{q}\,\vec{q}^{\,2}}{(2\pi)^3} 2\varphi_1(\vec{q})\varphi_2(\vec{q}) M_H \left\{ \frac{\omega_Q - \omega_q}{m_Q - m_q} + \frac{m_Q - m_q}{\omega_Q - \omega_q} + \frac{2\vec{q}^{\,2}(\omega_Q m_Q + \omega_q m_q)}{(\omega_Q m_q + \omega_q m_Q)^2} \right\}. \tag{27}$$

In order to solve numerically the relativistic Salpeter equation, we use three different groups of input parameters (i.e., parameters for the potential and the masses of quarks), as shown in Table 2, from the best fit values [21]:

$$a = e = 2.7183, \quad \alpha = 0.06 \text{ GeV}, \quad V_0 = -0.60 \text{ GeV}, \quad \lambda = 0.2 \text{ GeV}^2, \quad \Lambda_{\text{QCD}} = 0.26 \text{ GeV} \quad \text{and}$$
$$m_b = 5.224 \text{ GeV}, \quad m_c = 1.7553 \text{ GeV}, \quad m_s = 0.487 \text{ GeV}, \quad m_d = 0.311 \text{ GeV}, \quad m_u = 0.305 \text{ GeV}.$$

With these three input parameter sets, we now solve the full Salpeter equation and obtain the masses and wave functions of the ground $0^-$ states. We list the calculated mass spectra of some $0^-$ states as well as the measured experimental values in Table 3. Then, by using the obtained wave function of heavy meson, we calculated $\mu_\pi^2$ from Eq. (27), as shown in Table 3. As can be seen from Table 3, if we change the values of the input parameters (sets 1–3) used in solving the Salpeter equation, we find that the obtained values of $\mu_\pi^2$ are almost unchanged (especially for $B_d$, $B_u$, $D_d$ and $D_u$ mesons) when these parameters give a reasonably good fit of mass spectra. Therefore, we notice that our results for $\mu_\pi^2$ are quite insensitive to the model parameters within the instantaneous BS method. We also note that the average kinetic energies of the heavy quark in different mesons differ significantly even when the heavy quark is the same, e.g., the value of $\mu_\pi^2$ of the heavy quark is significantly larger in $B_s$ meson ($\approx 0.33$ GeV$^2$)

Table 2
Three sets (1–3) of input parameters. $\lambda$ is in the unit of GeV$^2$, others are in the unit of GeV

| Set | $\alpha$ | $V_0$ | $\lambda$ | $\Lambda_{\text{QCD}}$ | $m_b$ | $m_c$ | $m_s$ | $m_d$ | $m_u$ |
|---|---|---|---|---|---|---|---|---|---|
| (1) | 0.060 | −0.60 | 0.20 | 0.26 | 5.224 | 1.7553 | 0.487 | 0.311 | 0.305 |
| (2) | 0.055 | −0.40 | 0.19 | 0.24 | 5.130 | 1.660 | 0.428 | 0.285 | 0.278 |
| (3) | 0.063 | −0.787 | 0.21 | 0.275 | 5.3136 | 1.845 | 0.557 | 0.352 | 0.3465 |

Table 3
Mass spectra and $\mu_\pi^2$, for heavy mesons in $0^-$ states with three sets (1–3) of input parameters. 'Ex' means the results from experiments [22] and 'ER' is the error of experimental values. 'Th' means the results from our theoretical estimate

| | $B_c$ | $B_s$ | $B_d$ | $B_u$ | $\eta_c$ | $D_s$ | $D_d$ | $D_u$ |
|---|---|---|---|---|---|---|---|---|
| $M$ GeV(Ex) | 6.4 | 5.3696 | 5.2794 | 5.2790 | 2.9797 | 1.9685 | 1.8693 | 1.8645 |
| ER of Ex | ±0.4 | ±0.0024 | ±0.0005 | 0.0005 | ±0.0015 | ±0.0006 | ±0.0005 | ±0.0005 |
| $M$ GeV(Th)(1) | 6.296 | 5.3654 | 5.2804 | 5.2778 | 2.9791 | 1.9688 | 1.8687 | 1.8655 |
| $M$ GeV(Th)(2) | 6.304 | 5.3670 | 5.2804 | 5.2762 | 2.9795 | 1.9691 | 1.8699 | 1.8650 |
| $M$ GeV(Th)(3) | 6.292 | 5.3656 | 5.2806 | 5.2788 | 2.9799 | 1.9690 | 1.8673 | 1.8650 |
| $\mu_\pi^2$ GeV$^2$(1) | 0.828 | 0.329 | 0.245 | 0.242 | 0.615 | 0.259 | 0.200 | 0.198 |
| $\mu_\pi^2$ GeV$^2$(2) | 0.802 | 0.32 | 0.248 | 0.244 | 0.596 | 0.249 | 0.199 | 0.196 |
| $\mu_\pi^2$ GeV$^2$(3) | 0.856 | 0.344 | 0.251 | 0.248 | 0.636 | 0.273 | 0.207 | 0.205 |

Table 4
The calculated uncertainties (in per cents) if we allow changes of all input parameters simultaneously within 5% of the central values

| | $B_c$ | $B_s$ | $B_d$ | $B_u$ | $\eta_c$ | $D_s$ | $D_d$ | $D_u$ |
|---|---|---|---|---|---|---|---|---|
| $\Delta M/M$ | ±6.5 | ±6.0 | ±5.8 | ±5.8 | ±7.2 | ±7.5 | ±7.3 | ±7.2 |
| $\Delta\mu_\pi^2/\mu_\pi^2$ | ±13.5 | ±11.0 | ±10.5 | ±11.0 | ±9.5 | ±10.5 | ±10.6 | ±10.8 |

than in $B_d$ ($\approx 0.25$ GeV$^2$) or $B_u$ meson ($\approx 0.24$ GeV$^2$). The difference of about 0.08 GeV$^2$ is not a value which can be ignored compared with the value of $\mu_\pi^2$ itself. The bigger value of $\mu_\pi^2$ inside $B_s$ meson than inside $B_d$ or $B_u$ means that $b$ quark has a smaller residual momentum in $B_d$ or $B_u$ than in $B_s$. This implies that $b$ quark is bounded more deeply in $B_d$ or $B_u$ than in $B_s$ meson. In other words, the kinetic energy of the same $b$ quark in heavy meson is more restrained by a light partner quark than by a heavy one, which is consistent with the running behavior of $\alpha_s$. Since our calculation of the average kinetic energy of the heavy quark has used the relativistic wave functions obtained from the full Salpeter equation, our results of the average kinetic energy $\mu_\pi^2$ are relativistic. Note that our results are quite different from the previously estimated ones of the potential model [13–15]. This shows that the relativistic corrections are quite large, and cannot be ignored.

In Table 4, we also show the calculated theoretical uncertainties for our results of the mass and average kinetic energy when we allow variations of all the input parameters simultaneously within 5% range of the central values. Our results are very close to the theoretical results of Matsuki and Morii [16], which included the second order correction of $1/m_Q$. In comparison, our result for $B_{u,d}$

$$\mu_\pi^2 \approx 0.22\text{–}0.26\ \text{GeV}^2 \quad \text{(our estimate)}$$

is very close to the recently experimentally derived CLEO values of

$$\mu_\pi^2 = 0.25 \pm 0.05 \quad [23]$$

and

$$\mu_\pi^2 = 0.24 \pm 0.11 \quad [24].$$

In conclusion, we calculated the average kinetic energy of the heavy quark inside $B$ or $D$ meson by means of the instantaneous Bethe–Salpeter method. We solved the relativistic Salpeter equation and obtained the relativistic wave function and mass of $0^-$ state. Then we used the relativistic wave function to calculate the average kinetic energy of the heavy quark inside the heavy $0^-$ state. We obtained $\mu_\pi^2(=-\lambda_1) \approx 0.24(B^0, B^\pm)$, $0.20(D^0, D^\pm)$, $0.33(B_s)$, $0.26(D_s)$, $0.83(B_c)$ and $0.62(\eta_c)$ GeV$^2$.

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