THE ANALYTIC HIERARCHY PROCESS—WHAT IT IS AND HOW IT IS USED

R. W. SAATY

4922 Ellsworth Avenue, Pittsburgh, PA 15213, U.S.A.

Abstract—Here we introduce the Analytic Hierarchy Process as a method of measurement with ratio scales and illustrate it with two examples. We then give the axioms and some of the central theoretical underpinnings of the theory. Finally, we discuss some of the ideas relating to this process and its ramifications. In this paper we give special emphasis to departure from consistency and its measurement and to the use of absolute and relative measurement, providing examples and justification for rank preservation and reversal in relative measurement.

1. INTRODUCTION

The Analytic Hierarchy Process (AHP) is a general theory of measurement. It is used to derive ratio scales from both discrete and continuous paired comparisons. These comparisons may be taken from actual measurements or from a fundamental scale which reflects the relative strength of preferences and feelings. The AHP has a special concern with departure from consistency, its measurement and on dependence within and between the groups of elements of its structure. It has found its widest applications in multicriteria decision making, planning and resource allocation and in conflict resolution [1–6]. In its general form the AHP is a nonlinear framework for carrying out both deductive and inductive thinking without use of the syllogism by taking several factors into consideration simultaneously and allowing for dependence and for feedback, and making numerical tradeoffs to arrive at a synthesis or conclusion. T. L. Saaty developed the AHP in 1971–1975 while at the Wharton School (University of Pennsylvania, Philadelphia, Pa).

This paper is written in an expository style to make it more accessible to the nonmathematician interested in the subject. Our purpose is three-fold: first to introduce the AHP through two hierarchically structured examples (in contrast with more elaborate feedback–network structures); next to give the axioms and some of the central theoretical underpinnings of the theory; and finally, to discuss some of the ideas relating to the theory and its application that are most relevant in using the AHP. In particular, we write about its generalizations and about where to use absolute or relative measurement.

For a long time people have been concerned with the measurement of both physical and psychological events. By physical we mean the realm of what is fashionably known as the tangibles as it relates to some kind of objective reality outside the individual conducting the measurement. By contrast, the psychological is the realm of the intangibles as it relates to subjective ideas and beliefs of the individual about himself or herself and the world of experience. The question is whether there is a coherent theory that can deal with both these worlds of reality without compromising either. The AHP is a method that can be used to establish measures in both the physical and social domains.

In using the AHP to model a problem one needs a hierarchic or a network structure to represent that problem and pairwise comparisons to establish relations within the structure. In the discrete case these comparisons lead to dominance matrices and in the continuous case to kernels of Fredholm operators [7], from which ratio scales are derived in the form of principal eigenvectors, or eigenfunctions, as the case may be. These matrices, or kernels, are positive and reciprocal, e.g. $a_{ij} = 1/a_{ji}$. In particular, special efforts have been made to characterize these matrices, and several of the more theoretical papers in this issue address questions and offer new information about them. Because of the need for a variety of judgments, there has also been considerable work done to characterize the process of synthesizing diverse judgments [8].

2. TWO EXAMPLES

In general a hierarchical model of some societal problem might be one that descends from a focus (an overall objective), down to criteria, down further to subcriteria which are subdivisions of the criteria and finally to the alternatives from which the choice is to be made. Our first example is an application of the AHP developed by Hämäläinen and Seppäläinen [9] to solve a large-scale socio-technical decision problem with intangible criteria in Finland.

The parliament of Finland was faced with making a decision about what type of power plant to build. Members of parliament were concerned with how the new power plant would affect Finland's national economy; the health, safety and environment for Finnish citizens; and how political factors such as Finnish relations with the U.S.S.R. affect the type of plant to be adopted. Their goal was to build a power plant that would best serve the overall welfare of the nation. Each of the main criteria was further decomposed into subcriteria, followed by the alternatives: the different kinds of power plants. A simple hierarchy of the problem is shown in Fig. 1.

There has been extensive work on how to structure hierarchies for practical problems. Two general types of hierarchies are the forward and the backward process hierarchies. All problems have been found to fall into one or the other of these two categories. Planning combines them in an iterative fashion [4]. The elements of a hierarchy are grouped in clusters according to homogeneity (see Axiom 2 in Section 3) and a level may consist of one or several homogeneous clusters. The elements in each level may be regarded as constraints, refinements or decompositions of the elements above. In a "complete" hierarchy, as in the example of "choosing the best college" given below, all the elements in one level have all the elements in the succeeding level as descendants. In this case the levels are single homogeneous clusters. Otherwise a hierarchy is "incomplete", as in the Finnish energy example below.
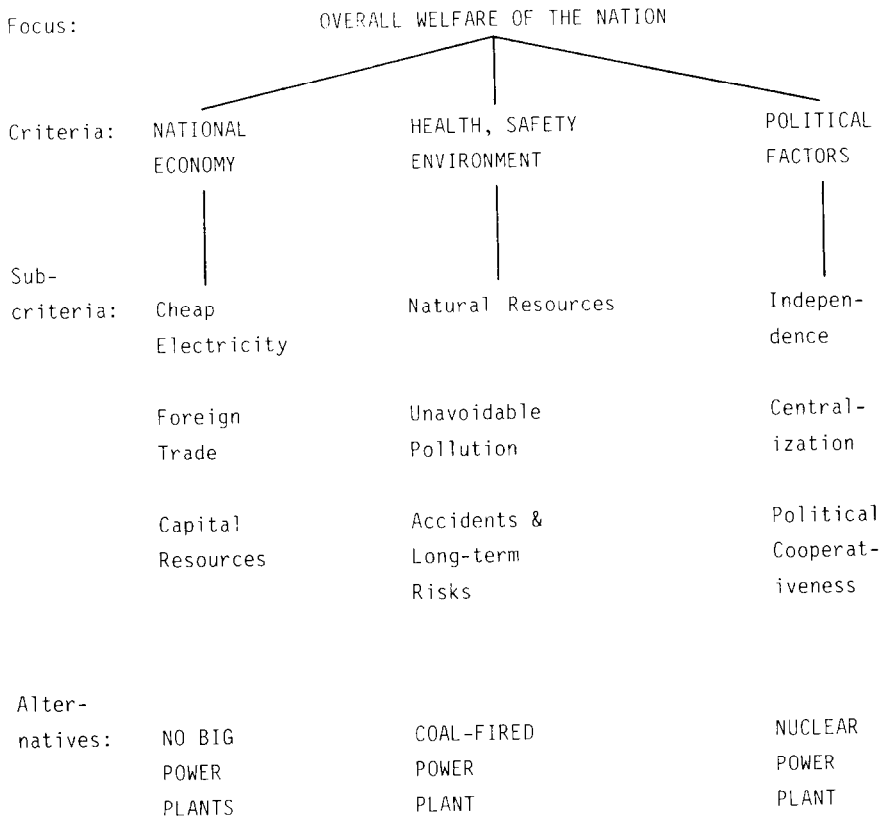

Fig. 1. Hierarchy of the Finnish energy decision.

We can make a few observations about this hierarchy. Obviously, it is simple: more elements could be added, at any level, and more levels. The question is, "How much should one include in a hierarchy?" A general rule is that the hierarchy should be complex enough to capture the situation, but small and nimble enough to be sensitive to changes. For the members of the Finnish Parliament the hierarchy given above captured the degree of complexity that they could address as a political unit.

Pairwise comparisons are fundamental in the use of the AHP. The members of parliament must first establish priorities for their main criteria by judging them in pairs for their relative importance, thus generating a pairwise comparison matrix. Judgments which are represented by numbers from the fundamental scale below are used to make the comparisons. The number of judgments needed for a particular matrix of order n, the number of elements being compared, is $n(n-1)/2$ because it is reciprocal and the diagonal elements are equal to unity. The paper by Harker [10, this issue, pp. 353–360] gives conditions under which it is possible to use fewer judgments and still obtain accurate results.

The next step is for the members of parliament to compare the subcriteria that belong to each of the main criteria, thus constructing three more pairwise comparison matrices for level 3. Then the three alternatives are compared with respect to each of the subcriteria, leading to nine pairwise comparison matrices for level 4. The final step is to weight or synthesize the results to obtain the final priorities of the three power plants.

We now give a second example with more details of how these operations are conducted.

The author has a teenage son who graduated from high school last year with good grades and high SAT scores. We used an AHP model to select the college of his choice. He was accepted by Swarthmore College, Northwestern University, the University of Michigan, Vanderbilt University and Carnegie–Mellon University (two blocks from his home). Since the cost of each is about the same, it was not included as a criterion in the hierarchy. He has an artistic temperament, writing right-handed in a hooked position, which is said to prove that he is "right-brained". The result of the analysis was that he chose Northwestern University, whose reputation and academic environment are slightly less preferred than that of Swarthmore College, because it appealed more to his nature. At Northwestern, he selected a dormitory room facing Lake Michigan, combining a convivial 4-year vacation site with study.

To summarize his outlook: as far as LOCATION was concerned, the farther away from home a school is perceived to be, the better (but not necessarily linearly); REPUTATION (how the school is rated) was slightly more important than ACADEMICS (personalized attention, small classes vs large ones); and AMBIENCE (how happy he felt at the place) was a little more important than either ACADEMICS or LOCATION. The hierarchy is shown below in Fig. 2.

Table 1. The fundamental scale

Intensity of importance on an absolute scale	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective
3	Moderate importance of one over another	Experience and judgment strongly favor one activity over another
5	Essential or strong importance	Experience and judgment strongly favor one activity over another
7	Very strong importance	An activity is strongly favored and its dominance demonstrated in practice
9	Extreme importance	The evidence favoring one activity over another is of the highest possible order of affirmation
2,4,6,8	Intermediate values between the two adjacent judgments	When compromise is needed
Reciprocals	If activity i has one of the above numbers assigned to it when compared with activity j, then j has the reciprocal value when compared with i	
Rationals	Ratios arising from the scale	If consistency were to be forced by obtaining n numerical values to span the matrix

n the elements being compared are closer together than indicated by the scale, one can use the scale 1.1, 1.2, ..., ill finer, one can use an appropriate even finer refinement.

What question should one ask?

On examining the matrices below, we note that a pair of elements (i, j) in a level of the hierarchy are compared with respect to a parent element in the level immediately above as a common property or criterion used to judge as to which one has it more and by how much. The typical way to phrase a question to fill an entry in the matrix of comparisons is: when considering two elements, i on the left side of the matrix and j on the top, which has the property more, or which one satisfies the criterion more, i.e. which one is considered more important under that criterion and how much more (using the fundamental scale values from Table 1)? This gives us a_{ij} (or a_{ji}). The reciprocal value is then automatically entered for the transpose.

The question asked in making a pairwise comparison can influence the judgments provided and hence also the priorities. It must be made clear from the start what the focus of the hierarchy is and how the elements in the second level either serve to fulfill that focus or are its consequence, and so on down the hierarchy for each parent element and its descendants.

We have the following for the matrix of pairwise comparisons of the criteria with respect to the overall focus:

FOCUS	LOCATION	AMBIENCE	REPUTATION	ACADEMICS
LOCATION	1	1/7	1/5	1/5
AMBIENCE	7	1	2	3
REPUTATION	5	1/2	1	1
ACADEMICS	5	1/3	1	1

In this matrix the first nontrivial comparison is (LOCATION, AMBIENCE). The question is: "How much more is LOCATION preferred over AMBIENCE?" AMBIENCE is actually preferred very strongly (7 times) over LOCATION, so the reciprocal value 1/7 is entered in the (1, 2) position. The value 7 is automatically entered in the transpose position (2, 1) for (AMBIENCE, LOCATION). As another illustration, AMBIENCE is judged to be between equal and moderately more important than REPUTATION and hence the value 2 (2 times) is entered in the (2, 3) position with the reciprocal (1/2) automatically entered in the (3, 2) position and so on.

A matrix is said to be consistent if $a_{ij} a_{jk} = a_{ik}$, $\forall i, j, k$. Note that this matrix is inconsistent. For example, (AMBIENCE, REPUTATION) = 2, while (REPUTATION, ACADEMICS) = 1, therefore to be consistent (AMBIENCE, ACADEMICS) = (AMBIENCE, REPUTATION) × (REPUTATION, ACADEMICS) = (2) × (1) = 2. But it has the value 3. Thus the value entered in
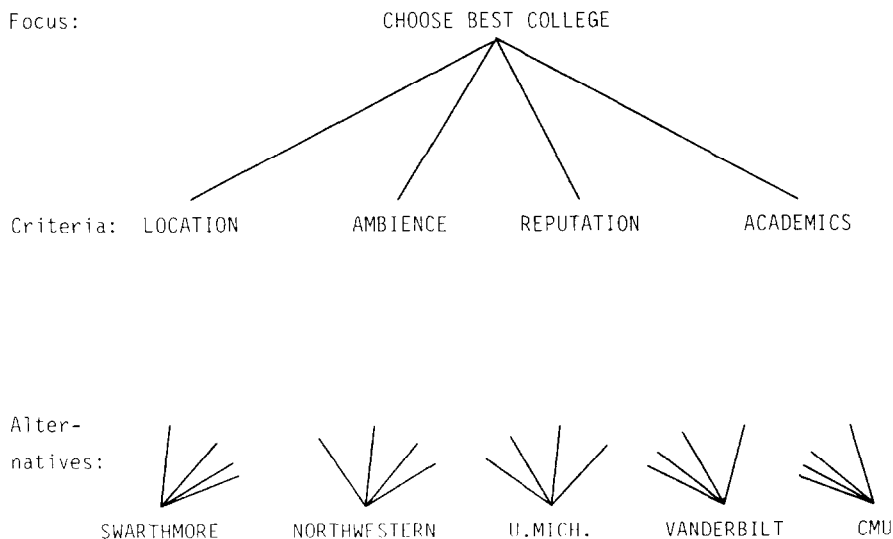

Fig. 2. Hierarchy for choosing a college.

the (2, 4) position is larger than it should be to be consistent, but we then enter the reciprocal value 1/3 in the (4, 2) position and note that 1/3 < 1/2. The relevance of this observation is that while one value exceeds the corresponding consistent value, its reciprocal is less than the reciprocal of the consistent value, and hence there is a tendency to compensate. When a positive reciprocal matrix of order n is consistent, the principal eigenvalue has the value n. When it is inconsistent, the principal eigenvalue exceeds n and its departure from n serves as a measure of inconsistency by forming a ratio (called the consistency ratio, C.R.) of this difference to the average of the corresponding differences from n of the principal eigenvalues of a large number of matrices of randomly chosen judgments (see Section 4).

The next step is to derive the scale of priorities (or weights). It has been shown that this scale is obtained by solving for the principal eigenvector of the matrix and then normalizing the result. This is called the *local* derived scale before weighting by the priority of its parent criterion (which for the second-level elements is always equal to unity, the weight of the focus). After weighting, it is called the *global* derived scale. It has also been shown that the principal eigenvector is the only way to obtain the derived scale that makes use of all the dominance information given in the matrix when the latter is inconsistent. In computing the principal eigenvector all possible intransitivities and chains of intransitivity enter into the calculations as the matrix is raised to powers [11].

We obtain for the vector of relative weights:

(LOCATION, AMBIENCE, REPUTATION, ACADEMICS) = (0.053, 0.491, 0.238, 0.213)

with an inconsistency ratio of 0.02 (see Section 4 for further details).

The next step is to set up matrices of paired comparisons for the schools in level 3 compared with respect to the criteria in level 2. There are four such matrices. In the illustrative matrix, given below, we compare SWARTHMORE, NORTHWESTERN, U. MICHIGAN, VANDERBILT and CMU with respect to their perceived desirability according to LOCATION:

LOCATION	SWARTH	NORTHW	U. MICH	VANDERB	CMU	Wt
SWARTH	1	1/4	1/3	1/3	7	0.115
NORTHW	4	1	2	3	7	0.402
U. MICH	3	1/2	1	3	6	0.283
VANDERB	3	1/3	1/3	1	4	0.163
CMU	1/7	1/7	1/6	1/4	1	0.037

C.R. = 0.092

Again the local derived scale for the schools, for criterion (1) LOCATION, after solving for the principal eigenvector, is listed in column 4 under weights:

(1) (SWARTH, NORTHW, U. MICH, VANDERB, CMU) = (0.115, 0.402, 0.283, 0.163, 0.037).

We now give the local derived scales for the other three criteria, (2) AMBIENCE, (3) REPUTATION and (4) ACADEMICS, omitting the pairwise comparison matrices:

(2) (SWARTH, NORTHW, U. MICH, VANDERB, CMU) = (0.034, 0.539, 0.250, 0.121, 0.056);

(3) (SWARTH, NORTHW, U. MICH, VANDERB, CMU) = (0.521, 0.235, 0.147, 0.038, 0.059);

(4) (SWARTH, NORTHW, U. MICH, VANDERB, CMU) = (0.564, 0.209, 0.132, 0.040, 0.055).

The software package supporting the AHP, called Expert Choice, for the IBM PC, was used to make these calculations and can guide the decision maker to improve his inconsistency if needed. When the consistency ratio exceeds 0.10 appreciably the judgments often need reexamination. To synthesize the overall priority scale we multiply as follows:

$$\begin{pmatrix} 0.115 & 0.034 & 0.521 & 0.564 \\ 0.402 & 0.539 & 0.235 & 0.209 \\ 0.284 & 0.250 & 0.147 & 0.132 \\ 0.163 & 0.121 & 0.038 & 0.040 \\ 0.037 & 0.056 & 0.059 & 0.055 \end{pmatrix} \begin{pmatrix} 0.053 \\ 0.491 \\ 0.238 \\ 0.218 \end{pmatrix} = \begin{pmatrix} 0.270 \\ 0.387 \\ 0.201 \\ 0.086 \\ 0.055 \end{pmatrix}.$$

This gives a final priority of 0.387 for NORTHWESTERN, the most preferred school, with SWARTHMORE as runner-up with a priority of 0.270. The overall consistency of these judgments is obtained as a ratio of sums of weighted inconsistency indices to the corresponding sums of weighted random indices (see Section 4).

In the next section we examine some of the mathematical foundations of the AHP.

3. THE AXIOMS OF THE AHP

Three principles guide one in problem solving using the AHP [12]: decomposition, comparative judgments and synthesis of priorities as embodied in the previous example.

The decomposition principle is applied by structuring a simple problem with the elements in a level being independent from those in succeeding levels, working downward from the focus in the top level, to criteria bearing on the focus in the second level, followed by subcriteria in the third level, and so on, from the more general (and sometimes uncertain) to the more particular and concrete. Saaty [12] makes a distinction between two types of dependence which he calls functional and structural. The former is the familiar contextual dependence of elements on other elements in performing their function, whereas the latter is the dependence of the priority of elements on the priority and number of other elements. Absolute measurement, sometimes called scoring, is used when it is desired to ignore such structural dependence among elements, while relative measurement is used otherwise.

The principle of comparative judgments is applied to construct pairwise comparisons of the relative importance of elements in some given level with respect to a shared criterion or property in the level above, giving rise to the kind of matrix encountered above and its corresponding principal eigenvector.

The third principle is that of synthesizing the priorities. In the AHP priorities are synthesized from the second level down by multiplying local priorities by the priority of their corresponding criterion in the level above and adding, for each element in a level according to the criteria it affects. (The second-level elements are multiplied by unity, the weight of the single top-level goal). This gives the composite or global priority of that element, which in turn is used to weight the local priorities of the elements in the level below compared to each other with it as the criterion, and so on to the bottom level.

Four axioms govern the AHP and utilize the notion of paired comparisons as a primitive.

Let $\mathfrak{A}$ be a finite set of n elements called alternatives. Let $\mathfrak{C}$ be a set of properties or attributes with respect to which elements in $\mathfrak{A}$ are compared. We will refer to the elements of $\mathfrak{C}$ as criteria. A criterion is a primitive.

We perform binary comparisons on the elements in $\mathfrak{A}$ according to a criterion in $\mathfrak{C}$. Let $>_C$ be a binary relation on $\mathfrak{A}$ representing "more preferred than" with respect to a criterion C in $\mathfrak{C}$. Let $\sim_C$ be the binary relation "indifferent to" with respect to a criterion C in $\mathfrak{C}$. Hence, given two elements $A_i, A_j \in A$, either $A_i >_C A_j$ or $A_j >_C A_i$ or $A_i \sim_C A_j$, $\forall C \in \mathfrak{C}$. We use $A_i \gtrsim_C A_j$ to indicate more preferred or indifferent. A given family of binary relations $>_C$ with respect to a criterion C in $\mathfrak{C}$ is a primitive.

Let $\mathfrak{P}$ be the set of mappings from $\mathfrak{A} \times \mathfrak{A}$ to $\mathbf{R}^+$ (the set of positive reals). Let $f: \mathfrak{C} \to \mathfrak{P}$. Let $P_C \in f(C)$ for $C \in \mathfrak{C}$. P_C assigns a positive real number to every pair $(A_i, A_j) \in \mathfrak{A} \times \mathfrak{A}$. Let $P_C(A_i, A_j) \equiv a_{ij} \in \mathbf{R}^+$, $A_i, A_j \in \mathfrak{A}$. For each $C \in \mathfrak{C}$, the triple $(\mathfrak{A} \times \mathfrak{A}, \mathbf{R}^+, P_C)$ is a *fundamental* or *primitive* scale. A fundamental scale is a mapping of objects to a numerical system.

Definition. For A_i, $A_j \in \mathfrak{A}$ and $C \in \mathfrak{C}$,

$$A_i >_C A_j \quad \text{iff} \quad P_C(A_i, A_j) > 1,$$

$$A_i \sim_C A_j \quad \text{iff} \quad P_C(A_i, A_j) = 1.$$

If $A_i >_C A_j$, we say that A_i dominates A_j with respect to $C \in \mathfrak{C}$. Thus, P_C represents the intensity or strength of preference for one alternative over another.

Axiom 1 (the reciprocal axiom)

For all A_i, $A_j \in \mathfrak{A}$ and $C \in \mathfrak{C}$,

$$P_C(A_i, A_j) = 1/P_C(A_j, A_i).$$

This axiom says that the comparison matrices we construct are formed of paired reciprocal comparisons, for if one stone is judged to be five times heavier than another, then the other must perforce be one-fifth as heavy as the first. It is this simple but powerful relationship that is the basis of the AHP.

Let $A = (a_{ij}) \equiv (P_C(A_i, A_j))$ be the set of paired comparisons of the alternatives with respect to a criterion $C \in \mathfrak{C}$. By the definition of P_C and Axiom 1, A is a positive reciprocal matrix. The object is to obtain a scale of relative dominance (or rank order) of the alternatives from the paired comparisons given in A.

We will now show how to derive the relative dominance of a set of alternatives from a pairwise comparison matrix A. Let $R_{M(n)}$ be the set of $(n \times n)$ positive reciprocal matrices $A = (a_{ij}) \equiv (P_C(A_i, A_j))$, $\forall C \in \mathfrak{C}$. Let $[0, 1]^n$ be the n-fold Cartesian product of $[0, 1]$ and let $W: R_{M(n)} > [0, 1]^n$ for $A \in R_{M(n)}$, $W(A)$ is an n-dimensional vector whose components lie in the interval $[0, 1]$. The triple $(R_{M(n)}, [0, 1]^n, W)$ is a derived scale. A derived scale is a mapping between two numerical relational systems.

An important aspect of the AHP is the idea of consistency. If one has a scale for a property possessed by some objects and measures that property in them, then their relative weights with respect to that property are fixed. In this case there is no judgmental inconsistency (although if one has a physical scale and applies it to objects in pairs and *then* derives the relative standing of the objects on the scale from the pairwise comparison matrix, it is likely that inaccuracies will have occurred in the act of applying the physical scale and again there would be inconsistency). But when comparing with respect to a property for which there is no established scale or measure, we are trying to derive a scale through comparing the objects two at a time. Since the objects may be involved in more than one comparison and we have no standard scale, but are assigning relative values as a matter of judgment, inconsistencies may well occur. In the AHP consistency is defined in the following way, and we are able to measure inconsistency (see Section 4).

Definition. The mapping P_C is said to be consistent iff

$$P_C(A_i, A_j)P_C(A_j, A_k) = P_C(A_i, A_k), \qquad \forall i, j, k.$$

Similarly, the matrix A is consistent iff $a_{ij}a_{jk} = a_{ik}$, $\forall i, j, k$.

We now turn to the hierarchic Axioms 2–4, and related definitions.

In a partially ordered set, we define $x < y$ to mean that $x < y$ and $x \neq y$; y is said to cover (dominate) x. If $x < y$, then $x < t < y$ is possible for no t. We use the notation $x^- = \{y \mid x$ covers $y\}$ and $x^+ = \{y \mid y$ covers $x\}$, for any element x in an ordered set.

Let H be a finite partially ordered set. Then H is a hierarchy if it satisfies the following conditions:

(a) there is a partition of H into sets L_k, $k = 1, \ldots, h$, for some h where $L_1 = \{b\}$, b is a single element;

(b) $x \in L_k$ implies $x^- \in L_{k+1}$, $k = 1, \ldots, h-1$;

(c) $x \in L_k$ implies $x^+ \in L_{k-1}$, $k = 2, \ldots, h$.

The notions of fundamental and derived scales can be extended to $x \in L_k$ and $x^- \subseteq L_{k+1}$, replacing C and A, respectively. The derived scale resulting from comparing the elements in x^-

with respect to x is called a local derived scale or the local priorities for the elements in x^-.

Definition. Given a positive real number $\rho \geqslant 1$, a nonempty set $x^- \subseteq L_{k+1}$ is said to be ρ-homogeneous with respect to $x \in L_k$ if for every pair of elements $y_1, y_2 \in x^-$, $1/\rho \leqslant P_C(y_1, y_2) \leqslant \rho$. In particular the reciprocal axiom implies that $P_C(y_i, y_i) = 1$.

Axiom 2 (the homogeneity axiom)

Given a hierarchy H, $x \in H$ and $x \in L_k$, $x^- \subseteq L_{k+1}$ is ρ-homogeneous for $k = 1, \ldots, h - 1$.

Homogeneity is essential for meaningful comparisons, as the mind cannot compare widely disparate elements. For example, we cannot compare a grain of sand with an orange according to size. When the disparity is great, elements should be placed in separate clusters of comparable size, or in different levels altogether.

Given L_k, $L_{k+1} \subseteq H$, let us denote the local derived scale for $y \in x^-$ and $x \in L_k$ by $\psi_{k+1}(y/x)$, $k = 2, 3, \ldots, h - 1$. Without loss of generality we may assume that $\psi_{k+1}(y/x) = 1$. Consider the matrix $\psi_k(L_k/L_{k-1})$ whose columns are local derived scales of elements in L_k with respect to elements in L_{k-1}.

Definition. A set A is said to be *outer dependent* on a set C if a fundamental scale can be defined on A with respect to every $C \in \mathfrak{C}$.

The process of relating elements (e.g. alternatives) in one level of the hierarchy according to the elements of the next higher level (e.g. criteria) expresses the dependence (what is called *outer dependence*) of the lower elements on the higher so that comparisons can be made between them. The steps are repeated upward in the hierarchy through each pair of adjacent levels to the top element, the focus or goal.

The elements in a level may also depend on one another with respect to a property in another level. Input–output of industries is an example of the idea of inner dependence, formalized as follows.

Definition. Let $\mathfrak{A}$ be outer dependent on $\mathfrak{C}$. The elements in $\mathfrak{A}$ are said to be *inner dependent* with respect to $C \in \mathfrak{C}$ if for some $A \in \mathfrak{A}$, $\mathfrak{A}$ is outer dependent on A.

Axiom 3

Let H be a hierarchy with levels $L_1, L_2, \ldots, L_h$. For each L_k, $k = 1, 2, \ldots, h - 1$:

  (1) L_{k+1} is outer dependent on L_k;
  (2) L_{k+1} is not inner dependent with respect to all $x \in L_k$;
  (3) L_k is not outer dependent on L_{k+1}.

Axiom 4 (the axiom of expectations)

$$C \subset H - L_h, \qquad A = L_h.$$

This axiom is merely the statement that thoughtful individuals who have reasons for their beliefs should make sure that their ideas are adequately represented in the model. All alternatives, criteria and expectations (explicit and implicit) can be and should be represented in the hierarchy. This axiom does not assume rationality. People are known at times to harbor irrational expectations and such expectations can be accommodated.

Based on the concepts in Axiom 3 we can now develop a weighting function. For each $x \in H$, there is a suitable weighting function (whose nature depends on the phenomenon being hierarchically structured):

$$w_x: x^- \to [0, 1] \text{ such that } \sum_{y \in x^-} w_x(y) = 1.$$

Note that $h = 1$ is the last level for which x^- is not empty.

The sets L_i are the levels of the hierarchy, and the function w_x is the priority function of the elements in one level with respect to the objective x. We observe that even if $x^- \neq L_k$ (for some level L_k), w_x may be defined for all of L_k by setting it equal to zero for all elements in L_k not in x.

The weighting function is one of the more significant contributions towards the application of hierarchy theory.

Definition. A hierarchy is complete if, $\forall x \subset L_k,\; x^+ \subset L_{k-1}$.
We can state the central question:

> *Basic Problem.* Given any element $x \in L_\alpha$, and subset $S \subset L_\beta$, $(\alpha < \beta)$, how do we define a function $w_{x,S} : S \to [0,1]$ which reflects the properties of the priority functions on the levels L_k, $k = \alpha, \ldots, \beta - 1$. Specifically, what is the function $w_{b,L_h} : L_h \to [0,1]$?

In less technical terms, this can be paraphrased thus:

> Given a social (or economic) system with a major objective b, and the set L_h of basic activities, such that the system can be modeled as a hierarchy with largest element b and lowest level L_h. What are the priorities of the elements of any level and in particular those of L_h with respect to b?

From the standpoint of optimization, to allocate a resource to the elements any interdependence may take the form of input–output relations such as, for example, the interflow of products between industries. A high priority industry may depend on flow of material from a low priority industry. In an optimization framework, the priority of the elements enables one to define the objective function to be maximized, and other hierarchies supply information regarding constraints, e.g. input–output relations.

We now present a method to solve the Basic Problem. Assume that $Y = \{y_1, \ldots, y_{m_k}\} \subset L_k$ and that $X = \{x_1, \ldots, x_{m_{k+1}}\} \subset L_{k+1}$. Without loss of generality we may assume that $X = L_{k+1}$, and that there is an element $z \in L_k$ such that $y_i \in z^-$. Then consider the priority functions

$$w_z : Y \to [0,1] \quad \text{and} \quad w_{y_j} : X \to [0,1] \, j = 1, \ldots, m_k.$$

Construct the priority function of the elements in X with respect to z, denoted $\mathbf{w}$, $\mathbf{w} : X \to [0,1]$, by

$$\mathbf{w}(x_i) = \sum_{j=1}^{m_k} w_{y_j}(x_i) w_z(y_j), \qquad i = 1, \ldots, m_{k+1}.$$

It is obvious that this is no more than the process of weighting the influence of the element y_j on the priority of x_i by multiplying it with the importance of x_i with respect to z.

The algorithms involved will be simplified if one combines the $w_{y_j}(x_i)$ into a matrix B by setting $b_{ij} = w_{y_j}(x_i)$. If one further sets $w_i = \mathbf{w}(x_i)$ and $w'_j = w_z(y_j)$, then the above formula becomes

$$w_i = \sum_{j=1}^{m_k} b_{ij} w'_j, \qquad i = 1, \ldots, n_{k+1}.$$

Thus, one may speak of the priority vector $\mathbf{w}$ and, indeed, of the priority matrix B of the $(k+1)$th level; this gives the final formulation,

$$\mathbf{w} = B\mathbf{w}'.$$

The following theorem is easy to prove.

Theorem. Let H be a complete hierarchy with largest element b and h levels. Let B_k be the priority matrix of the kth level, $k = 1, \ldots, h$. If $\mathbf{w}'$ is the priority vector of the pth level with respect to some element z in the $(p-1)$th level, then the priority vector $\mathbf{w}$ of the qth level $(p < q)$ with respect to z is given by

$$\mathbf{w} = B_q B_{q-1} \ldots B_{p+1} \mathbf{w}'.$$

Thus, the priority vector of the lowest level with respect to the element b is given by

$$\mathbf{w} = B_h B_{h-1} \ldots B_2 b_1$$

if L_1 has a single element, $b_1 = 1$. Otherwise, b_1 is a prescribed vector. We note that the pairwise comparison process takes into consideration nonlinearities. Such nonlinearities are captured by the composition weighting process.

Network systems

Often alternatives depend on criteria and criteria on alternatives and there should be a cycle connecting the two which is more accurately studied with the network feedback approach. The AHP has been generalized to deal with feedback as shown below, although people generally prefer to simplify and arrange their thinking in terms of a linear hierarchy even if the answers are only approximate.

A network is a set of nodes (each of which consists of a set of elements) and a set of arcs which indicate the order of interaction among the components. The priorities of the elements in each node are components of the principal eigenvector of the matrix of pairwise comparisons of the relative impact of these elements with respect to an element or node with which they interact. The interaction is indicated by an arc of the network. All such eigenvectors define what is known as a

supermatrix of impact priorities. By weighting the eigenvectors corresponding to each component by the priority of that component in the system, the supermatrix is transformed into a stochastic matrix. The limiting impact priorities are obtained by computing large powers of this matrix. Formally we have the following.

Definition. A partially ordered set S is a network system if

    (a)   there is a partition of S into sets C_k, $k = 1, \ldots, s$;
    (b)   there is an ordering on C_k, $k = 1, \ldots, s$ such that $x \subseteq C_k$ implies either x^- or x^+
          is in $C_{k_j}^k$ for some k_j or both $x^- \subseteq C_{k_j}$, $x^+ \subseteq C_{k_j}$ for one or more k_j;
    (c)   For each $x \in S$, there is a suitable weighting function

$$w_x: x^- \to [0, 1] \text{ such that } \sum_{y \in x^-} w_x(y) = 1$$

and for $C_k \subseteq S$, $k = 1, \ldots, s$, there is a weighting function

$$w_{(k)}: C_k^- \to [0, 1],$$

where $C_k^- = \{C_h \mid C_k \text{ covers } C_h\}$.

We shall now turn to the calculation procedures for the weights and for the inconsistency index.

4. THE EIGENVECTOR SOLUTION FOR WEIGHTS AND CONSISTENCY

There is an infinite number of ways to derive the vector of priorities from the matrix (a_{ij}). But emphasis on consistency leads to an eigenvalue formulation.

If a_{ij} represents the importance of alternative i over alternative j and a_{jk} represents the importance of alternative j over alternative k then a_{ik}, the importance of alternative i over alternative k, must equal $a_{ij}a_{jk}$ for the judgments to be consistent. If we do not have a scale at all, or do not have it conveniently, as in the case of some measuring devices, we cannot give the precise values of w_i/w_j but only an estimate. Our problem becomes $A'w' = \lambda_{\max}w'$, where $\lambda_{\max}$ is the largest or principal eigenvalue of $A' = (a'_{ij})$ the perturbed value of $A = (a_{ij})$ with $a'_{ji} = 1/a'_{ij}$ forced. To simplify the notation we shall continue to write $Aw = \lambda_{\max}w$, where A is the matrix of pairwise comparisons.

The solution is obtained by raising the matrix to a sufficiently large power then summing over the rows and normalizing to obtain the priority vector $w = (w_1, \ldots, w_n)$. The process is stopped when the difference between components of the priority vector obtained at the kth power and at the $(k + 1)$th power is less than some predetermined small value.

An easy way to get an approximation to the priorities is to normalize the geometric means of the rows. This result coincides with the eigenvector for $n \leqslant 3$. A second way to obtain an approximation is by normalizing the elements in each column of the judgment matrix and then averaging over each row.

We would like to caution the reader that for important applications one should only use the eigenvector derivation procedure because approximations can lead to rank reversal in spite of the

closeness of the result to the eigenvector [13]. It is easy to prove that for an arbitrary estimate $\mathbf{x}$ of the priority vector,

$$\lim_{k \to \infty} \frac{1}{\lambda_{\max}^k} A^k \mathbf{x} = c\mathbf{w}$$

where c is a positive constant and $\mathbf{w}$ is the principal eigenvector of A. This may be interpreted roughly to say that if we begin with an estimate and operate on it successively by $A/\lambda_{\max}$ to get new estimates, the result converges to a constant multiple of the principal eigenvector.

A simple way to obtain the exact value (or an estimate) of $\lambda_{\max}$ when the exact value (or an estimate) of $\mathbf{w}$ is available in normalized form is to add the columns of A and multiply the resulting vector with the vector $\mathbf{w}$. The resulting number is $\lambda_{\max}$ (or an estimate). This follows from

$$\sum_{j=1}^{n} a_{ij} w_j = \lambda_{\max} w_i$$

and

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} w_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} \right) w_j = \sum_{i=1}^{n} \lambda_{\max} w_i = \lambda_{\max}.$$

The problem is now, how good is the principal eigenvector estimate $\mathbf{w}$? Note that if we obtain $\mathbf{w} = (w_1, \ldots, w_n)$ by solving this problem, the matrix whose entries are w_i/w_j is a consistent matrix which is our consistent estimate of the matrix A. The original matrix A itself need not be consistent. In fact, the entries of A need not even be transitive; i.e. A_1 may be preferred to A_2 and A_2 to A_3 but A_3 may be preferred to A_1. What we would like is a measure of the error due to inconsistency. It turns out that A is consistent iff $\lambda_{\max} = n$ and that we always have $\lambda_{\max} \geqslant n$. This suggests using $\lambda_{\max} - n$ as an index of departure from consistency. But

$$\lambda_{\max} - n = - \sum_{i=2}^{n} \lambda_i, \qquad \lambda_{\max} = \lambda_1,$$

where λ_i, $i = 1, \ldots, n$ are the eigenvalues of A. We adopt the average value $(\lambda_{\max} - n)/(n - 1)$, which is the (negative) average of λ_i, $i = 2, \ldots, n$ (some of which may be complex conjugates).

It is interesting to note that $2(\lambda_{\max} - n)/(n - 1)$ is the variance of the error incurred in estimating a_{ij}. This can be shown by writing

$$a_{ij} = (w_i/w_j)\varepsilon_{ij}, \varepsilon_{ij} > 0 \quad \text{and} \quad \varepsilon_{ij} = 1 + \delta_{ij}, \delta_{ij} > -1,$$

and substituting in the expression for $\lambda_{\max}$. It is δ_{ij} that concerns us as the error component and its value $|\delta_{ij}| < 1$ for an unbiased estimator. Normalizing the principal eigenvector yields a unique estimate of a ratio scale underlying the judgments.

The C.I. of a matrix of comparisons is given by C.I. $= (\lambda_{\max} - n)/(n - 1)$. The consistency ratio (C.R.) is obtained by comparing the C.I. with the appropriate one of the following set of numbers, each of which is an average random consistency index (R.I.) derived from a sample of size 500, of a randomly generated reciprocal matrix using the scale $1/9, 1/8, \ldots, 1, \ldots 8, 9$ to see if it is about 0.10 or less. If it is not less than 0.10, study the problem and revise the judgments:

n	1	2	3	4	5	6	7	8	9	10
Random consistency index (R.I.)	0	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49

Why tolerate 10% inconsistency? The priority of consistency to obtain a coherent explanation of a set of facts must differ by an order of magnitude from the priority of inconsistency which is an error in the measurement of consistency. Thus, on a scale from 0–1, inconsistency should not exceed 0.10 by very much. Note that the requirement of 10% should not be made much smaller such as 1% or 0.1%. The reason is that inconsistency itself is important, for without it new knowledge which changes preference order cannot be admitted. Assuming all knowledge to be consistent contradicts experience which requires continued adjustment in understanding. Thus the objective of developing a wide-ranging consistent framework depends on admitting some inconsistency.

This also accounts for why the number of elements compared should be small. If the number of elements is large, their relative priorities would be small and error can distort these priorities considerably. If the number of items is small and the priorities are comparable a small error does not affect the order of magnitude of the answers and hence the relative priorities would be about the same. For this to happen, the items must be < 10 so their values on the whole would be $> 10\%$ each and hence remain relatively unaffected by 1% error for example.

The consistency index (C.I.) for an entire hierarchy is defined by

$$C_{\mathrm{H}} = \sum_{j=1}^{h} \sum_{i=1}^{n_{i_{j+1}}} w_{ij}\,\mu_{i,j+1},$$

where $w_{ij} = 1$ for $j = 1$, and $n_{i_{j+1}}$ is the number of elements of the $(j + 1)$th level with respect to the ith criterion of the jth level.

Let $|C_k^-|$ be the number of elements of C_k^-, and let $w_{(k)(h)}$ be the priority of the impact of the hth component on the kth component, i.e. $w_{(k)(h)} = w_{(k)}(C_h)$ or $w_{(k)}: C_h \to w_{(k)(h)}$.

If we label the components of a system along lines similar to those we followed for a hierarchy, and denote by w_{jk} the limiting priority of the jth element in the kth component, we have

$$C_{\mathrm{S}} = \sum_{k=1}^{s} \sum_{j=1}^{n_k} w_{jk} \sum_{h=1}^{|C_k^-|} w_{(k)(h)}\mu_k(j,h),$$

where $\mu_k(j,h)$ is the C.I. of the pairwise comparison matrix of the elements in the kth component with respect to the jth element in the hth component.

5. ABSOLUTE AND RELATIVE MEASUREMENT AND STRUCTURAL INFORMATION

The following is largely taken from a paper by Saaty [6]. Cognitive psychologists have recognized for some time that there are two kinds of comparisons, absolute and relative. In absolute comparisons alternatives are compared with a standard in one's memory that has been developed through experience; in relative comparisons alternatives are compared in pairs according to a common attribute. The AHP has been used with both types of comparisons to derive ratio scales of measurement. We call such scales absolute and relative measurement scales, respectively. Relative measurement in the AHP is well-developed and its use has already been illustrated in the school selection example in this paper. Here is a brief description of absolute measurement. Incidentally the software package Expert Choice [14] also includes this method of measurement under the name of "ratings".

Absolute measurement (sometimes called scoring) is applied to rank the alternatives in terms of the criteria or else in terms of ratings (or intensities) of the criteria; e.g. excellent (A), very good (B), good (C), average (D), below average (E), poor and very poor (F). After setting priorities on the criteria (or subcriteria, if there are any) pairwise comparisons are also performed on the ratings themselves to set priorities for them under each criterion. Finally, alternatives are scored by checking off their rating under each criterion and summing these ratings for all the criteria. This produces a ratio scale score for the alternative. The scores thus obtained of the alternatives can be normalized.

Using absolute measurement, no matter how many new alternatives are introduced, or old ones deleted, the ranks of the alternatives cannot reverse. This idea has been used to rank cities in the U.S.A. according to nine criteria as judged by six different people [6]. Another example of an appropriate use for absolute measurement is that of schools admitting students. Most schools set their criteria for admission independently from the performance of the current crop of students seeking admission. Their priorities are then used to determine whether a given student meets the standard set for qualification. In that case absolute measurement should be used to determine which students qualify for admission.

Absolute measurement needs standards, often set by society for its convenience, and sometimes having little to do with the values and objectives of the judge making the comparisons. In completely new decision problems or in old problems where no standards have been established, we must continue to use relative measurement comparing alternatives in pairs to identify the best. The question now is: "What happens to rank when using relative measurement and alternatives are added or deleted?"

When relative measurement is used to, for example, buy a car, even when the priorities of the criteria are set in advance independently of the alternatives, the car that qualifies in the end depends on the number of cars examined. Adding a new car to the collection being examined may cause reversal in the rank of the original cars. This phenomenon can be accounted for by considering the normalization operation as a structural criterion which has to do with information generated in the measurement process. With relative measurement the priority of such a criterion changes when new alternatives are added or old ones deleted and hence the priorities of the old alternatives, which depend on all the criteria, including this structural criterion, would change and a different rank order may occur among the old alternatives. The analogy can be made with any mathematical model, e.g. linear programming, when a new variable or a new constraint is added. There is no necessary relation between the solution of the new problem and the old one.

As it should, rank is unaffected when only one criterion is involved and the judgments are consistent. More generally, one can show that with consistency, the rank order of two alternatives is unaffected when the judgment values of one dominate those of the other in every pairwise comparison matrix under the criteria. However, the final rank can change even when the judgments are consistent when an alternative dominates another under one criterion but is dominated by it under another. The following example illustrates the idea of rank reversal due to structural criteria.

In an investment decision let us use only two criteria, RETURN and LOW RISK to determine where it is best to invest. Let us for simplicity assume that they are equally important so that their priorities are 0.5 and 0.5. We first take the two alternatives TAX FREE BONDS (A) and SECURITIES (B), pairwise compare them according to preference first under RETURN and then under LOW RISK, and obtain their composite weights. We have:

RETURN	A	B	Priority		LOW RISK	A	B	Priority
A	1	3	0.75		A	1	1/2	0.33
B	1/3	1	0.25		B	2	1	0.67

The final ranks are

$$A: 0.5 \times 0.75 + 0.5 \times 0.33 = 0.54$$

and

$$B: 0.5 \times 0.25 + 0.5 \times 0.67 = 0.46.$$

A is preferred to B. On observing that the priorities of the alternatives under each criterion are obtained by adding the elements in either column and dividing by the total which is the normalization factor, we may rewrite the above arithmetic operations to indicate that normalization is applied to rescale the priorities of the criteria and use them to weight the priorities of the alternatives before they are normalized. Thus on writing $0.75 = 3/4$, $0.25 = 1/4$, $0.33 = 1/3$, $0.67 = 2/3$ and noting that 4 is the normalization factor for RETURN and 3 is the normalization factor for LOW RISK the above can be written as

$$A: (0.5 \times 1/4)3 + (0.5 \times 1/3)1 = 0.54$$

and

$$B: (0.5 \times 1/4)1 + (0.5 \times 1/3)2 = 0.46.$$

In other words, normalization may be regarded as an operation that transforms the weights of the criteria from the original scale of (0.5, 0.5) to the new scale (0.5/4, 0.5/3) which when normalized becomes

$$[0.125/(0.125 + 0.167), 0.167/(0.125 + 0.167)] = (0.43, 0.57),$$

which are the rescaled priorities of the criteria. That is, relative measurement may be regarded as an operation which always introduces a new criterion that operates on the existing criteria by modifying their priorities. Earlier we called this criterion a *structural criterion*. It is known in traditional decision theory that rank reversal can occur when a new criterion is introduced and here we have encountered a fundamentally new kind of criterion that is always present when we perform relative measurement. Now let us see how rank reversal occurs here by introducing the new alternative SAVINGS ACCOUNTS (C) with the resulting paired comparisons shown in the following two matrices:

RETURN	A	B	C	Priority
A	1	3	1/2	0.30
B	1/3	1	1/6	0.10
C	2	6	1	0.60

LOW RISK	A	B	C	Priority
A	1	1/2	4	0.31
B	2	1	8	0.08 0.62
C	1/4	1/8	1	

As before, but now without the details, we have

$$A: 0.5 \times 0.3 + 0.5 \times 0.31 = (0.5 \times 1/10)3 + (0.5 \times 1/13)4 = 0.30,$$

$$B: 0.5 \times 0.1 + 0.5 \times 0.62 = (0.5 \times 1/10)1 + (0.5 \times 1/13)8 = 0.36$$

and

$$C: 0.5 \times 0.6 + 0.5 \times 0.08 = (0.5 \times 1/10)6 + (0.5 \times 1/13)1 = 0.34.$$

Thus the presence of alternative C has caused rank reversal between A and B and now B is preferred to C which is preferred to A. In this case we would have to ignore the first ranking and use the second. Let us note that this process of rank reversal does not contradict any existing proven fact and we have no historical precedent in some well-developed multicriteria theory to check it against. In fact the only known theory which is concerned with rank reversal under even a single criterion is utility theory. There when rank reversal occurs under one criterion the new alternatives which cause it are called relevant, otherwise they are called irrelevant. The relative measurement approach of the AHP requires that in order for elements to be comparable in pairs, it is essential that they all be homogeneous, which means that they must be relevant to begin with. There is no escaping the fact that to deal with myriads of intangibles we need relative measurement and that when we use it rank reversal may occur. It is likely that rank reversal due to structural criteria happens in real life without being recognized. In a recent unpublished work, T. Saaty has provided the following principle with regard to rank preservation and reversal:

"There is greater uncertainty about rank optimality with increasing rate of change of information."

Group judgments?

When dealing with group judgments, Saaty has proposed that any rule to combine the judgments of several individuals should also satisfy the reciprocal property. A proof that the geometric mean, which makes no requirement on who should vote first, satisfies this condition was later generalized in a paper by Aczél and Saaty [8] and by Aczél and Alsina [15, this issue, pp. 311–320].

Group judgment differences can be resolved through a consistency check. When several people propose radically different judgments in certain positions of the matrix these can be tested with other judgments on which there is wide agreement by solving the problem separately for each controversial judgment and measuring the consistency. The judgment yielding the highest consistency

in the overall problem is retained. The following consistency comparison for each individual's judgments with those of the scale vector **w** derived from group judgments has been proposed:

$$\sum_{i,j=1}^{n} b_{ij} w_j / w_i - n^2 \sim 0.1.$$

Probabilistic judgments have been studied extensively by Vargas [16]. In particular he showed that when the judgments are given by a γ-distribution the derived vector belongs to a Dirichlet distribution with a β-distribution of each component.

6. TOPICS FOR INVESTIGATION

There are a number of areas that need further investigation and others in which ground needs to be broken. The reader is referred to the AHP literature for applications and theoretical developments in most of these areas:

(1) Generalization of the hierarchy and systems networks to manifolds.

(2) Deeper and more extensive research on continuous judgments.

(3) Test different group decision-making approaches on the same problem and search for common elements. Develop A, B, C guidelines for group participation in decision making.

(4) Investigate the relationship of the principal eigenvector to the Weber–Fechner power law.

(5) Develop applications of the AHP in game theory, particularly with respect to negotiation.

(6) Investigate the relationship of the AHP and optimization. Can the general optimization problem be solved using the AHP?

(7) Implement psychological studies to show how people's strength of feeling can be adequately represented by numerical scales.

(8) Study the sensitivity of priorities to the number of criteria and, more generally, to the size of the hierarchy.

(9) Sample opinions on how satisfied clients are with AHP outcomes.

(10) Formulate more cases using the AHP in resource allocation, planning, cost–benefit analysis and conflict resolution.

(11) Is there power in hierarchic formulation and judgments to make better predictions? How can it be tested?

(12) The AHP and risk analysis: put forth a definitive theory about the use of scenarios in risk analysis.

(13) Investigate the relationship between the AHP and artificial intelligence.

(14) Develop communication and causal languages using the AHP.

REFERENCES

There are five books in English on the AHP, some of which have been translated into other languages, as indicated in the references. There have been issues of two journals solely devoted to articles on the AHP, this special issue of *Mathematical Modelling* and Volume 20, Number 6 (1986) of *Socio-Economic Planning Sciences*.

F. Zahedi has published a comprehensive survey article on the AHP [17] with up-to-date references on its literature. S. Xu (Tianjin University, China) has also published a complete reference list on the AHP [18]. One book has been written on the subject by a Japanese author and two by Chinese authors.

The subject has also been presented in a chapter in the book by S. I. Gass, *Decision Making, Models and Algorithms, A First Course*, (Wiley, New York, 1985) and in a chapter in the book by D. Anderson, D. Sweeney and T. Williams, *Quantitative Methods for Business*, 3rd edn (West Publishing, St Paul, 1986).

There is a software program available, Expert Choice [14], that runs on the IBM PC, the IBM XT, the IBM AT and compatible computers. It requires 256K memory and one double-sided disk drive. There is also a version of Expert Choice available for the Japanese IBM personal computer (with the Kanji keyboard).

1. T. L. Saaty, *The Analytic Hierarchy Process*. McGraw-Hill, New York (1980). (This book has been translated into

Chinese by S. Xu *et al.*; information is available from them at the Inst. of Systems Engineering, Tianjin Univ., Tianjin, China.) (A translation into Russian by R. Vachnadze is currently underway.)

2. T. L. Saaty and L. G. Vargas, *The Logic of Priorities, Applications in Business, Energy, Health, Transportation.* Kluwer-Nijhoff, The Hague (1981).

3. T. L. Saaty, *Decision Making for Leaders*, Wadsworth, Belmont, Calif. (1982). (Also reprinted as a paperback and available from the author at 322 Mervis Hall, Univ. Pittsburgh, Pittsburgh, PA 15260.) (Available in French under the title *Decider face a la complexite.* Entreprise Moderne D'Edition, Paris, 1984).

4. T. L. Saaty and K. P. Kearns, *Analytical Planning.* Pergamon Press, Oxford (1985).

5. T. L. Saaty and J. Alexander, *A New Logic for Conflict Resolution.* In preparation.

6. T. L. Saaty, Absolute and relative measurement with the AHP. The most livable cities in the United States. *Socio-econ. Plann. Sci.* **20**(6), 327–331 (1986).

7. T. L. Saaty and L. Vargas, Stimulus–response with reciprocal kernels: the rise and fall of sensation. *J. math. Sci.* (1987).

8. J. Aczél and T. L. Saaty, Procedures for synthesizing ratio judgments. *J. math. Psychol.* **27**(1), 93–102 (1983).

9. R. P. Hämäläinen and T. O. Seppäläinen, The analytic network process in energy policy planning. *Socio-econ. Plann. Sci.* **20**(6), 399–405 (1986).

10. P. T. Harker, Alternative modes of questioning in the Analytic Hierarchy Process. *Mathl Modelling* **9**, 353–360 (1987).

11. T. L. Saaty, Inconsistency and rank preservation. *J. math. Psychol.* **28**(2), 205–214 (1984).

12. T. L. Saaty, Axiomatic foundation of the Analytic Hierarchy Process. *Mgmt Sci.* **32**(7), 841–855 (1986).

13. T. L. Saaty, Rank generation, preservation and reversal in the Analytic Hierarchy Process. *Decis. Sci.* (July 1987).

14. Expert Choice, software package. Decision Support Software, McLean, Va.

15. J. Aczél and C. Alsina, Synthesizing judgements: a functional equations approach. *Mathl Modelling* **9**, 311–320 (1987).

16. L. Vargas, Reciprocal matrices with random coefficients. *J. math. Modelling* **3**, 69–81 (1982).

17. F. Zahedi, The Analytic Hierarchy Process—a survey of the method and its applications. *Interfaces* **16**(4), 96–108 (1986).

18. S. Xu, References on the Analytic Hierarchy Process. Inst. of Systems Engineering, Tianjin Univ., Tianjin, China (1986).