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# ON CONVOLUTION TAILS

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In proving limit theorems for some stochastic processes, the following classes of distribution functions were introduced by Chover–Ney–Wainger and Chistyakov $F$ belongs to $\mathscr{S}(\gamma)$ if and only if:

(i) $\lim_{x\to\infty} \overline{F^{(2)}}(x)/\bar{F}(x) = c < \infty$,
(ii) $\lim_{x\to\infty} \bar{F}(x-y)/\bar{F}(x) = e^{\gamma y}$ for all $y$ real,
(iii) $\int_0^\infty e^{\gamma y}\, dF(y) < \infty$.

Some new results on $\mathscr{S}(\gamma)$ are presented. The class $\mathscr{S}(\gamma)$ is strictly smaller than the class of $F$ for which the distribution function $\int_0^x e^{\gamma y}\, dF(y)/\int_0^\infty e^{\gamma y}\, dF(y)$ belongs to $\mathscr{S}(0)$, although several papers assume the two classes coincide. Consequences of the one-way inclusion in renewal theory and random walks are investigated.

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|---|---|
| Compound Poisson | random walks |
| renewal theory | subexponentiality |
| tails of probability measures | |

## 1. Introduction

We consider distribution functions on $[0, \infty[$ with unbounded support. We write $F^{(n)}$ for the $n$th-convolution of $F$ with itself, $\bar{F} = 1 - F$ for the tail of $F$, $\overline{F^{(n)}} = 1 - F^{(n)}$, and lower case letters $f, g$ always denote Laplace–Stieltjes transforms of the corresponding distribution functions $F, G$.

**Definition 1.1.** A distribution function $F$ on $[0, \infty[$ *belongs to* $\mathscr{S}(\gamma)$ with $\gamma \geq 0$ iff

(i) $\lim_{x\to\infty} \overline{F^{(2)}}(x)/\bar{F}(x) = c < \infty$,
(ii) $\lim_{x\to\infty} \bar{F}(x-y)/\bar{F}(x) = e^{\gamma y}$, for all $y$ real,
(iii) $f(-\gamma) = \int_0^\infty e^{\gamma x}\, dF(x) < \infty$.

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These classes of functions were introduced independently by Chistyakov [2] and Chover–Ney–Wainger [3, 4] to obtain detailed information about the Yaglom limit in a subcritical, age-dependent branching process for which the Malthusian parameter does not exist. Other applications include renewal theory [16], random walks [18], queues [10, 13] and infinite divisibility [7, 8].

In [16] the following related classes were investigated.

**Definition 1.2.** A distribution function $F$ on $[0, \infty[$ *belongs to* $\mathcal{T}(\gamma)$ iff

(i) $f(-\gamma)<\infty$,

(ii) the so-called $\gamma$-transform $F_\gamma$ defined by

$$F_\gamma(x)=f(-\gamma)^{-1}\int_0^x \mathrm{e}^{\gamma y}\,\mathrm{d}F(y)$$

belongs to $\mathcal{S}(0)$.

The class $\mathcal{S}(0)$ of subexponential distribution functions is usually denoted by $\mathcal{S}$. An extensive study of $\mathcal{S}$ and its applications is given in [6] and [8]. In [16] it was stated that, for all $\gamma\geqslant 0$, $\mathcal{S}(\gamma)=\mathcal{T}(\gamma)$. However, as mentioned in [8, Remark 2], the proof of this result is incomplete: in the present paper, we prove that for $\gamma>0$, $\mathcal{S}(\gamma)\neq\mathcal{T}(\gamma)$ (see Section 3). This has many ramifications, on which we shall comment in Section 5. For reasons that will become clear at the end of the paper, we shall mainly focus on properties of $\mathcal{S}(\gamma)$, and in Section 2 we bring together most of the important theorems about it. In Section 4 these results will be applied to the compound Poisson case, indicating very naturally how both classes give rise to different limit theorems. The compound Poisson example covers most of the known applications.

## 2. Properties of $\mathcal{S}(\gamma)$ functions

**Lemma 2.1** ([3]). *Using the notation of Definition* 1.1 *for all $F$ in $\mathcal{S}(\gamma)$ we have that $c=2f(-\gamma)$.*

This result is by no means trivial, its proof depending heavily on Banach algebra techniques. Using Lemma 2.1 we can rewrite Definition 1.1.

**Definition 2.2.** *$F$ belongs to $\mathcal{S}(\gamma)$* iff

(i) $\lim_{x\to\infty}\overline{F^{(2)}}(x)/\bar{F}(x)=2f(-\gamma)<\infty$,

(ii) $\lim_{x\to\infty}\bar{F}(x-y)/\bar{F}(x)=\mathrm{e}^{\gamma y}$ for all $y$ real.

The class of d.f.'s satisfying Definition 2.2(ii) will be denoted by $\mathcal{L}(\gamma)$. We refer to Embrechts–Goldie [6] for some closure properties of $\mathcal{L}(\gamma)$. The convergence in (ii) is automatically uniform on $y$-compacta (see [17, 12] or more recently [1, appendix]).

**Lemma 2.3** ([16, p. 1002 bottom lines]). *If* $\lim_{x\to\infty} \overline{F^{(2)}}(x)/\bar{F}(x) = c < \infty$ *and* $F \in \mathscr{L}(\gamma)$, *then* $f(-\gamma) < \infty$.

**Lemma 2.4.** *Suppose* $F \in \mathscr{L}(\gamma)$ *and* $\varepsilon > 0$, *then*
(i) $\lim_{x\to\infty} e^{(\gamma+\varepsilon)x}\bar{F}(x) = \infty$,
(ii) $\int_0^\infty e^{(\gamma+\varepsilon)y}\, dF(y) = \infty$.

**Proof.** (i) $F \in \mathscr{L}(\gamma)$ is equivalent to regular variation, with index $-\gamma$, of $\bar{F}(\log x)$. By a standard property [12, p. 18], $\lim_{x\to\infty} x^{\gamma+\varepsilon}\bar{F}(\log x) = \infty$, and (i) follows.
(ii) Since $\int_x^\infty e^{(\gamma+\varepsilon)y}\, dF(y) \geq e^{(\gamma+\varepsilon)x}\bar{F}(x)$, (ii) is immediate.

From (ii) we see that the assumptions $f(-\gamma) < \infty$ and $F \in \mathscr{L}(\gamma)$ imply that $-\gamma$ is the left abscissa of convergence of $f$. By itself, the assumption $f(-\gamma) < \infty$ implies $\lim_{x\to\infty} e^{\gamma x}\bar{F}(x) = 0$.

An explicit example of an $\mathscr{S}(\gamma)$ function is

$$G_\gamma(x) = \begin{cases} \int_0^x (e^{\sqrt{\gamma}}/(2\sqrt{\pi}))t^{-3/2} e^{-1/(4t)-\gamma t}\, dt, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

(For details, see [2, p. 646].) On the other hand the gamma distribution with parameters $a$ and $b$,

$$F'(x) = b^{-a}/\Gamma(a) x^{a-1} e^{-x/b} I_{[0,\infty[}(x),$$

with $f(s) = (1+bs)^{-a}$, $\operatorname{Re} s > -1/b$, satisfies $F \in \mathscr{L}(1/b)$ but $\overline{F^{(2)}}(x)/\bar{F}(x)$ does not converge to a finite limit. Note that $f(-1/b) = \infty$ (cf. Lemma 2.3).

**Lemma 2.5** ([4]). *Suppose* $F \in \mathscr{S}(\gamma)$ *and* $n \geq 1$ *an integer, then* $\lim_{x\to\infty} \overline{F^{(n)}}(x)/\bar{F}(x) = nf(-\gamma)^{n-1}$.

**Lemma 2.6** ([14]). *If* $F \in \mathscr{S}(\gamma)$ *and* $\varepsilon > 0$, *then there exists a constant* $K < \infty$ *such that for all* $n \geq 1$ *integer and for all* $x$ *positive,*

$$\overline{F^{(n)}}(x)/\bar{F}(x) \leq K(f(-\gamma)+\varepsilon)^n.$$

**Theorem 2.7.** *Suppose* $F \in \mathscr{S}(\gamma)$ *and that for a distribution function* $G$ *on* $[0, \infty[$, $\lim_{x\to\infty} \bar{G}(x)/\bar{F}(x) = c$ *where* $0 < c < \infty$. *Then* $G \in \mathscr{S}(\gamma)$.

**Proof.** The assumptions give $G \in \mathscr{L}(\gamma)$. Because

$$f(-\gamma) = 1 + \gamma \int_0^\infty e^{\gamma y}\bar{F}(y)\, dy < \infty,$$

it is easy to see that $g(-\gamma)$ is also finite.

Fix $v>0$ and $x>2v$. Put $X$, $Y$ independent random variables, with common distribution function $G$. Then

$$\{X+Y>x\}=\{X\leqslant v, X+Y>x\}\cup\{v<X\leqslant x-v, X+Y>x\}$$
$$\cup\{Y>v, X>x-v\}\cup\{Y\leqslant v, X+Y>x\}.$$

The events on the right-hand side being disjoint we conclude

$$\frac{\overline{G^{(2)}}(x)}{\bar{G}(x)}=2\int_0^v \frac{\bar{G}(x-y)}{\bar{G}(x)}\,\mathrm{d}G(y)+\int_v^{x-v}\frac{\bar{G}(x-y)}{\bar{G}(x)}\,\mathrm{d}G(y)+\frac{\bar{G}(x-v)}{\bar{G}(x)}\bar{G}(v). \tag{1}$$

Writing $I(x, v)=\int_v^{x-v}(\bar{G}(x-y)/\bar{G}(x))\,\mathrm{d}G(y)$, we shall show that

$$\lim_{v\to\infty}\limsup_{x\to\infty} I(x, v)=0. \tag{2}$$

Using $G\in\mathscr{L}(\gamma)$, the rest of the right-hand side of (1) converges to $2\int_0^v \mathrm{e}^{\gamma y}\,\mathrm{d}G(y)+\mathrm{e}^{\gamma v}\bar{G}(v)$ as $x\to\infty$, and the latter converges to $2g(-\gamma)$ as $v\to\infty$, so that, when (2) is established, the proof will be complete. Fix $\varepsilon$ satisfying $0<\varepsilon<c$, and $x_0$ such that $c-\varepsilon\leqslant\bar{G}(x)/\bar{F}(x)\leqslant c+\varepsilon$ for all $x\geqslant x_0$. Then, for $v\geqslant x_0$ and $x>2v$,

$$\begin{aligned}
I(x, v)&\leqslant -(c+\varepsilon)(c-\varepsilon)^{-1}\int_v^{x-v}(\bar{F}(x-y)/\bar{F}(x))\,\mathrm{d}\bar{G}(y)\\
&=-\frac{c+\varepsilon}{c-\varepsilon}\left(\frac{\bar{F}(v)\bar{G}(x-v)}{\bar{F}(x)}-\frac{\bar{F}(x-v)\bar{G}(v)}{\bar{F}(x)}-\int_{x-v}^{v}\frac{\bar{G}(x-t)}{\bar{F}(x)}\,\mathrm{d}\bar{F}(t)\right)\\
&\leqslant-\frac{c+\varepsilon}{c-\varepsilon}\left(\bar{F}(v)\frac{\bar{G}(x-v)}{\bar{G}(x)}\frac{\bar{G}(x)}{\bar{F}(x)}-\frac{\bar{F}(x-v)}{\bar{F}(x)}\bar{G}(v)+(c+\varepsilon)\int_{x-v}^{v}\frac{\bar{F}(x-t)}{\bar{F}(x)}\,\mathrm{d}F(t)\right)\\
&\leqslant-(c+\varepsilon)(c-\varepsilon)^{-1}(c\,\mathrm{e}^{\gamma v}\bar{F}(v)-\mathrm{e}^{\gamma v}\bar{G}(v)-(c+\varepsilon)H(v))+\mathrm{o}(1)
\end{aligned}$$

as $x\to\infty$, where

$$H(v)=\limsup_{x\to\infty}\int_v^{x-v}(\bar{F}(x-y)/\bar{F}(x))\,\mathrm{d}F(y).$$

By replacing $G$ by $F$ in (1) one finds $\lim_{v\to\infty}H(v)=0$, and so (2) is proved, as required.

A useful inequality is the following.

**Lemma 2.8.** *If $F\in\mathscr{L}(\gamma)$, then, for all $n\geqslant 1$ integer,*

$$\liminf_{x\to\infty}\overline{F^{(n)}}(x)/\bar{F}(x)\geqslant nf(-\gamma)^{n-1}.$$

*(This result holds even if $f(-\gamma)=\infty$.)*

**Proof.** For $x \geqslant A$,

$$\overline{F^{(n)}}(x) = \int_0^\infty \bar{F}(x - t \wedge A)\,\mathrm{d}F^{(n-1)}(t) + \int_0^\infty I_{[0,x-A[}(t)\overline{F^{(n-1)}}(x-t)\,\mathrm{d}F(t) \quad (3)$$

where $t \wedge A$ denotes minimum of $t$ and $A$. Let $I_n$ denote $\liminf_{x\to\infty} \overline{F^{(n)}}(x)/\bar{F}(x)$. Divide (3) by $\bar{F}(x)$ and use Fatou's lemma:

$$I_n \geqslant \int_0^\infty \lim_{x\to\infty} (\bar{F}(x - t\wedge A)/\bar{F}(x))\,\mathrm{d}F^{(n-1)}(t)$$

$$+ \int_0^\infty \liminf_{x\to\infty}\left( I_{[0,x-A[}(t) \frac{\overline{F^{(n-1)}}(x)}{\bar{F}(x)} \frac{\overline{F^{(n-1)}}(x-t)}{\overline{F^{(n-1)}}(x)}\right) \mathrm{d}F(t).$$

By Embrechts–Goldie [6, Theorem 3(b)] we know that $F \in \mathscr{L}(\gamma)$ implies $F^{(n-1)} \in \mathscr{L}(\gamma)$, and so the lim inf in the last integral reduces to $I_{n-1}\,\mathrm{e}^{\gamma t}$. Thus

$$I_n \geqslant \int_0^\infty \mathrm{e}^{\gamma(t\wedge A)}\,\mathrm{d}F^{(n-1)}(t) + I_{n-1}f(-\gamma) \to f(-\gamma)^{n-1} + I_{n-1}f(-\gamma) \quad \text{as } A\to\infty.$$

The result follows by induction.

The following lemma provides a useful criterion for proving $\mathscr{S}(\gamma)$-membership.

**Lemma 2.9.** *Suppose $F \in \mathscr{L}(\gamma)$ and for some $n \geqslant 2$,*

$$\limsup_{x\to\infty} \left(\overline{F^{(n)}}(x)/\bar{F}(x)\right) \leqslant nf(-\gamma)^{n-1} < \infty.$$

*Then $F \in \mathscr{S}(\gamma)$.*

**Proof.** Let $S_n$ denote $\limsup_{x\to\infty} \overline{F^{(n)}}(x)/\bar{F}(x)$. Divide (3) by $\bar{F}(x)$ and take lim sup, then the first term on the right-hand side converges (by dominated convergence)

$$nf(-\gamma)^{n-1} \geqslant S_n = \lim_{x\to\infty} \int_0^\infty \frac{\bar{F}(x - t\wedge A)}{\bar{F}(x)}\,\mathrm{d}F^{(n-1)}(t)$$

$$+ \limsup_{x\to\infty} \int_0^{x-A} \frac{\overline{F^{(n-1)}}(x-t)}{\bar{F}(x)}\,\mathrm{d}F(t).$$

Therefore

$$nf(-\gamma)^{n-1} - \int_0^\infty \mathrm{e}^{\gamma(t\wedge A)}\,\mathrm{d}F^{(n-1)}(t) \geqslant$$

$$\geqslant \limsup_{x\to\infty} \frac{\overline{F^{(n-1)}}(x)}{\bar{F}(x)} \int_0^{x-A} \frac{\overline{F^{(n-1)}}(x-t)}{\overline{F^{(n-1)}}(x)}\,\mathrm{d}F(t)$$

$$\geqslant \limsup_{x\to\infty} \frac{\overline{F^{(n-1)}}(x)}{\bar{F}(x)} \int_0^{A} \frac{\overline{F^{(n-1)}}(x-t)}{\overline{F^{(n-1)}}(x)}\,\mathrm{d}F(t)$$

$$= S_{n-1} \lim_{x\to\infty} \int_0^{A} \frac{\overline{F^{(n-1)}}(x-t)}{\overline{F^{(n-1)}}(x)}\,\mathrm{d}F(t)$$

(the latter limit exists, by dominated convergence, because $F^{(n-1)}\in\mathcal{L}(\gamma)$)

$$=S_{n-1}\int_0^A \mathrm{e}^{\gamma t}\,\mathrm{d}F(t).$$

Let $A\to\infty$, whence $S_{n-1}\le(n-1)f(-\gamma)^{n-2}$. Repeating the process we find eventually that $S_2\le 2f(-\gamma)$, and so $F\in\mathcal{S}(\gamma)$ by Lemma 2.8.

A crucial point for proving limit theorems using $\mathcal{S}(\gamma)$ functions is the convolution-roots closure of $\mathcal{S}(\gamma)$. The following theorem is the best known result in that direction.

**Theorem 2.10.** *If $F\in\mathcal{L}(\gamma)$ and, for some positive integer $k$, $F^{(k)}\in\mathcal{S}(\gamma)$, then $F\in\mathcal{S}(\gamma)$.*

**Proof.** From Lemma 2.5 we know that $\overline{F^{(2k)}}(x)/\overline{F^{(k)}}(x)\to 2f(-\gamma)^k$. Moreover, all positive-integer convolution powers of $F$ are in $\mathcal{L}(\gamma)$ (see [6, Theorem 3(b)]). Now for fixed $A$ and any $x\ge A$,

$$2\int_0^A \frac{\overline{F^{(k)}}(x-t)}{\overline{F^{(k)}}(x)}\,\mathrm{d}F^{(k)}(t)+\int_A^{x-A}\frac{\overline{F^{(k)}}(x-t)}{\overline{F^{(k)}}(x)}\,\mathrm{d}F^{(k)}(t)+\frac{\overline{F^{(k)}}(x-A)}{\overline{F^{(k)}}(x)}\overline{F^{(k)}}(A)=$$

$$=\frac{\overline{F^{(2k)}}(x)}{\overline{F^{(k)}}(x)}\to 2f(-\gamma)^k\quad\text{as } x\to\infty.$$

On the left-hand side the first term converges, using dominated convergence, to $2\int_0^A \mathrm{e}^{\gamma t}\,\mathrm{d}F^{(k)}(t)$ and the third term converges to $\mathrm{e}^{\gamma A}\overline{F^{(k)}}(A)$. Thus

$$\int_A^{x-A}\frac{\overline{F^{(k)}}(x-t)}{\overline{F^{(k)}}(x)}\,\mathrm{d}F^{(k)}(t)\to 2\int_A^\infty \mathrm{e}^{\gamma t}\,\mathrm{d}F^{(k)}(t)-\mathrm{e}^{\gamma A}\overline{F^{(k)}}(A). \tag{4}$$

Fix $u>0$ so that $F^{(k-1)}(u)>0$. Then, for $x\ge 4u$, we can split up $\overline{F^{(k)}}(x)$ to get

$$1=\int_0^{2u}\frac{\bar{F}(x-t)}{\overline{F^{(k)}}(x)}\,\mathrm{d}F^{(k-1)}(t)+\int_0^{2u}\frac{\overline{F^{(k-1)}}(x-t)}{\overline{F^{(k)}}(x)}\,\mathrm{d}F(t)$$

$$+\int_{2u}^{x-2u}\frac{\overline{F^{(k-1)}}(x-t)}{\overline{F^{(k)}}(x)}\,\mathrm{d}F(t)+\overline{F^{(k-1)}}(2u)\bar{F}(x-2u)/\overline{F^{(k)}}(x)$$

$$=I_1(x)+I_2(x)+I_3(x)+I_4(x),\quad\text{say.} \tag{5}$$

We bound $I_2$, $I_3$, $I_4$ in turn.

Provided the integrand of $I_2$ is dominated by some function of $t$ which is integrable with respect to $\mathrm{d}F(t)$ over $]0, 2u]$, we may by Fatou's lemma say that

$$\begin{aligned}\limsup_{x\to\infty} I_2(x) &\le \int_0^{2u} \limsup_{x\to\infty} \frac{\overline{F^{(k-1)}}(x-t)}{\overline{F^{(k)}}(x)}\,\mathrm{d}F(t)\\ &= \int_0^{2u} \limsup_{x\to\infty} \frac{\overline{F^{(k-1)}}(x-t)}{\overline{F^{((k-1)k)}}(x-t)} \lim_{x\to\infty} \frac{\overline{F^{((k-1)k)}}(x-t)}{\overline{F^{(k)}}(x-t)}\\ &\qquad \times \lim_{x\to\infty} \frac{\overline{F^{(k)}}(x-t)}{\overline{F^{(k)}}(x)}\,\mathrm{d}F(t),\end{aligned}$$

where, in the latter integrand, the middle factor is $(k-1)f(-\gamma)^{k(k-2)}$ because $F^{(k)} \in \mathscr{S}(\gamma)$, using Lemma 2.5 (i.e., the limit exists and has this value) and the limit in the third factor exists and is $\mathrm{e}^{\gamma t}$ because $F^{(k)} \in \mathscr{L}(\gamma)$. The first factor (the lim sup) is at most $1/(kf(-\gamma)^{(k-1)^2})$ because $F^{(k-1)} \in \mathscr{L}(\gamma)$, using Lemma 2.8. The required domination of the integrand of $I_2$ is obtainable using these results, and so the use of Fatou's lemma is justified and we have

$$\limsup_{x\to\infty} I_2(x) \le ((k-1)/(kf(-\gamma))) \int_0^{2u} \mathrm{e}^{\gamma t}\,\mathrm{d}F(t). \tag{6}$$

For $I_3$, let $S_{k-1}, X_k, S'_k$ be mutually independent random variables with distribution functions $F^{(k-1)}, F, F^{(k)}$ respectively, and set $S_k = S_{k-1} + X_k$. Then

$$\begin{aligned}&F^{(k-1)}(u) \int_{2u}^{x-2u} \overline{F^{(k-1)}}(x-t)\,\mathrm{d}F(t) \le\\ &\le F^{(k-1)}(u) \int_{2u}^{x-2u} \overline{F^{(k)}}(x-t)\,\mathrm{d}F(t)\\ &= \mathbf{P}(S_{k-1} \le u)\mathbf{P}(2u < X_k \le x-2u,\, x < X_k + S'_k)\\ &\le \mathbf{P}(S_{k-1} \le u < X_k \le x-2u,\, x < X_k + S'_k)\\ &\le \mathbf{P}(u < S_k \le x-u,\, x < S_k + S'_k),\end{aligned}$$

so that

$$\begin{aligned}F^{(k-1)}(u) I_3(x) &\le \int_u^{x-u} (\overline{F^{(k)}}(x-t)/\overline{F^{(k)}}(x))\,\mathrm{d}F^{(k)}(t)\\ &\to 2\int_u^{\infty} \mathrm{e}^{\gamma t}\,\mathrm{d}F^{(k)}(t) - \mathrm{e}^{\gamma u}\overline{F^{(k)}}(u)\end{aligned} \tag{7}$$

by (4), as $x \to \infty$. Lastly,

$$\begin{aligned}\limsup_{x\to\infty} I_4(x) &\le \overline{F^{(k)}}(2u) \limsup_{x\to\infty} \overline{F^{(k)}}(x-2u)/\overline{F^{(k)}}(x)\\ &= \overline{F^{(k)}}(2u)\,\mathrm{e}^{2\gamma u}.\end{aligned} \tag{8}$$

Then from (5), (6), (7) and (8)

$$\liminf_{x\to\infty} I_1(x) \geq 1 - \frac{k-1}{kf(-\gamma)}\int_0^{2u} \mathrm{e}^{\gamma t}\,\mathrm{d}F(t) - \frac{2}{\overline{F^{(k-1)}}(u)}\int_u^{\infty} \mathrm{e}^{\gamma t}\,\mathrm{d}F^{(k)}(t)$$

$$-\overline{F^{(k)}}(2u)\,\mathrm{e}^{2\gamma u} \to 1/k \quad \text{as } u\to\infty. \tag{9}$$

Now let $x_n \to \infty$ be any sequence such that $\overline{F^{(k)}}(x)/\bar{F}(x) \to r \leq \infty$. Then

$$I_1(x_n) = \frac{\bar{F}(x_n)}{\overline{F^{(k)}}(x_n)}\int_0^{2u} \frac{\bar{F}(x_n - t)}{F(x_n)}\,\mathrm{d}F^{(k-1)}(t)$$

$$\to (1/r)\int_0^{2u} \mathrm{e}^{\gamma t}\,\mathrm{d}F^{(k-1)}(t) \tag{10}$$

as $n\to\infty$, by dominated convergence. Therefore, the right-hand side of (10) is at least the right-hand side of (9) and letting $u\to\infty$ we conclude that $f(-\gamma)^{k-1}/r \geq 1/k$, so that

$$\limsup_{x\to\infty} \overline{F^{(k)}}(x)/\bar{F}(x) \leq kf(-\gamma)^{k-1}.$$

The proof is ended with an appeal to Lemma 2.9.

The basic assumption $F^{(k)} \in \mathscr{S}(\gamma)$ already implies that $F^{(k)} \in \mathscr{L}(\gamma)$; however, we cannot prove that it always follows that $F \in \mathscr{L}(\gamma)$. Therefore we made the following conjecture [6]:

"If for any integer $k \geq 2$, $F^{(k)} \in \mathscr{L}(\gamma)$, then $F \in \mathscr{L}(\gamma)$".

## 3. The counterexample

**Theorem 3.1.** *For every $\gamma > 0$, $\mathscr{S}(\gamma) \subsetneq \mathscr{T}(\gamma)$.*

**Proof.** (i) Suppose $F \in \mathscr{S}(\gamma)$ and denote $H(x) = F_\gamma(x)$ (see Definition 1.2(ii)). Now

$$\bar{H}(x) = f(-\gamma)^{-1}\int_x^{\infty} \mathrm{e}^{\gamma y}\,\mathrm{d}F(y)$$

$$= f(-\gamma)^{-1}\,\mathrm{e}^{\gamma x}\bar{F}(x) + \gamma f(-\gamma)^{-1}\int_x^{\infty} \mathrm{e}^{\gamma y}\bar{F}(y)\,\mathrm{d}y$$

$$\sim \gamma f(-\gamma)^{-1}\int_x^{\infty} \mathrm{e}^{\gamma y}\bar{F}(y)\,\mathrm{d}y \quad \text{as } x\to\infty,$$

using the slow variation of $u^{\gamma}\bar{F}(\log u)$. One easily proves that

$$H^{(2)}(x) = f(-\gamma)^{-2}\int_0^{x} \mathrm{e}^{\gamma y}\,\mathrm{d}F^{(2)}(y)$$

and so

$$\overline{H^{(2)}}(x) \sim \gamma f(-\gamma)^{-2} \int_x^\infty e^{\gamma y} \overline{F^{(2)}}(y)\, dy,$$

using the slow variation of $\overline{F^{(2)}}(\log u)u^\gamma$. Thus

$$\lim_{x\to\infty} \overline{H^{(2)}}(x)/\bar{H}(x) = f(-\gamma)^{-1} \lim_{x\to\infty} \overline{F^{(2)}}(x)/\bar{F}(x) = 2,$$

whence $F \in \mathcal{T}(\gamma)$.

(ii) $\mathcal{S}(\gamma) \neq \mathcal{T}(\gamma)$. Let $c = \sum_{n=1}^\infty n^{-2} e^{-2^n}$ and let $F$ be atomic with atoms of mass $c^{-1} n^{-2} e^{-2^n}$ at the points $2^n$, $n = 1, 2, \ldots$. Then

$$f(-1) = \int_0^\infty e^x \, dF(x) = c^{-1} \sum_{n=1}^\infty n^{-2} = \pi^2/(6c).$$

So, with $\gamma = 1$, $F_\gamma$ is atomic with atoms of mass $6/(\pi^2 n^2)$ at points $2^n$, $n = 1, 2, \ldots$. Now, for $2^{n-1} \leq x < 2^n$,

$$1 \leq \bar{F}_1(x)/\bar{F}_1(2x) = \sum_{j=n}^\infty j^{-2} \Big/ \sum_{j=n+1}^\infty j^{-2} \to 1 \quad \text{as } x \to \infty.$$

Thus $\bar{F}_1$, being monotone, is slowly varying, so $F_1 \in \mathcal{S}$. However, $F$ does not belong to any $\mathcal{S}(\gamma)$ for $\gamma > 0$, because $\bar{F}(2^n + 1)/\bar{F}(2^n) = 1$ for all $n$. Indeed, $\lim_{x\to\infty} \bar{F}(x+t)/\bar{F}(x)$ fails to exist for any $t \neq 0$, as one may see directly or from the consideration that the limit would have to be an exponential function, as a monotone solution of the Hamel functional equation.

The example above is similar to the one given by Feller [9, Example 32] showing that a function with a slowly varying partial moment need not be regularly varying itself. It follows from the proof that there exist $\mathcal{T}(\gamma)$ functions for which $\bar{F}(x-y)/\bar{F}(x)$ does not converge, this being one of the main disadvantages of these classes. Theorem 3.1 also shows how to interpret the following statement of Chover–Ney–Wainger [4, p, 664]:

> "One way to construct densities whose distributions are in $\mathcal{S}(\gamma)$ for $\gamma > 0$ is to multiply densities whose distributions are in $\mathcal{S}$ by negative exponentials. Thus, if $G$ is absolutely continuous and $G'(t) \sim t^{-b} e^{-\gamma t}$, $b > 1$, then $G$ is such a distribution".

## 4. Example: the compound Poisson case

In this paragraph, we always suppose $\lambda > 0$ and, for a certain distribution function $G$ on $[0, \infty[$, $F$ to be a $(\lambda, G)$-compound Poisson distribution,

$$F(x) = e^{-\lambda} \sum_{n=0}^\infty (\lambda^n/n!) G^{(n)}(x). \tag{11}$$

Often one is interested in relating the behaviour of $\bar{F}$ and $\bar{G}$ for large $x$. This was done for subexponential distribution functions in [8], generalizing the regular variation case dealt with in [5, 9, 14]. In getting exponential rates of decay, the classes $\mathscr{S}(\gamma)$ and $\mathscr{T}(\gamma)$ provide nice limit theorems.

**Theorem 4.1.** *Suppose $\gamma > 0$, then the following statements are equivalent*:

(i) $F \in \mathscr{T}(\gamma)$,

(ii) $G \in \mathscr{T}(\gamma)$,

(iii) $\lim_{x\to\infty} \bar{F}_\gamma(x)/\bar{G}_\gamma(x) = \lambda g(-\gamma)$.

**Proof.** Using the fact that $(M * N)_\gamma = M_\gamma * N_\gamma$, whenever these $\gamma$-transforms exist ($*$ denotes convolution), one easily verifies that, for all $x$ positive,

$$F_\gamma(x) = \mathrm{e}^{-\lambda g(-\gamma)} \sum_{n=0}^{\infty} ((\lambda g(-\gamma))^n/n!) G_\gamma^{(n)}(x). \tag{12}$$

But then the result immediately follows from Embrechts–Goldie–Veraverbeke [8, Theorem 3], indeed

$$\begin{aligned} F \in \mathscr{T}(\gamma) &\Leftrightarrow F_\gamma \in \mathscr{S} \quad \text{(Definition 1.2)} \\ &\Leftrightarrow G_\gamma \in \mathscr{S} \quad \text{([8, Theorem 3] and (12))} \\ &\Leftrightarrow G \in \mathscr{T}(\gamma) \quad \text{(Definition 1.2)} \\ &\Leftrightarrow \bar{F}_\gamma(x)/\bar{G}_\gamma(x) \to \lambda g(-\gamma) \quad \text{([8, Theorem 3] and (12))}, \end{aligned}$$

finishing the proof.

Whereas the $\mathscr{T}(\gamma)$ result follows directly from the known $\mathscr{S}$ theorem, the $\mathscr{S}(\gamma)$ case is much more involved and needs a proof along the same lines as in the $\mathscr{S}$ case. However, we have to impose some extra conditions: this is a general feature!

**Theorem 4.2.** (*We use the above notation.*)

(i) *If $F \in \mathscr{S}(\gamma)$ and for a certain, positive integer $k$, such that $0 < \lambda/k < \log(2f^{-1/k}(-\gamma))$, we have $F^{(1/k)} \in \mathscr{L}(\gamma)$, then $G \in \mathscr{S}(\gamma)$ and moreover $\bar{F}(x) \sim \lambda f(-\gamma)\bar{G}(x)$ as $x \to \infty$.*

(ii) *If $G \in \mathscr{S}(\gamma)$, then $F \in \mathscr{S}(\gamma)$ and indeed $\bar{F}(x) \sim \lambda f(-\gamma)\bar{G}(x)$.*

(iii) *Suppose $\bar{F}(x) \sim \lambda f(-\gamma)\bar{G}(x)$ as $x \to \infty$ and $F \in \mathscr{L}(\gamma)$, then $F, G \in \mathscr{S}(\gamma)$.*

**Proof.** (i) Consider, for $x \geq 0$,

$$R(x) = (\mathrm{e}^\lambda - 1)^{-1} \sum_{n=1}^{\infty} (\lambda^n/n!) G^{(n)}(x). \tag{13}$$

Then

$$(\mathrm{e}^\lambda - 1)\,\mathrm{e}^{-\lambda} R(x) = F(x) - \mathrm{e}^{-\lambda} U_0(x) \tag{14}$$

where $U_0(x)=0$ $(x<0)$, $=1$ $(x\geq 0)$. So for $x>0$,

$$(1-e^{-\lambda})\bar{R}(x)=\bar{F}(x). \tag{15}$$

Taking Laplace–Stieltjes transforms in (14) we find

$$r(s)=(e^{\lambda g(s)}-1)/(e^{\lambda}-1), \quad \operatorname{Re} s \geq -\gamma. \tag{16}$$

Assume in the first part of the proof $f(-\gamma)<2$ and $0<\lambda<\log(2/f(-\gamma))$. Using (16) the latter is equivalent with $0<(e^{\lambda}-1)r(-\gamma)<1$. So for all $x\geq 0$,

$$\lambda G(x)=-\sum_{n=1}^{\infty} n^{-1}(1-e^{\lambda})^n R^{(n)}(x). \tag{17}$$

By (15) and Theorem 2.7, $R\in\mathscr{S}(\gamma)$, whence using (for $R$) Lemma 2.6 and dominated convergence in (17), we get

$$\lim_{x\to\infty} \lambda\bar{G}(x)/\bar{R}(x)=(e^{\lambda}-1)(1-(1-e^{\lambda})r(-\gamma))^{-1}.$$

From $R\in\mathscr{S}(\gamma)$ and Theorem 2.7 we conclude $G\in\mathscr{S}(\gamma)$ and moreover $\lim_{x\to\infty}\bar{F}(x)/\bar{G}(x)=\lambda f(-\gamma)$.

Now fix $\lambda>0$ arbitrarily. We can find a positive integer $k$ such that $0<\lambda/k<\log(2f^{-1/k}(-\gamma))$, i.e., $f(-\gamma)<2^k$ and $\lambda<\log(2^k f^{-1}(-\gamma))$.

By assumption, such a $k$ exists for which $F^{(1/k)}\in\mathscr{L}(\gamma)$. Consider

$$H(x)=e^{-\lambda/k}\sum_{n=0}^{\infty}((\lambda/k)^n/n!)G^{(n)}(x).$$

Then $H^{(k)}=F\in\mathscr{S}(\gamma)$ and $H=F^{(1/k)}\in\mathscr{L}(\gamma)$, so Theorem 2.10 yields $H\in\mathscr{S}(\gamma)$.

Moreover, $h(-\gamma)<2$ and $0<\lambda/k<\log(2/h(-\gamma))$, and therefore the first part of the proof applies.

(ii) This is easy, using Lemmas 2.5 and 2.6, Theorem 2.7 and dominated convergence.

(iii) Since $F\in\mathscr{L}(\gamma)$, we also have $G\in\mathscr{L}(\gamma)$. Using Lemma 2.8 we find for all integer $n$

$$\liminf_{x\to\infty}\overline{G^{(n)}}(x)/\bar{G}(x)\geq n(g(-\gamma))^{n-1}<\infty.$$

So

$$\bar{F}(x)/\bar{G}(x)=(e^{-\lambda}\lambda^2/2)(\overline{G^{(2)}}(x)/\bar{G}(x)+e^{-\lambda}\sum_{n\neq 2}(\lambda^n/n!)\overline{G^{(n)}}(x)/\bar{G}(x)),$$

from which it follows easily that

$$\limsup_{x\to\infty}\overline{G^{(2)}}(x)/\bar{G}(x)\leq 2g(-\gamma),$$

whence $G\in\mathscr{S}(\gamma)$, consequently $F\in\mathscr{S}(\gamma)$.

A solution of the above mentioned conjecture on convolution-roots closure of $\mathscr{L}(\gamma)$ would enable us to drop the technical assumption in part (i) of the theorem. However, the extra assumption $F \in \mathscr{L}(\gamma)$ in part (iii) seems to be much deeper and is intimately related to the question of the relation between the convergence of $\overline{F^{(2)}}(x)/\bar{F}(x)$ and that of $\bar{F}(x-y)/\bar{F}(x)$. An example of a distribution function for which $\overline{F^{(2)}}(x)/\bar{F}(x)$ converges but $\bar{F}(x-y)/\bar{F}(x)$ does not would elucidate this problem.

The main difference between Theorems 4.1 and 4.2 is the 'Mercerian' statement (iii); indeed in Theorem 4.1 one compares integrated tails $\bar{F}_\gamma/\bar{G}_\gamma$, whereas in Theorem 4.2 one has much more precise information, namely on the tails $\bar{F}/\bar{G}$ itself. This is one of the reasons why we focus on $\mathscr{S}(\gamma)$, rather than on $\mathscr{T}(\gamma)$.

These considerations also clarify the general remark on the importance of $\mathscr{S}(\gamma)$ as a Mercerian class, given in [16, p. 1010].

## 5. Some applications

In both [16] and [18] the classes $\mathscr{S}(\gamma)$ were used in order to derive limit theorems for distribution functions with exponential tail decay. However, the results were based on the misapprehension that $\mathscr{S}(\gamma) = \mathscr{T}(\gamma)$. It is the aim of this paragraph to present the correct statements, using the previous theory on $\mathscr{S}(\gamma)$ and $\mathscr{T}(\gamma)$.

In [16] the context is transient renewal theory. Let $F$ be a defective distribution function such that $F(0+) = 0$ and for all $x$ positive $F(x) < F(\infty) = \alpha$ where $0 < \alpha < 1$. The renewal function associated with $F$ is given by $U(x) = \sum_{n=1}^{\infty} F^{(n)}(x)$, which can be rewritten as

$$R(x) = (1-\alpha)\alpha^{-1} \sum_{n=1}^{\infty} \alpha^n G^{(n)}(x), \tag{18}$$

where the nondefective distribution functions $R$ and $G$ are defined by $R(x) = (1-\alpha)\alpha^{-1} U(x)$ and $G(x) = \alpha^{-1} F(x)$.

The relation (18) being compound geometric, we can prove the following in a similar way as in Theorem 4.1 (but now the corresponding $\mathscr{S}$ result is [8, Corollary 3]).

**Corollary 5.1.** *Assume* $f(-\gamma) = \int_0^\infty e^{\gamma y}\, dF(y) < 1$, *where* $\gamma \geq 0$. *The following statements are equivalent*:

(i) $F^{-1}(\infty)F(x) \in \mathscr{T}(\gamma)$.

(ii) $U^{-1}(\infty)U(x) \in \mathscr{T}(\gamma)$.

(iii) $\displaystyle\lim_{x\to\infty} \frac{U_\gamma(\infty) - U_\gamma(x)}{F_\gamma(\infty) - F_\gamma(x)} = (1 - f(-\gamma))^{-2}$,

*where* $U_\gamma(x) = \int_0^x e^{\gamma y}\, dU(y)$, $F_\gamma(x) = \int_0^x e^{\gamma y}\, dF(y)$.

This result corrects [16, Corollary 6]. Observe that (iii) is not equivalent to

$$\text{(iv)}\quad \lim_{x\to\infty} \frac{1-U^{-1}(\infty)U(x)}{1-F^{-1}(\infty)F(x)} = (1-f(-\gamma))^{-1}.$$

We know that (iv) plus the extra condition $F^{-1}(\infty)F(x) \in \mathscr{L}(\gamma)$ yields $F^{-1}(\infty)F(x) \in \mathscr{S}(\gamma)$ which follows as in Theorem 4.2(iii). Further information on $\mathscr{S}(\gamma)$ functions in this set up can be obtained from Theorem 4.2.

In [18] the context is random walk theory. Suppose $F$ is a distribution function on the real line. Given $X_1, X_2, \ldots$ a sequence of i.i.d. random variables, distributed according to $F$, we denote $N = \min\{n>0 \mid S_n = \sum_{j=1}^{n} X_j > 0\}$ and $\bar{N} = \min\{n>0 \mid S_n \leq 0\}$. Writing $f, f_+$ and $f_-$ for the Fourier–Stieltjes transforms of $X$, $S_N$ and $S_{\bar{N}}$ we get the Wiener–Hopf equation $1-f(t) = (1-f_+(t))(1-f_-(t))$. The distribution functions of $S_N$ and $S_{\bar{N}}$ are denoted by $F_+$ and $F_-$, whereas $F_0(x) = F(x)U_0(x)$ and for all $x$ for which the integrals exist

$$g_-(x) = \int_{-\infty}^{0+} e^{xy}\, dF_-(y), \qquad g_+(x) = \int_{0+}^{\infty} e^{-xy}\, dF_+(y).$$

According to the constants $A = \sum_{n=1}^{\infty} n^{-1}\mathbf{P}(S_n \leq 0)$, $B = \sum_{n=1}^{\infty} n^{-1}\mathbf{P}(S_n > 0)$ there are three cases to consider:

(i) $A<\infty$ (hence $B=\infty$) so $F_+(\infty)=1$ and $F_-(0+) = 1-e^{-A}$.

(ii) $B<\infty$ (hence $A=\infty$) so $F_+(\infty) = 1-e^{-B}$ and $F_-(0+)=1$.

(iii) $A=B=\infty$, then $F_+(\infty) = F_-(\infty) = 1$.

In [18] eq. (14) reads, for $x$ positive,

$$\frac{\bar{F}(x)}{F_+(\infty)-F_+(x)} = 1 - \int_{-\infty}^{0+} \frac{F_+(\infty)-F_+(x-y)}{F_+(\infty)-F_+(x)}\, dF_-(y). \tag{19}$$

Using dominated convergence and Proposition 2.7 we find the following.

**Corollary 5.2.** *Suppose $\gamma>0$ and $F_+ \in \mathscr{S}(\gamma)$, then*

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{F_+(\infty)-F_+(x)} = 1-g_-(\gamma)$$

*and $F_0 \in \mathscr{S}(\gamma)$.*

*If $F_+ \in \mathscr{S}$ and $A<\infty$ (i.e., $g_-(0) = 1-e^{-A} < 1$), then $F_0 \in \mathscr{S}$ and*

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{F_+(\infty)-F_+(x)} = 1-g_-(0) = e^{-A}.$$

Moreover we know that [18, eq. (18)]

$$\bar{F}_{+\gamma}(x) = \frac{f(-i\gamma)}{(1-g_-(\gamma))g_+(-\gamma)} \int_{-\infty}^{0+} \bar{F}_\gamma(x-y)\, dG(y) \quad \text{for } \gamma>0, \tag{20}$$

$$\bar{F}_+(x) = (1-g_-(0))^{-1} \int_{-\infty}^{0+} \bar{F}(x-y)\, dG(y) \quad \text{for } \gamma=0, A<\infty. \tag{21}$$

In the above $G$ is defined for all $\gamma \geq 0$ as

$$G(x) = (1 - g_-(\gamma)) \sum_{n=0}^{\infty} B^{(n)}(x), \qquad B(\mathrm{d}x) = \mathrm{e}^{\gamma x}\, \mathrm{d}F_-(x).$$

**Corollary 5.3** (from (21)). *Suppose $A$ finite and $\gamma \geq 0$, then $F_0 \in \mathcal{S}(\gamma)$ implies $F_+ \in \mathcal{S}(\gamma)$ and* $\lim_{x\to\infty}(\bar{F}_+(x)/\bar{F}(x)) = (1 - g_-(\gamma))^{-1}$.

**Corollary 5.4** (from (20)). *Suppose $A$ infinite and $\gamma > 0$, then $F_0 \in \mathcal{S}(\gamma)$ implies $F_+ \in \mathcal{T}(\gamma)$ and*

$$\lim_{x\to\infty} (\bar{F}_{+\gamma}(x)/\bar{F}_\gamma(x)) = \frac{f(-\mathrm{i}\gamma)}{(1 - g_-(\gamma))g_+(-\gamma)}. \tag{22}$$

Corollaries 5.2, 5.3 and 5.4 correct the corresponding statements in [18, Theorem 1]. Finally, we should remark that it is mainly (20) which causes the problems; indeed, as shown in Corollary 5.4, if the right tail of $F$ belongs to $\mathcal{S}(\gamma)$, then we know that $F_\gamma$ belongs to $\mathcal{S}$ and hence dominated convergence in (20) yields $F_{+\gamma} \in \mathcal{S}$; so $F_+$ belongs to $\mathcal{T}(\gamma)$ but not necessarily to $\mathcal{S}(\gamma)$. Moreover, the limit relation holds between the integrated tails. The corresponding $\mathcal{T}(\gamma)$ theorem would read as follows:

$$F_0 \in \mathcal{T}(\gamma) \text{ implies } F_+ \in \mathcal{T}(\gamma) \text{ and (22) holds,}$$

this result being stronger than Corollary 5.4 in as much that it yields the same conclusion on the non-empty set $\mathcal{T}(\gamma)\backslash\mathcal{S}(\gamma)$. However, we have no information whatsoever on the tail behaviour of the latter functions.

In an analogous way one should reformulate [18, Theorem 2(iii), remark and corollary on p. 36]. For further comments on the importance of these results, we refer to [18].

## 6. Remarks

**6.1.** Chover–Ney–Wainger [14] used the notation $\mathcal{S}(d)$, $d \geq 1$, whenever
 (i) $\lim_{x\to\infty} \overline{F^{(2)}}(x)/\bar{F}(x) = c < \infty$,
 (ii) $\lim_{x\to\infty} \bar{F}(x-y)/\bar{F}(x) = \mathrm{e}^{\gamma y}$ for a certain $\gamma$ positive,
 (iii) $f(-\gamma) = d < \infty$.
However, we think that labelling these classes according to the left abscissa of convergence for $f$ is much more appealing.

**6.2.** The proof of [16, Theorem 1(i)] is incomplete (see [8, Remark 5.2]), and we do not know whether its statement,

> "$F$ belongs to $\mathcal{S}$ iff $\lim_{x\to\infty} \overline{F^{(2)}}(x)/\bar{F}(x) = c$, finite and
> for all $\varepsilon > 0$, $f(-\varepsilon) = \infty$"

holds. In the discrete case, a related problem was posed by Rudin [11, remark on p. 984].

**6.3.** Using the $\mathscr{S}(\gamma)$, $\mathscr{T}(\gamma)$ theory, we could build on Theorems 4.1 and 4.2 to obtain more general results on the tail behaviour of infinitely divisible distribution functions. Again, the $\mathscr{T}(\gamma)$ theorem follows quite readily from the original $\mathscr{S}$ result in [8].

**Theorem 6.4.** *Suppose $F$ is infinitely divisible on $[0, \infty[$ and $\gamma > 0$ such that $f(-\gamma)$ is finite. If $\nu$ is the Lévy-measure of $F$, we set $\mu = \nu(1, \infty)$ and $Q(x) = \mu^{-1}\nu(x, \infty)$, $x \geq 1$. The following are equivalent:*

(i) $F \in \mathscr{T}(\gamma)$.

(ii) $Q \in \mathscr{T}(\gamma)$.

(iii) $\lim_{x\to\infty} \bar{F}_\gamma(x)/\bar{Q}_\gamma(x) = \mu q(-\gamma)$.

The $\mathscr{S}(\gamma)$ version however is overloaded with technical assumptions and will not be given. Theorem 6.4 is sharp in as much that for every infinitely divisible $F$ on $[0, \infty[$, Steutel proved [15] the following:

"$F$ is either degenerate or $\bar{F}(x) = \exp\{-\mathrm{O}(x \log x)\}, \quad x \to \infty$".


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## References


[1] N.H. Bingham and C.M. Goldie, Extensions of regular variation Part I: Uniformity and quantifiers, University of London, Westfield College, preprint, 1979

[2] V.P. Chistyakov, A theorem on sums of independent positive random variables and its applications to branching processes, Theory Probab. Appl. 9 (1964) 640–648.

[3] J. Chover, P. Ney and S. Wainger, Functions of probability measures, J. Analyse Math. 26 (1973) 255–302.

[4] J. Chover, P. Ney and S. Wainger, Degeneracy properties of subcritical branching processes, Ann. Probab. 1 (1973) 663–673.

[5] J.W. Cohen, Some results on regular variation for distributions in queueing and fluctuation theory, J. Appl. Probab. 10 (1973) 343–353.

[6] P. Embrechts and C.M. Goldie, On closure and factorization properties of subexponential and related distributions, J. Austral. Math. Soc. Ser. A 29 (1980) 243–256.

[7] P. Embrechts and C.M. Goldie, Comparing the tail of an infinitely divisible distribution with integrals of its Lévy measure, Ann. Probab. 9 (1981) 468–481.

[8] P. Embrechts, C.M. Goldie and N. Veraverbeke, Subexponentiality and infinite divisibility, Z. Wahrsch. Verw. Gebiete 49 (1979) 335–347.

[9] W. Feller, An introduction to probability theory and its applications, Vol. II (Wiley, New York, 2nd ed., 1971).
[10] A.G. Pakes, On the tails of waiting time distributions, J. Appl. Probab. 15 (1975) 555–564.
[11] W. Rudin, Limits of ratios of tails of measures, Ann. Probab. 1 (1973) 982–984.
[12] E. Seneta, Regularly varying functions, Lecture Notes in Math. 508 (Springer, New York, 1976).
[13] W. Smith, On the tails of queueing-time distributions, Mimeo Series No. 830, Inst. of Stat., Univ. North Carolina at Chapel Hill, 1972.
[14] A.J. Stam, Regular variation of the tail of a subordinated probability distribution, Adv. Appl. Probab. 5 (1972) 308–327.
[15] F.W. Steutel, On the tails of infinitely divisible distributions, Z. Wahrsch. Verw. Gebiete 28 (1974) 273–276.
[16] J.L. Teugels, The class of subexponential distributions, Ann. Probab. 3 (1975) 1000–1011.
[17] T. van Aardenne-Ehrenfest, N.G. de Bruijn and J. Korevaar, A note on slowly oscillating functions, Nieuw Arch. Wisk. (2) 23 (1949) 77–86.
[18] N. Veraverbeke, Asymptotic behaviour of Wiener–Hopf factors of a random walk, Stochastic Process. Appl. 5 (1977) 27–37.