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On the definition of the intuitionistic fuzzy subgroups[☆]

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ABSTRACT

In this paper, a new kind of intuitionistic fuzzy subgroup theory, which is different from that of Ma, Zhan and Davvaz (2008) [22,23], is presented. First, based on the concept of cut sets on intuitionistic fuzzy sets, we establish the neighborhood relations between a fuzzy point x_a and an intuitionistic fuzzy set A. Then we give the definitions of the grades of x_a belonging to A, x_a quasi-coincident with A, x_a belonging to and quasi-coincident with A and x_a belonging to or quasi-coincident with A, respectively. Second, by applying the 3-valued Lukasiewicz implication, we give the definition of (α, β)-intuitionistic fuzzy subgroups of a group G for $\alpha, \beta \in \{\in, q, \in \wedge q, \in \vee q\}$, and we show that, in 16 kinds of (α, β)-intuitionistic fuzzy subgroups, the significant ones are the $(\in, \in)$-intuitionistic fuzzy subgroup, the $(\in, \in \vee q)$-intuitionistic fuzzy subgroup and the $(\in \wedge q, \in)$-intuitionistic fuzzy subgroup. We also show that A is a $(\in, \in)$-intuitionistic fuzzy subgroup of G if and only if, for any $a \in (0, 1]$, the cut set A_a of A is a 3-valued fuzzy subgroup of G, and A is a $(\in, \in \vee q)$-intuitionistic fuzzy subgroup (or $(\in, \in \vee q)$-intuitionistic fuzzy subgroup) of G if and only if, for any $a \in (0, 0.5]$(or for any $a \in (0.5, 1]$), the cut set A_a of A is a 3-valued fuzzy subgroup of G. At last, we generalize the $(\in, \in)$-intuitionistic fuzzy subgroup, $(\in, \in \vee q)$-intuitionistic fuzzy subgroup and $(\in \wedge q, \in)$-intuitionistic fuzzy subgroup to intuitionistic fuzzy subgroups with thresholds, i.e., $(s, t]$-intuitionistic fuzzy subgroups. We show that A is a $(s, t]$-intuitionistic fuzzy subgroup of G if and only if, for any $a \in (s, t]$, the cut set A_a of A is a 3-valued fuzzy subgroup of G. We also characterize the $(s, t]$-intuitionistic fuzzy subgroup by the neighborhood relations between a fuzzy point x_a and an intuitionistic fuzzy set A.

1. Introduction

Since the concept of fuzzy group was introduced by Rosenfeld in 1971 [1], the theories and approaches on different fuzzy algebraic structures developed rapidly. Anthony and Sherwood [2] gave the definition of fuzzy subgroup based on t-norm. Yuan and Lee [3] defined the fuzzy subgroup and fuzzy subring based on the theory of falling shadows. Liu [4] gave the definition of fuzzy invariant subgroups. By far, two books on fuzzy algebra have been published [5,6].

It is worth pointing out that Bhakat and Das [7,8] gave the concepts of (α, β)-fuzzy subgroups by using the "belong to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point x_a and a fuzzy set A, and introduced the concept of $(\in, \in \vee q)$-fuzzy subgroup. Yuan et al. [9] gave the definition of a fuzzy subgroup with thresholds from the aspect of multi-implication, which generalized the Rosenfeld's fuzzy subgroup and $(\in, \in \vee q)$-fuzzy subgroup to $(\lambda, \mu]$-fuzzy subgroup.

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Davvaz et al. [10–16] further generalized the results in [7–9]. Yuan et al. [15,16] applied the idea and approach in [7–9] into the researches of convex fuzzy subset and fuzzy topology.

K. Atanassov [17] introduced the concept of intuitionistic fuzzy sets in 1986. Since then, many researchers have investigated this topic such as intuitionistic fuzzy group [18] and intuitionistic fuzzy topology [19,20]. It is well known that the intuitionistic fuzzy set and the interval-valued fuzzy set are equivalent [21], and consequently the results about interval-valued fuzzy sets can be generalized to the intuitionistic fuzzy sets. In [22], Davvaz and Zhan,et al., presented the interval-valued (α, β)-fuzzy H_v-submodules. In [23], Ma and Zhan, et al., studied $(\in, \in \vee q)$-interval-valued fuzzy ideals of BCI-algebras. Davvaz et al. [22] and Ma et al. [23] built a method to study (α, β)-interval-valued fuzzy algebras. However, because of complexity of interval-valued fuzzy sets, main results in [22,23] are true only when the following conditions hold:

(1) Condition(E): $\bar{F}(x) \leq [0.5, 0.5]$ or $[0.5, 0.5] < \bar{F}(x)$ for all $x \in X$;
(2) Any two element of $D[0, 1] = \{[a^-, a^+] \mid 0 \leq a^- \leq a^+ \leq 1\}$ are comparable.

It is easily seen that the two conditions as above do not hold for all interval-valued fuzzy sets. If the two conditions are deleted, then main results in [22,23] may not be true. Therefore, a natural question to ask is if there exist a method to study (α, β)-intuitionistic fuzzy algebras with no conditions attached. Clearly, in order to answer this question, the neighborhood relations between a fuzzy point x_a and an intuitionistic fuzzy set A should be built.

In this paper, using cut sets on intuitionistic fuzzy sets presented in [24], the neighborhood relations between a fuzzy point and an intuitionistic fuzzy set are introduced, which are generalizations of neighborhood relations between an element and a set in set theory. Then, based on these neighborhood relations, we give the definitions of (α, β)-intuitionistic fuzzy subgroups of a group G differently from that of [22,23]. Also, we show that the significant ones obtained in this manner are the $(\in, \in)$-intuitionistic fuzzy subgroup, the $(\in, \in \vee q)$-intuitionistic fuzzy subgroup and the $(\in \wedge q, \in)$-intuitionistic fuzzy subgroup. Furthermore, as a generalization of the three intuitionistic fuzzy subgroups as above, we put forward the $(s, t]$-intuitionistic fuzzy subgroup. We prove that an intuitionistic fuzzy subset over a group is a $(s, t]$-intuitionistic fuzzy subgroup if and only if its a-cut set($a \in (s, t]$) is a 3-valued fuzzy subgroup.

The rest of this paper is organized as follows. In Section 2, we give some definitions and notations. In Section 3, based on the concept of cut sets on intuitionistic fuzzy sets presented in [24], we establish the neighborhood relations between a fuzzy point and an intuitionistic fuzzy set. In Section 4, we give the definition of (α, β)-intuitionistic fuzzy subgroup over a group G. In Section 5, we give the definition of $(s, t]$-intuitionistic fuzzy subgroup and prove that an intuitionistic fuzzy subset over a group is a $(s, t]$-intuitionistic fuzzy subgroup if and only if its a-cut set ($a \in (s, t]$) is a 3-valued fuzzy subgroup. Also, we characterize $(s, t]$-intuitionistic fuzzy subgroup by the neighborhood relations between a fuzzy point and an intuitionistic fuzzy set.

2. Preliminaries

Definition 2.1 ([7]). Let $A : X \to [0, 1]$ be a mapping. If there exist $a \in (0, 1]$ and $x \in A$ such that

$$A(y) = \begin{cases} a, & y = x \\ 0, & y \neq x, \end{cases}$$

then A is called a fuzzy point, and denoted by x_a.

Definition 2.2 ([17]). Let X be a set and $\mu_A : X \to [0, 1]$ and $\nu_A : X \longrightarrow [0, 1]$ be two mappings. If

$$\mu_A(x) + \nu_A(x) \leq 1, \quad \forall x \in X,$$

then we call $A = (X, \mu_A, \nu_A)$ an intuitionistic fuzzy subset over X, and denote $A(x) = (\mu_A(x), \nu_A(x))$.

Definition 2.3 ([1]). Let $A : G \to [0, 1]$ be a fuzzy subset over group G. If for any $x, y \in G$,

$$A(xy) \geq A(x) \wedge A(y), \qquad A(x^{-1}) \geq A(x),$$

then we call A a fuzzy subgroup of G.

Remark 2.1. In this paper, if A is a fuzzy subgroup of G and $\{A(x)|x \in G\} \subset \{0, \frac{1}{2}, 1\}$, then A is called a 3-valued fuzzy subgroup of G.

Theorem 2.1 ([5]). *A is a fuzzy subgroup of G if and only if for any $a \in (0, 1]$, $A_a = \{x|x \in G, A(x) \geq a\}$ is a subgroup of G.*

Definition 2.4 ([7]). Let A be a fuzzy subset over G and x_a be a fuzzy point.

(1) If $A(x) \geq a$, then we say x_a belong to A, and denote $x_a \in A$.
(2) If $A(x) + a \geq 1$, then we say x_a is quasi-coincident with A, and denote $x_a qA$.
(3) $x_a \in \wedge qA \Leftrightarrow x_a \in A$ and $x_a qA$.
(4) $x_a \in \vee qA \Leftrightarrow x_a \in A$ or $x_a qA$.

Definition 2.5 ([22,23]). Let $D[0,1]=\{[a^-,a^+]|0\leq a^-\leq a^+\leq 1\}$ and X be a set. Then

1. The mapping $\bar{F}:X\to D[0,1], x\mapsto [F^-(x),F^+(x)]$ is called an interval-valued fuzzy subset.
2. Let $x\in X, \bar{t}=[t^-,t^+]\in D[0,1]$. If the interval-valued fuzzy subset $\bar{G}$ satisfies

$$\bar{G}(y)=\begin{cases}\bar{t}, & y=x\\ [0,0], & y\neq x,\end{cases}$$

then $\bar{G}$ is called an interval-valued fuzzy point, and is denoted by $x_{\bar{t}}$.
3. Let $\bar{F}$ be an interval-valued fuzzy subset of X and $x_{\bar{t}}$ be an interval-valued fuzzy point. We call $x_{\bar{t}}$ belong to(or resp., is quasi-coincident with) $\bar{F}$, written by $x_{\bar{t}}\in\bar{F}$ (resp. $x_{\bar{t}}q\bar{F}$), if $\bar{F}(x)\geq\bar{t}$ (resp. $\bar{F}(x)+\bar{t}>[1,1]$); If $x_{\bar{t}}\in\bar{F}$ or $x_{\bar{t}}q\bar{F}$, then we write $x_{\bar{t}}\in\vee q\bar{F}$; If $\bar{F}(x)<\bar{t}$(resp. $\bar{F}(x)+\bar{t}\leq[1,1]$), then we write $x_{\bar{t}}\bar{\in}\bar{F}$(resp. $x_{\bar{t}}\bar{q}\bar{F}$); The symbol $\overline{\in\vee q}$ means that $\in\vee q$ does not hold.

Definition 2.6 ([24]). Let $A=(X,\mu_A,\nu_A)$ be an intuitionistic fuzzy subset over X, and $a\in[0,1]$.

(1) We call

$$A_a(x)=\begin{cases}1, & \mu_A(x)\geq a;\\ \frac{1}{2}, & \mu_A(x)<a\leq 1-\nu_A(x);\\ 0, & a>1-\nu_A(x),\end{cases}$$

and

$$A_{\underline{a}}(x)=\begin{cases}1, & \mu_A(x)>a;\\ \frac{1}{2}, & \mu_A(x)\leq a<1-\nu_A(x);\\ 0, & a\geq 1-\nu_A(x)\end{cases}$$

the a-the upper cut set and a-strong upper cut set of A, respectively.

(2) We call

$$A^a(x)=\begin{cases}1, & \nu_A(x)\geq a;\\ \frac{1}{2}, & \nu_A(x)<a\leq 1-\mu_A(x);\\ 0, & a>1-\mu_A(x),\end{cases}$$

and

$$A^{\underline{a}}(x)=\begin{cases}1, & \nu_A(x)>a;\\ \frac{1}{2}, & \nu_A(x)\leq a<1-\mu_A(x);\\ 0, & a\geq 1-\mu_A(x)\end{cases}$$

the a-lower cut set and a-strong lower cut set of fuzzy set A, respectively.

(3) We call

$$A_{[a]}(x)=\begin{cases}1, & \mu_A(x)+a\geq 1;\\ \frac{1}{2}, & \nu_A(x)\leq a<1-\mu_A(x);\\ 0, & a<\nu_A(x),\end{cases}$$

and

$$A_{[\underline{a}]}(x)=\begin{cases}1, & \mu_A(x)+a>1;\\ \frac{1}{2}, & \nu_A(x)<a\leq 1-\mu_A(x);\\ 0, & a\leq \nu_A(x)\end{cases}$$

the a-upper Q-cut set and a-strong upper Q-cut set of fuzzy set A, respectively.

(4) We call

$$A^{[a]}(x)=\begin{cases}1, & \nu_A(x)+a\geq 1;\\ \frac{1}{2}, & \mu_A(x)\leq a<1-\nu_A(x);\\ 0,\ a<\mu_A(x), & \end{cases}$$

and

$$A^{[\underline{a}]}(x) = \begin{cases} 1, & \nu_A(x) + a > 1; \\ \dfrac{1}{2}, & \mu_A(x) < a \leq 1 - \nu_A(x); \\ 0, & a \leq \mu_A(x) \end{cases}$$

the a-lower Q-cut set and a-strong lower Q-cut set of fuzzy set A, respectively.

It is obvious that $A_{[\underline{a}]}(x) = A_{\underline{1-a}}(x)$.

Property 2.1 ([24]). (1) $A_{\underline{a}} \subset A_a$; (2) $a < b \Rightarrow A_{\underline{a}} \supseteq A_{\underline{b}}$.

Definition 2.7 ([24]). Let $3^X = \{A \mid A : X \to \{0, \frac{1}{2}, 1\}$ is a mapping $\}$. For $A \in 3^X$ and $a \in [0, 1]$, let $a \circ A$ be an intuitionistic fuzzy subset of X and for any $x \in X$,

$$(a \circ A)(x) = \begin{cases} (0, 1), & A(x) = 0; \\ (a, 1 - a), & A(x) = 1; \\ (0, 1 - a), & A(x) = \dfrac{1}{2}. \end{cases}$$

Then we have the following decomposition theorem of intuitionistic fuzzy sets.

Theorem 2.2 ([24]). *Let $A = (X, \mu_A, \nu_A)$ be an intuitionistic fuzzy set, then*

$$A = \bigcup_{a \in [0,1]} a \circ A_a = \bigcup_{a \in [0,1]} a \circ A_{\underline{a}}.$$

3. The neighborhood relations between a fuzzy point and an intuitionistic fuzzy set

Let x_a be a fuzzy point and A be a fuzzy subset of X, then we have that

(i) $A(x) \geq a$ (i.e., $x_a \in A$) or $A(x) < a$ (i.e., $x_a \overline{\in} A$);
(ii) $a + A(x) > 1$ (i.e., $x_a q A$) or $a + A(x) \leq 1$ (i.e., $x_a \overline{q} A$).

If $x_{\bar{t}}$ is an interval-valued fuzzy point and $\bar{F}$ is an interval-valued fuzzy subset of X, then any one of the following cases may not hold:

(a) $x_{\bar{t}} \in \bar{F}$; (b) $x_{\bar{t}} \overline{\in} \bar{F}$; (c) $x_{\bar{t}} q \bar{F}$; (d) $x_{\bar{t}} \overline{q} \bar{F}$.

For example, Let $X = \{x\}$, $\bar{t} = [0.3, 0.6]$ and $\bar{F}(x) = [0.4, 0.5]$, then $(a) - (b)$ do not hold. Therefore, among discussions in [22,23], authors emphasize all interval-valued fuzzy subsets of X must satisfy the following conditions:

(I) Condition(E): $\bar{F}(x) \leq [0.5, 0.5]$ or $[0.5, 0.5] < \bar{F}(x)$ for all $x \in X$.
(II) Any two elements of $D[0, 1]$ are comparable.

If the two conditions are deleted, then many results in [22,23] may not true. In order to solve the problem, we first build the neighborhood relations between a fuzzy point x_a and an intuitionistic fuzzy set $A = (X, \mu_A, \nu_A)$ based on Definition 2.6.

Definition 3.1. (1) Let $[x_a \in A]$ and $[x_a q A]$ represent the grades of membership of $x_a \in A$ and $x_a q A$, respectively, and

$$[x_a \in A] = A_a(x), \qquad [x_a q A] = A_{[\underline{a}]}(x).$$

(2) $[x_a \in \wedge q A]$ represents the grade of membership of $x_a \in A$ and $x_a q A$, $[x_a \in \vee q A]$ represents the grade of $x_a \in A$ or $x_a q A$, and

$$[x_a \in \wedge q A] = [x_a \in A] \wedge [x_a q A] = A_a(x) \wedge A_{[\underline{a}]}(x),$$
$$[x_a \in \vee q A] = [x_a \in A] \vee [x_a q A] = A_a(x) \vee A_{[\underline{a}]}(x).$$

(3) $[x_a \overline{\in} A]$ represents the grade of nonmembership of $x_a \in A$, $[x_a \overline{q} A]$ represents the grade of nonmembership of $x_a q A$, and

$$[x_a \overline{\in} A] = A^a(x), \qquad [x_a \overline{q} A] = A^{[\underline{a}]}(x).$$

(4)

$$[x_a \overline{\in \wedge q} A] = [x_a \overline{\in} \vee \overline{q} A] = [x_a \overline{\in} A] \vee [x_a \overline{q} A] = A^a(x) \vee A^{[\underline{a}]}(x),$$
$$[x_a \overline{\in \vee q} A] = [x_a \overline{\in} \wedge \overline{q} A] = [x_a \overline{\in} A] \wedge [x_a \overline{q} A] = A^a(x) \wedge A^{[\underline{a}]}(x).$$

Then we have the following property.

Table 1
The table of truth value of Lukasiewicz implication.

$\to$	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

Property 3.1. (1) $[x_a\overline{\in}A]=[x_a\in A^c]$, $[x_a\overline{q}A]=[x_a qA^c]$.
(2) $[x_a\overline{\in}\wedge\overline{q}A]=[x_a\in\wedge qA^c]$, $[x_a\overline{\in}\vee\overline{q}A]=[x_a\in\vee qA^c]$.
(3) $\left[x_a\in(\bigcap_{t\in T}A_t)\right]=\bigwedge_{t\in T}[x_a\in A_t]$, $\left[x_a q(\bigcup_{t\in T}A_t)\right]=\bigvee_{t\in T}[x_a qA_t]$.
(4) $\left[x_a\overline{\in}(\bigcup_{t\in T}A_t)\right]=\bigwedge_{t\in T}[x_a\overline{\in}A_t]$, $\left[x_a\overline{q}(\bigcap_{t\in T}A_t)\right]=\bigvee_{t\in T}[x_a\overline{q}A_t]$.

Remark 3.1. (i) Property 3.1 (3) is a generalization of the cases in the classical sets "$x\in\bigcap_{t\in T}A_t\Leftrightarrow\forall t\in T, x\in A_t$" and "$x\in\bigcup_{t\in T}A_t\Leftrightarrow\exists t\in T, x\in A_t$".

(ii) Property 3.1 (4) is a generalization of the cases in the classical sets "$x\overline{\in}\bigcup_{t\in T}A_t\Leftrightarrow\forall t\in T, x\overline{\in}A_t$" and "$x\overline{\in}\bigcap_{t\in T}A_t\Leftrightarrow\exists t\in T, x\overline{\in}A_t$".

4. (α,β)-intuitionistic fuzzy subgroup

In this section, we will redefine (α,β)-intuitionistic fuzzy subgroup in different way with [22,23].

Let $\to$ denote the implication of Lukasiewicz in triple valued logic. Then we have the following table of truth value.

Let G be a group and $\alpha,\beta\in\{\in,q,\in\wedge q,\in\vee q\}$. For $a\in[0,1]$, $x\in G$, x_a is a fuzzy point. By Definition 3.1 we know that $[x_a\alpha A]\in\{0,\frac{1}{2},1\}$. Then we have

Definition 4.1. Let G be a group, $A=(G,\mu_A,\nu_A)$ be an intuitionistic fuzzy subset of G and $\alpha,\beta\in\{\in,q,\in\wedge q,\in\vee q\}$. If for any $x,y\in G$ and $s,t\in(0,1]$

$$(1)\ ([x_s\alpha A]\wedge[y_t\alpha A]\to[x_sy_t\beta A])=1;\tag{1}$$

$$(2)\ ([x_s\alpha A]\to[x_s^{-1}\beta A])=1,\tag{2}$$

then A is called a (α,β)-intuitionistic fuzzy subgroup of G, where $x_sy_t=(xy)_{s\wedge t}$, $x_s^{-1}=(x^{-1})_s$.

From Table 1, for $p_1,p_2\in\{0,\frac{1}{2},1\}$, we have

$$(p_1\to p_2)=1\Leftrightarrow p_1\le p_2.\tag{3}$$

Then we have the following equivalent definition

Definition 4.2. Let G be a group, $A=(G,\mu_A,\nu_A)$ be an intuitionistic fuzzy subgroup of G and $\alpha,\beta\in\{\in,q,\in\wedge q,\in\vee q\}$. We call A a (α,β)-intuitionistic fuzzy subgroup of G if for any $x,y\in G$ and $s,t\in(0,1]$

$$(3)\ [x_sy_t\beta A]\ge[x_s\alpha A]\wedge[y_t\alpha A]\tag{4}$$

$$(4)\ [x_s^{-1}\beta A]\ge[x_s\alpha A].\tag{5}$$

Clearly, the Definition 4.1 and the Definition 4.2 are the generalizations of the concept on (α,β)-fuzzy subgroups in [7]. In Definition 4.2, α can be chosen one from four kinds of relations, and β also can be chosen one from four kinds of relations. Thus there are 16 kinds of (α,β)-intuitionistic fuzzy subgroups in all. Next, we will discuss the properties of these 16 kinds of (α,β)-intuitionistic fuzzy subgroups.

Theorem 4.1. *Let A be a (α,β)-intuitionistic fuzzy subgroup of G. If $\alpha\ne\in\wedge q$, then $A_{\underline{0}}$ is a 3-valued fuzzy subgroup of G, i.e., $\forall x,y\in G$*

$$A_{\underline{0}}(xy)\ge A_{\underline{0}}(x)\wedge A_{\underline{0}}(y),\qquad A_{\underline{0}}(x^{-1})\ge A_{\underline{0}}(x).\tag{6}$$

Proof. (I) First, we prove that $A_{\underline{0}}(x)\wedge A_{\underline{0}}(y)=1\Rightarrow A_{\underline{0}}(xy)=1$.

Let $A_{\underline{0}}(x)\wedge A_{\underline{0}}(y)=1$. Denote $t=\mu_A(x)\wedge\mu_A(y)$, then there exists $s\in(0,1)$ such that $0<1-s<t=\mu_A(x)\wedge\mu_A(y)$. Thus, $[x_t\in A]=A_t(x)=1$, $[y_t\in A]=A_t(y)=1$, $[x_sqA]=A_{[s]}(x)=1$ and $[y_sqA]=A_{[s]}(y)=1$.

(i) If $\alpha=\in$ or $\alpha=\in\vee q$, then $[x_t\alpha A]=[y_t\alpha A]=1$. Thus, for $\beta\in\{\in,q,\in\wedge q,\in\vee q\}$, we have $1\ge[x_ty_t\beta A]\ge[x_t\alpha A]\wedge[y_t\alpha A]=1$, i.e., $[(xy)_t\beta A]=1$. Hence, $A_t(xy)=1$ or $A_{[t]}(xy)=1$, which implies that $\mu_A(xy)\ge t>0$ or $\mu_A(xy)>1-t\ge 0$. Therefore, $A_{\underline{0}}(xy)=1$.

(ii) If $\alpha = q$, then $[x_s\alpha A] = [y_s\alpha A] = 1$. Thus, for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$, we have $[x_sy_s\beta A] = 1$. Hence, $A_s(xy) = 1$ or $A_{[\underline{s}]}(xy) = 1$, which implies that $\mu_A(xy) \geq s > 0$ or $\mu_A(xy) > 1 - s \geq 0$. Therefore, $A_{\underline{0}}(xy) = 1$.

(II) Second, we show that $A_{\underline{0}}(x) \wedge A_{\underline{0}}(y) = \frac{1}{2} \Rightarrow A_{\underline{0}}(xy) \geq \frac{1}{2}$.

Let $A_{\underline{0}}(x) \wedge A_{\underline{0}}(y) = \frac{1}{2}$. Then $\nu_A(x) < 1$ and $\nu_A(y) < 1$. Let $s, t \in (0, 1)$ such that $\nu_A(x) \vee \nu_A(y) < 1 - t < s < 1$. Then

$$[x_t \in A] = A_t(x) \geq \frac{1}{2}, \qquad [y_t \in A] = A_t(y) \geq \frac{1}{2},$$
$$[x_sqA] = A_{[\underline{s}]}(x) \geq \frac{1}{2}, \qquad [y_sqA] = A_{[\underline{s}]}(y) \geq \frac{1}{2}.$$

(i) If $\alpha = \in$ or $\alpha = \in \vee q$, then $[x_t\alpha A] \wedge [y_t\alpha A] \geq \frac{1}{2}$. Thus, for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$, we have $[x_ty_t\beta A] \geq [x_t\alpha A] \wedge [y_t\alpha A] \geq \frac{1}{2}$. Hence, $A_t(xy) \geq \frac{1}{2}$ or $A_{[\underline{t}]}(xy) \geq \frac{1}{2}$, which implies that $\nu_A(xy) \leq 1 - t < 1$ or $\nu_A(xy) < t < 1$. Therefore, $A_{\underline{0}}(xy) \geq \frac{1}{2}$.

(ii) If $\alpha = q$, then $[x_s\alpha A] \wedge [y_s\alpha A] \geq \frac{1}{2}$. Thus, for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$, we have $[x_sy_s\beta A] \geq \frac{1}{2}$. Hence, $A_s(xy) \geq \frac{1}{2}$ or $A_{[\underline{s}]}(xy) \geq \frac{1}{2}$, which implies that $\nu_A(xy) \leq 1 - s < 1$ or $\nu_A(xy) < s < 1$. Therefore, $A_{\underline{0}}(xy) \geq \frac{1}{2}$.

By $A_{\underline{0}}(x), A_{\underline{0}}(y), A_{\underline{0}}(xy) \in \{0, \frac{1}{2}, 1\}$ and the proof of (I) and (II), we know that $A_{\underline{0}}(x) \wedge A_{\underline{0}}(y) \geq A_{\underline{0}}(xy)$.

By the similar reasoning, we have $A_{\underline{0}}(x^{-1}) \geq A_{\underline{0}}(x)$.

Therefore, $A_{\underline{0}}$ is a 3-valued fuzzy subgroup of G. □

Theorem 4.2. *Let $A = (G, \mu_A, \nu_A)$ be a (α, β)-intuitionistic fuzzy subgroup of G. For $x \in G$, let $A(x) = (\mu_A(x), \nu_A(x))$. If $\mu_A(x) > 0$, then for $(\alpha, \beta) \in \{(\in, q), (\in, \in \wedge q), (\in \vee q, q), (\in \vee q, \in \wedge q), (q, \in), (q, \in \wedge q), (\in \vee q, \in)\}$, we have $A(x) = (1, 0)$.*

Proof. First, we prove $\mu_A(x) > 0 \Rightarrow \mu_A(e) > 0$, where e is the identity element of G.

In fact, from $\mu_A(x) > 0$ we know that $A_{\underline{0}}(x) = 1$, then by Theorem 4.1 we have

$$A_{\underline{0}}(x^{-1}) \geq A_{\underline{0}}(x) = 1.$$

Thus

$$A_{\underline{0}}(e) = A_{\underline{0}}(xx^{-1}) \geq A_{\underline{0}}(x) \wedge A_{\underline{0}}(x^{-1}) = 1.$$

Therefore, $\mu_A(e) > 0$.

Second, we show that $\mu_A(e) = 1$.

Otherwise, $0 < \mu_A(e) < 1$. Then there exist $s, t \in (0, 1)$ such that

$$0 < s < (1 - \mu_A(e)) \wedge \mu_A(e) \leq (1 - \mu_A(e)) \vee \mu_A(e) < t < 1,$$

thus $[e_s \in A] = 1$ and $[e_tqA] = 1$.

If $(\alpha, \beta) \in \{(\in, q), (\in, \in \wedge q), (\in \vee q, q), (\in \vee q, \in \wedge q)\}$, then $[e_s\alpha A] = 1$. Thus $[e_s\beta A] = [e_s^{-1}\beta A] = 1$, which implies that $[e_sqA] = 1$, i.e., $\mu_A(e) > 1 - s$. This is a contradiction to $\mu_A(e) < 1 - s$.

If $(\alpha, \beta) \in \{(q, \in), (q, \in \wedge q), (\in \vee q, \in)\}$, then $[e_t\alpha A] = 1$. Thus $[e_t\beta A] = [e_t^{-1}\beta A] = 1$, which implies that $[e_t \in A] = 1$, i.e., $\mu_A(e) \geq t$. This is a contradiction to $\mu_A(e) < t$.

Therefore, we have $\mu_A(e) = 1$.

At last, we show that $\mu_A(x) = 1$.

Otherwise, $0 < \mu_A(x) < 1$. Then there exist $s, t \in (0, 1)$ such that

$$0 < s < (1 - \mu_A(x)) \wedge \mu_A(x) \leq (1 - \mu_A(x)) \vee \mu_A(x) < t < 1.$$

Thus $[x_s \in A] = 1$ and $[x_tqA] = 1$. On the other hand, from $\mu_A(e) = 1$ we know that $[e_s \in A] = 1$ and $[e_tqA] = 1$.

If $(\alpha, \beta) \in \{(\in, q), (\in, \in \wedge q), (\in \vee q, q), (\in \vee q, \in \wedge q)\}$, then $[x_s\alpha A] = 1$ and $[e_s\alpha A] = 1$. Thus $[x_se_s\beta A] = 1$, which implies that $[x_sqA] = 1$, i.e., $\mu_A(x) > 1 - s$. This is a contradiction to $\mu_A(x) < 1 - s$.

If $(\alpha, \beta) \in \{(q, \in), (q, \in \wedge q), (\in \vee q, \in)\}$, then $[x_t\alpha A] = [e_t\alpha A] = 1$. Thus $[x_te_t\beta A] = 1$, which implies that $[x_t \in A] = 1$, i.e., $\mu_A(x) \geq t$. This is a contradiction to $\mu_A(x) < t$.

Therefore, we have $\mu_A(x) = 1$.

On the other hand, by $\mu_A(x) + \nu_A(x) \leq 1$, we have that $\nu_A(x) = 0$. Hence, we have $A(x) = (1, 0)$. □

Theorem 4.3. *Let A be a (q, q)-intuitionistic fuzzy subgroup of G. Then*

(1) $\mu_A(x) > 0 \Rightarrow A(x) = (\mu_A(e), \nu_A(e))$;
(2) $\mu_A(x) = 0, \nu_A(x) \leq 1 \Rightarrow A(x) = (0, \nu_A(e))$.

Proof. (1) Let $\mu_A(x) > 0$. Then $\nu_A(x) < 1$.

(I) We prove that $\mu_A(e) \geq \mu_A(x)$ and $\nu_A(e) \leq \nu_A(x)$. In fact, let $s \in (0, 1)$ such that $\mu_A(x) > 1 - s$. Then $[x_s^{-1} qA] \geq [x_s qA] = 1$. Thus, $[e_s qA] = [x_s x_s^{-1} qA] \geq [x_s^{-1} qA] \wedge [x_s qA] \geq 1$, i.e., $[e_s qA] = 1$. Hence, $\mu_A(e) > 1 - s$. Then

$$\mu_A(e) \geq \vee\{1 - s | \mu_A(x) > 1 - s\} = \mu_A(x).$$

Let $t \in (0, 1)$ such that $\nu_A(x) < t$. Then $[x_t^{-1} qA] \geq [x_t qA] \geq \frac{1}{2}$. Thus $[e_t qA] = [x_t x_t^{-1} qA] \geq [x_t qA] \wedge [x_t^{-1} qA] \geq \frac{1}{2}$, i.e., $\nu_A(e) < t$. Then

$$\nu_A(e) \leq \wedge\{t | \nu_A(x) < t\} = \nu_A(x).$$

(II) We show that $\mu_A(x) = \mu_A(e)$ and $\nu_A(x) = \nu_A(e)$. Otherwise, we have $\mu_A(x) < \mu_A(e)$ or $\nu_A(x) > \nu_A(e)$. If $\mu_A(x) < \mu_A(e)$, then there exist $s, t \in (0, 1)$ such that $1 - \mu_A(e) < s < 1 - \mu_A(x) < t < 1$. So we have $[x_t qA] = [e_s qA] = 1$. Thus $[x_s qA] = [x_t e_s qA] \geq [x_t qA] \wedge [e_s qA] \geq 1$, i.e., $s > 1 - \mu_A(x)$. This is a contradiction to $s < 1 - \mu_A(x)$. Therefore, $\mu_A(x) = \mu_A(e)$.

If $\nu_A(x) > \nu_A(e)$, then there exist $s, t \in (0, 1)$ such that $\nu_A(e) < s < \nu_A(x) < t < 1$. So $[x_t qA] \geq \frac{1}{2}$ and $[e_s qA] \geq \frac{1}{2}$. Hence, $[x_s qA] = [x_t e_s qA] \geq [x_t qA] \wedge [e_s qA] \geq \frac{1}{2}$, i.e., $\nu_A(x) < s$. This is a contradiction to $\nu_A(x) > s$. So $\nu_A(x) = \nu_A(e)$.

Therefore, when $\mu_A(x) > 0$, we have $A(x) = (\mu_A(e), \nu_A(e))$.

(2) When $\mu_A(x) = 0$ and $\nu_A(x) < 1$, then by the similar reasoning with (1), we can show that $\nu_A(x) = \nu_A(e)$. Therefore, $A(x) = (0, \nu_A(e))$. □

Theorem 4.4. *Let A be a $(q, \in \vee q)$-intuitionistic fuzzy subgroup of G and*

$$H = \{x | x \in G, \mu_A(x) > 0\}, \qquad K = \{x | x \in G, \nu_A(x) < 1\}.$$

Then

(1) *If $\mu_A(x)$ is not a constant on H, then for any $x \in H$, $A(x) \geq (0, 5, 0.5)$, i.e., $\mu_A(x) \geq 0.5$, $\nu_A(x) \leq 0.5$.*
(2) *If $\nu_A(x)$ is not a constant on K, then for any $x \in K$, $\nu_A(x) \leq 0.5$.*

Proof. (1) First, we show that there exists $x' \in H$ such that $\mu_A(x') \geq 0.5$. Otherwise, $\forall x \in H$, $\mu_A(x) < 0.5$. Since $H = \{x | x \in G, \mu_A(x) > 0\} = \{x | A_{\underline{0}}(x) = 1\}$ is a subgroup of G, $e \in H$ and $x^{-1} \in H$ for any $x \in H$. Thus we have $\mu_A(e) < 0.5$ and $\mu_A(x^{-1}) < 0.5$. Next, we show that $\mu_A(e) \geq \mu_A(x)$. In fact, for $x \in H$, let $t \in (0.5, 1]$ such that $t > 0.5 > \mu_A(e) \vee \mu_A(x) \geq \mu_A(x) > 1 - t$. Then $[x_t qA] = 1$. Thus $[x_t^{-1} \in \vee qA] = 1$. By $\mu_A(x^{-1}) < t$, we have that $[x_t^{-1} qA] = 1$. Hence, $[e_t \in \vee qA] = [x_t x_t^{-1} \in \vee qA] \geq [x_t qA] \wedge [x_t^{-1} qA] = 1$. So $\mu_A(e) > 1 - t$. Therefore,

$$\mu_A(e) \geq \vee\{1 - t | \mu_A(x) > 1 - t\} = \mu_A(x).$$

On the other hand, $\mu_A(x)$ is not a constant on H, then there exists $x \in H$ such that $\mu_A(x) < \mu_A(e)$. Thus there exist $s, t' \in (0, 1)$ such that

$$t' > 1 - \mu_A(x) > s > 1 - \mu_A(e) > \mu_A(e) > \mu_A(x). \tag{7}$$

Then $[x_{t'} qA] = [e_s qA] = 1$. Thus $[x_s \in \vee qA] = [x_{t'} e_s \in \vee qA] \geq [x_{t'} qA] \wedge [e_s qA] = 1$, i.e., $\mu_A(x) \geq s$ or $s + \mu_A(x) > 1$. This is a contradiction to Eq. (7). Therefore, there exists $x' \in H$ such that $\mu_A(x') \geq 0.5$.

Second, we show that $\mu_A(e) \geq 0.5$. In fact, for any $t > 0.5$, $\mu_A(x') + t > 1$. Then $[(x')_t^{-1} \in \vee qA] \geq [x'_t qA] = 1$, which implies that $[(x')_t^{-1} qA] = 1$. So $[e_t \in \vee qA] \geq [x'_t qA] \wedge [(x')_t^{-1} qA] = 1$, i.e., $\mu_A(e) \geq t > 1 - t$ or $\mu_A(e) > 1 - t$, which also implies that $\mu_A(e) \geq \vee\{1 - t | t > 0.5\} = 0.5$.

At last, we show that $\mu_A(x) \geq 0.5$ for any $x \in H$. Otherwise, there exists $y \in H$ such that $\mu_A(y) < 0.5$. Then there exists $u, v \in (0, 1)$ such that

$$v > 1 - \mu_A(y) > u > \mu_A(y) \vee (1 - \mu_A(e)). \tag{8}$$

Thus, $[y_u \in \vee qA] = [y_v e_u \in \vee qA] \geq [y_v qA] \wedge [e_u qA] = 1$. So $\mu_A(y) \geq u$ or $\mu_A(y) + u > 1$. This is a contradiction to Eq. (8). Therefore, $\mu_A(x) \geq 0.5$ for any $x \in H$.

By $\mu_A(x) + \nu_A(x) \leq 1$, we know that $\nu_A(x) \leq 0.5$. Hence, $A(x) \geq (0.5, 0.5)$ for any $x \in H$.

(2) Clearly, $K = \{x \in G \mid A_{\underline{0}}(x) \geq \frac{1}{2}\}$. Because $A_{\underline{0}}$ is a 3-valued fuzzy subgroup of G, K is a subgroup of G. Then $x^{-1} \in K$ for any $x \in K$ and $e \in K$.

First, we show that there exists $x'' \in K$ such that $\nu_A(x'') \leq 0.5$.

Otherwise, $\nu_A(x) > 0.5$ for any $x \in K$. Thus $\nu_A(e) > 0.5$ and $\nu_A(x) > 0.5$ for any $x \in K$. Then we have that $\nu_A(e) \leq \nu_A(x), \forall x \in K$

In fact, let $t \in (0, 1)$ such that $0.5 < \nu_A(x) < t$, then $[x_t^{-1} \in \vee qA] \geq [x_t qA] \geq \frac{1}{2}$. By $\nu_A(x^{-1}) > 0.5$, we have that $[x_t^{-1} qA] \geq 1/2$. Thus $[e_t \in \vee qA] \geq [x_t^{-1} qA] \wedge [x_t qA] \geq \frac{1}{2}$. By $\nu_A(e) > 0.5$, we have that $[e_t qA] \geq \frac{1}{2}$, i.e., $\nu_A(e) < t$, which implies that $\nu_A(e) \leq \wedge\{t \mid \nu_A(x) < t\} = \nu_A(x)$.

Because $\nu_A(x)$ is not a constant on K, there exists $x \in K$ such that $\nu_A(e) < \nu_A(x)$. Then there exist $a, b \in (0, 1)$ such that $a > \nu_A(x) > b > \nu_A(e) > 1 - \nu_A(e) > 1 - \nu_A(x)$. Thus $[e_b qA] \geq \frac{1}{2}$ and $[x_a qA] \geq \frac{1}{2}$. Then $[x_b \in \vee qA] = [x_a e_b \in \vee qA] \geq [x_a qA] \wedge [e_b qA] \geq \frac{1}{2}$.

When $[x_b \in A] \geq \frac{1}{2}$, we have that $b \leq 1 - \nu_A(x)$, and this is a contradiction to $b > 1 - \nu_A(x)$.

When $[x_b qA] \geq \frac{1}{2}$, we have that $\nu_A(x) < b$, that is a contradiction to $\nu_A(x) > b$.

Therefore, there exists $x'' \in K$ such that $\nu_A(x'') \leq 0.5$.

Second, we show that $\nu_A(e) \leq 0.5$.

In fact, because $[x''_t qA] \geq \frac{1}{2}$ for any $t \in (0.5, 1]$, $[(x'')_t^{-1} \in \vee qA] \geq [x''_t qA] \geq \frac{1}{2}$ and consequently $[(x'')_t^{-1} qA] \geq \frac{1}{2}$. Thus $[e_t \in \vee qA] = [(x'')_t^{-1} x''_t \in \vee qA] \geq [(x'')_t^{-1} qA] \wedge [x''_t qA] \geq \frac{1}{2}$, which implies that $\nu_A(e) < t$. Thus $\nu_A(e) \leq \wedge \{t \mid t > 0.5\} = 0.5$.

At last, we show that $\nu_A(x) \leq 0.5$ for any $x \in K$.

Otherwise, there exists $y \in K$ such that $\nu_A(y) > 0.5$. Then there exist $c, d \in (0, 1)$ such that $c > \nu_A(y) > d > (1 - \nu_A(y)) \vee \nu_A(e)$. Then $[e_d qA] \geq \frac{1}{2}$ and $[y_c qA] \geq \frac{1}{2}$. Thus $[y_d \in \vee qA] = [y_c e_d \in \vee qA] \geq [y_c qA] \wedge [e_d qA] \geq \frac{1}{2}$.

When $[y_d \in A] \geq \frac{1}{2}$, we have that $d \leq 1 - \nu_A(y)$, this is a contradiction to $d > 1 - \nu_A(y)$;

When $[y_d qA] \geq \frac{1}{2}$, we have that $\nu_A(y) < d$, this is a contradiction to $d < \nu_A(y)$.

Therefore, $\nu_A(x) \leq 0.5$ for any $x \in K$. □

Theorem 4.5. *If A is a $(\in \vee q, \in \vee q)$-intuitionistic fuzzy subgroup of G, then A is a $(\in, \in \vee q)$-intuitionistic fuzzy subgroup of G*

Theorem 4.6. (1) *A is a $(\in, \in)$-intuitionistic fuzzy subgroup of G if and only if for any $x, y \in G$*

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y), \qquad \mu_A(x^{-1}) \geq \mu_A(x) \tag{9}$$

and

$$\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y), \qquad \nu_A(x^{-1}) \leq \nu_A(x); \tag{10}$$

(2) *A is a $(\in, \in \vee q)$-intuitionistic fuzzy subgroup of G if and only if for any $x, y \in G$*

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \wedge 0.5, \qquad \mu_A(x^{-1}) \geq \mu_A(x) \wedge 0.5 \tag{11}$$

and

$$\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y) \vee 0.5, \qquad \nu_A(x^{-1}) \leq \nu_A(x) \vee 0.5; \tag{12}$$

(3) *A is a $(\in \wedge q, \in)$-intuitionistic fuzzy subgroup of G if and only if for any $x, y \in G$*

$$\mu_A(xy) \vee 0.5 \geq \mu_A(x) \wedge \mu_A(y), \qquad \mu_A(x^{-1}) \vee 0.5 \geq \mu_A(x) \tag{13}$$

and

$$\nu_A(xy) \wedge 0.5 \leq \nu_A(x) \vee \nu_A(y), \qquad \nu_A(x^{-1}) \wedge 0.5 \leq \nu_A(x). \tag{14}$$

Proof. (1) "⇒" Let $t = \mu_A(x) \wedge \mu_A(y)$. Then $[x_t y_t \in A] \geq [x_t \in A] \wedge [y_t \in A] = 1$. Thus $\mu_A(xy) \geq t = \mu_A(x) \wedge \mu_A(y)$.

Let $s = \nu_A(xy)$. For $t > 1 - s$, we have $0 = [x_t y_t \in A] \geq [x_t \in A] \wedge [y_t \in A]$, then $[x_t \in A] = 0$ or $[y_t \in A] = 0$, i.e., $\nu_A(x) > 1 - t$ or $\nu_A(y) > 1 - t$. Thus $\nu_A(x) \vee \nu_A(y) > 1 - t$. So $\nu_A(x) \vee \nu_A(y) \geq \vee\{1 - t | 1 - t < s\} = s = \nu_A(xy)$.

By the similar reasoning, we have $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$.

"⇐" For any $x, y \in G$ and $s, t \in (0, 1]$, let $a = [x_s \in A] \wedge [y_t \in A]$.

Case 1. $a = 1$. Then $[x_s \in A] = 1$ and $[y_t \in A] = 1$. Thus $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \geq s \wedge t$. Hence $[x_s y_t \in A] = 1$.

Case 2. $a = \frac{1}{2}$. Then $[x_s \in A] \geq \frac{1}{2}$ and $[y_t \in A] \geq \frac{1}{2}$. Thus $1 - \nu_A(x) \geq s$ and $1 - \nu_A(y) \geq t$. So $1 - \nu_A(xy) \geq 1 - \nu_A(x) \vee \nu_A(y) = (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \geq s \wedge t$, which implies that $[x_s y_t \in A] \geq \frac{1}{2}$. Hence $[x_s y_t \in A] \geq [x_s \in A] \wedge [y_t \in A]$.

Similarly, we have $[x_s^{-1} \in A] \geq [x_s \in A]$.

Therefore, A is a $(\in, \in)$-intuitionistic fuzzy subgroup of G.

(2) "⇒" Let $t = \mu_A(x) \wedge \mu_A(y) \wedge 0.5$. Then $[x_t y_t \in \vee qA] \geq [x_t \in A] \wedge [y_t \in A] = 1$. Thus $\mu_A(xy) \geq t$ or $\mu_A(xy) > 1 - t \geq 0.5 \geq t$. Hence $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \wedge 0.5$.

Let $\nu_A(x) \vee \nu_A(y) \vee 0.5 = 1 - s$. Then $[x_s y_s \in \vee qA] \geq [x_s \in A] \wedge [y_s \in A] \geq \frac{1}{2}$. Thus, $s \leq 1 - \nu_A(xy)$ or $\nu_A(xy) < s \leq 1 - s$. Hence $\nu_A(xy) \leq 1 - s = \nu_A(x) \vee \nu_A(y) \vee 0.5$.

Similarly, we have $\mu_A(x^{-1}) \geq \mu_A(x) \wedge 0.5$ and $\nu_A(x^{-1}) \leq \nu_A(x) \vee 0.5$.

"⇐" For any $x, y \in G$ and $s, t \in (0, 1]$, let $a = [x_s \in A] \wedge [y_t \in A]$.

Case 1. $a = 1$. If $[x_s y_t \in \vee qA] \leq \frac{1}{2}$, then $\mu_A(x) \geq s$, $\mu_A(y) \geq t$, $\mu_A(xy) < s \wedge t$ and $\mu_A(xy) \leq 1 - s \wedge t$. Thus $0.5 > \mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \wedge 0.5$. So $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \geq s \wedge t$, which contradicts to $\mu_A(xy) < s \wedge t$. Thus, we have $[x_s y_t \in \vee qA] = 1$.

Case 2. $a = \frac{1}{2}$. Then $1 - \nu_A(x) \geq s$ and $1 - \nu_A(y) \geq t$. Thus $1 - \nu_A(x) \vee \nu_A(y) \geq s \wedge t$. If $[x_s y_t \in \vee qA] = 0$, then $s \wedge t > 1 - \nu_A(xy)$ and $\nu_A(xy) \geq s \wedge t$. Thus $\nu_A(xy) > 0.5$. So $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$ and $1 - \nu_A(xy) \geq 1 - \nu_A(x) \vee \nu_A(y) \geq s \wedge t$, which is a contradiction to $1 - \nu_A(xy) < s \wedge t$. Thus we have $[x_s y_t \in \vee qA] \geq \frac{1}{2}$.

Therefore, $[x_s y_t \in \vee qA] \geq [x_s \in A] \wedge [y_t \in A]$.

Similarly, we have $[x_s^{-1} \in \vee qA] \geq [x_s \in A]$.

(3) "⇒" If $\mu_A(xy) \vee 0.5 < t = \mu_A(x) \wedge \mu_A(y)$, then $\mu_A(x) \geq t > 0.5$, $\mu_A(y) \geq t > 0.5$ and $\mu_A(xy) < t$. Thus $1 = [x_t \in \wedge qA] \wedge [y_t \in \wedge qA] \leq [x_t y_t \in A]$. So $\mu_A(xy) \geq t$, which contradicts to $\mu_A(xy) < t$. Hence, $\mu_A(xy) \vee 0.5 \geq t = \mu_A(x) \wedge \mu_A(y)$.

If $\nu_A(xy) \wedge 0.5 > t = 1 - s = \nu_A(x) \vee \nu_A(y)$, then $s \leq 1 - \nu_A(x)$, $s \leq 1 - \nu_A(y)$, $\nu_A(xy) > t$ and $s > 0.5 > t$. Thus $\nu_A(x) \leq t < s$ and $\nu_A(y) \leq t < s$. So $[x_s y_s \in A] \geq [x_s \in \wedge qA] \wedge [y_s \in \wedge qA] \geq \frac{1}{2}$. So $s \leq 1 - \nu_A(xy)$, i.e., $\nu_A(xy) \leq 1 - s = t$, which contradicts to $\nu_A(xy) > t$. Hence $\nu_A(xy) \wedge 0.5 \leq \nu_A(x) \vee \nu_A(y)$.

Similarly, we have $\mu_A(x^{-1}) \vee 0.5 \geq \mu_A(x)$ and $\nu_A(x^{-1}) \wedge 0.5 \leq \nu_A(x)$.

"⇐" For any $x, y \in G$ and $s, t \in (0, 1]$, let $a = [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$.

Case 1. $a = 1$. Then $\mu_A(x) \geq s$, $\mu_A(x) > 1 - s$, $\mu_A(y) \geq t$ and $\mu_A(y) > 1 - t$. Thus $\mu_A(x) > 0.5$ and $\mu_A(y) > 0.5$. So $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \geq s \wedge t$, i.e., $[x_s y_t \in A] = 1$.

Case 2. $a = \frac{1}{2}$. Then $1 - \nu_A(x) \geq s > \nu_A(x)$ and $1 - \nu_A(y) \geq t > \nu_A(y)$. Thus, $\nu_A(x) < 0.5$ and $\nu_A(y) < 0.5$. So $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$ and $1 - \nu_A(xy) \geq (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \geq s \wedge t$, i.e., $[x_s y_t \in A] \geq \frac{1}{2}$.

Hence, $[x_s y_t \in A] \geq [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$.

Similarly, we have $[x_s^{-1} \in A] \geq [x_s \in \wedge qA]$.

Therefore, A is a $(\in \wedge q, \in)$-intuitionistic fuzzy subgroup of G. □

Theorem 4.7. (1) *A is a $(\in, \in)$-intuitionistic fuzzy subgroup of G if and only if for any $a \in [0, 1]$, A_a is a 3-valued fuzzy subgroup of G;*

(2) *A is a $(\in, \in \vee q)$-intuitionistic fuzzy subgroup of G if and only if for any $a \in (0, 0.5]$, A_a is a 3-valued fuzzy subgroup of G;*

(3) *A is a $(\in \wedge q, \in)$-intuitionistic fuzzy subgroup of G if and only if for any $a \in (0.5, 1]$, A_a is a 3-valued fuzzy subgroup of G.*

Proof. (1) "⇒" Because A is a $(\in, \in)$-intuitionistic fuzzy subgroup of G, then for any $a \in [0, 1]$ and $x \in G$,

$$[x_a y_a \in A] \geq [x_a \in A] \wedge [y_a \in A], \qquad [x_a^{-1} \in A] \geq [x_a \in A],$$

i.e., $A_a(xy) \geq A_a(x) \wedge A_a(y)$ and $A_a(x^{-1}) \geq A_a(x)$. So A_a is a 3-valued fuzzy subgroup of G.

"⇐" For any $x, y \in G$ and $s, t \in (0, 1]$, $[x_s y_t \in A] = A_{s\wedge t}(xy) \geq A_{s\wedge t}(x) \wedge A_{s\wedge t}(y) \geq A_s(x) \wedge A_t(y) = [x_s \in A] \wedge [y_t \in A]$ and $[x_s^{-1} \in A] = A_s(x^{-1}) \geq A_s(x) = [x_s \in A]$.

Therefore, A is a $(\in, \in)$-intuitionistic fuzzy subgroup of G.

(2) "⇒" Because A is a $(\in, \in \vee q)$-intuitionistic fuzzy subgroup, for any $a \in (0, 0.5]$ and $x \in G$, we have $[x_a y_a \in \vee qA] \geq [x_a \in A] \wedge [y_a \in A]$. So $A_a(xy) \vee A_{[\underline{a}]}(xy) \geq A_a(x) \wedge A_a(y)$. By $0 < a \leq 0.5$, we have that $a \leq 0.5 \leq 1 - a$. Thus $A_{[\underline{a}]}(xy) = A_{\underline{1-a}}(xy) \leq A_{\underline{a}}(xy) \leq A_a(xy)$. Hence $A_a(xy) \geq A_a(x) \wedge A_a(y)$. Similarly, we have $A_a(x^{-1}) \geq A_a(x)$. So A_a is a 3-valued fuzzy subgroup of G.

"⇐" Let $s, t \in (0, 1]$. If $s \wedge t \leq 0.5$, then $1 - s \wedge t \geq 0.5 \geq s \wedge t$. Thus $A_{[\underline{s\wedge t}]}(xy) \leq A_{s\wedge t}(xy)$. So $[x_s y_t \in \vee qA] = A_{s\wedge t}(xy) \vee A_{[\underline{s\wedge t}]}(xy) = A_{s\wedge t}(xy) \geq A_{s\wedge t}(x) \wedge A_{s\wedge t}(y) \geq A_s(x) \wedge A_t(y) = [x_s \in A] \wedge [y_t \in A]$.

If $s \wedge t > 0.5$, then let $a \in (0, 1)$ such that $1 - s \wedge t < a < 0.5 < s \wedge t$. Thus $A_{s\wedge t}(xy) \leq A_{[\underline{s\wedge t}]}(xy)$ and $A_{[\underline{s\wedge t}]}(xy) \geq A_a(xy)$. So $[x_s y_t \in \vee qA] = A_{s\wedge t}(xy) \vee A_{[\underline{s\wedge t}]}(xy) = A_{[\underline{s\wedge t}]}(xy) \geq A_a(xy) \geq A_a(x) \wedge A_a(y) \geq A_s(x) \wedge A_t(y) = [x_s \in A] \wedge [y_t \in A]$. Hence, $[x_s y_t \in \vee qA] \geq [x_s \in A] \wedge [y_t \in A]$.

Similarly, we have $[x_s^{-1} \in \vee qA] \geq [x_s \in A]$.

Therefore, A is a $(\in, \in \vee q)$-intuitionistic fuzzy subgroup of G.

(3) "⇒" Let $a \in (0.5, 1]$ and $x \in G$. Then $A_{[\underline{a}]}(x) \geq A_a(x)$. Thus $A_a(xy) = [x_a y_a \in A] \geq [x_a \in \wedge qA] \wedge [y_a \in \wedge qA] \geq A_a(x) \wedge A_{[\underline{a}]}(x) \wedge A_a(y) \wedge A_{[\underline{a}]}(y) \geq A_a(x) \wedge A_a(y)$.

Similarly, we have $A_a(x^{-1}) \geq A_a(x)$. So A_a is a 3-valued fuzzy subgroup of G.

"⇐" For any $x, y \in G$ and $s, t \in (0, 1]$, let $a = [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$.

Case 1. $a = 1$. Then $\mu_A(x) \geq s$, $\mu_A(x) > 1 - s$, $\mu_A(y) \geq t$ and $\mu_A(y) > 1 - t$. Thus $\mu_A(x) > 0.5$, $\mu_A(y) > 0.5$. So $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \geq s \wedge t$, i.e., $[x_s y_t \in A] = 1$.

Case 2. $a = \frac{1}{2}$. Then $1 - \nu_A(x) \geq s > \nu_A(x)$ and $1 - \nu_A(y) \geq t > \nu_A(y)$. Thus $\nu_A(x) < 0.5$ and $\nu_A(y) < 0.5$. So $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$ and $1 - \nu_A(xy) \geq (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \geq s \wedge t$. Hence $[x_s y_t \in A] \geq \frac{1}{2}$. So $[x_s y_t \in A] \geq [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$. Similarly, we have $[x_s^{-1} \in A] \geq [x_s \in \wedge qA]$.

Therefore, A is a $(\in \wedge q, \in)$-intuitionistic fuzzy subgroup of G. □

Theorem 4.8. *Let A be a $(\in \wedge q, \beta)$-intuitionistic fuzzy subgroup of G and $N = \{x | x \in G, \mu_A(x) > 0.5\}$, where $\beta \in \{q, \in \wedge q\}$. Then for any $x \in N$, $A(x) = (\mu_A(e), \nu_A(e))$, i.e., A is a constant on N.*

Proof. If A is a $(\in \wedge q, \beta)$-intuitionistic fuzzy subgroup of G, then A is a $(\in \wedge q, q)$-intuitionistic fuzzy subgroup of G. Thus, we only need to show that the theorem is true for $\beta = q$.

First, we can show that $\mu_A(e) > 0.5$ and $\mu_A(x^{-1}) > 0.5$ for any $x \in N$. In fact, for any $x \in N$, we have $[x_{0.5}^{-1} qA] \geq [x_{0.5} \in \wedge qA] = 1$, then $\mu_A(x^{-1}) > 0.5$. Thus, $[e_{0.5} qA] = [x_{0.5} x_{0.5}^{-1} qA] \geq [x_{0.5} \in \wedge qA] \wedge [x_{0.5}^{-1} \in \wedge qA] = 1$. So $\mu_A(e) > 0.5$.

Second, we show that $\mu_A(x) = \mu_A(e)$ for any $x \in N$.

If there exists $x \in N$ such that $0.5 < \mu_A(x) < \mu_A(e)$, then there exist $s, t \in (0, 1)$ such that $1 - \mu_A(e) < s < 1 - \mu_A(x) < t < 0.5 < \mu_A(x) < \mu_A(e)$. Thus $[x_s qA] = [x_t e_s qA] \geq [x_t \in \wedge qA] \wedge [e_s \in \wedge qA] = 1$. So $s + \mu_A(x) > 1$, which is a contradiction to $\mu_A(x) + s < 1$.

If there exists $x \in N$ such that $0.5 < \mu_A(e) < \mu_A(x)$, then there exists $t \in (0, 1)$ such that $1 - \mu_A(x) < t < 1 - \mu_A(e) < 0.5 < \mu_A(e) < \mu_A(x)$. Thus $[x_t^{-1}qA] \geq [x_t \in \wedge qA] = 1$. By $\mu_A(x^{-1}) > 0.5 > t$, we have that $[x_t^{-1} \in \wedge qA] = 1$. Hence $[e_t qA] = [x_t x_t^{-1} qA] \geq [x_t \in \wedge qA] \wedge [x_t^{-1} \in \wedge qA] = 1$, i.e., $t + \mu_A(e) > 1$, which contradicts to $t < 1 - \mu_A(e)$.

Therefore, we have $\mu_A(x) = \mu_A(e)$ for any $x \in N$.

At last, we show that $\nu_A(x) = \nu_A(e)$ for any $x \in N$.

In fact, for any $x \in N$, $\mu_A(x) > 0.5$, $\mu_A(x^{-1}) > 0.5$ and $\mu_A(e) > 0.5$. Then $\nu_A(x) < 0.5$, $\nu_A(x^{-1}) < 0.5$ and $\nu_A(e) < 0.5$. Let t satisfy $\nu_A(x) < t < 0.5$, then $[x_t^{-1}qA] \geq [x_t \in \wedge qA] \geq \frac{1}{2}$. Thus $\nu_A(x^{-1}) < t < 0.5$. So $[e_t qA] = [x_t x_t^{-1} qA] \geq [x_t \in \wedge qA] \wedge [x_t^{-1} \in \wedge qA] \geq \frac{1}{2}$. Hence $\nu_A(e) < t$ and $\nu_A(e) \leq \wedge\{t | \nu_A(x) < t < 0.5\} = \nu_A(x)$.

Next, we show that $\nu_A(x) = \nu_A(e)$ for any $x \in N$. Otherwise, let x' satisfy $\nu_A(x') > \nu_A(e)$. Then there exist $s, t \in (0, 1)$ such that $\nu_A(e) < s < \nu_A(x') < t < 0.5$. Thus $[x'_s qA] = [e_s x'_t qA] \geq [e_s \in \wedge qA] \wedge [x'_t \in \wedge qA] \geq \frac{1}{2}$, i.e., $\nu_A(x') < s$, which contradicts to $\nu_A(x') > s$.

Hence, $\nu_A(x) = \nu_A(e)$.

Therefore, we have $A(x) = (\mu_A(e), \nu_A(e))$ for any $x \in N$. □

Theorem 4.9. *A is a $(\in \wedge q, \in \vee q)$-intuitionistic fuzzy subgroup of G if and only if for any $x, y \in G$*

$$(1)\ \mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \wedge 0.5 \quad or \quad \mu_A(xy) \vee 0.5 \geq \mu_A(x) \wedge \mu_A(y); \tag{15}$$

$$(2)\ \mu_A(x^{-1}) \geq \mu_A(x) \wedge 0.5 \quad or \quad \mu_A(x^{-1}) \vee 0.5 \geq \mu_A(x); \tag{16}$$

$$(3)\ \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y) \vee 0.5 \quad or \quad \nu_A(xy) \wedge 0.5 \leq \nu_A(x) \vee \nu_A(y); \tag{17}$$

$$(4)\ \nu_A(x^{-1}) \leq \nu_A(x) \vee 0.5 \quad or \quad \nu_A(x^{-1}) \wedge 0.5 \leq \nu_A(x). \tag{18}$$

Proof. "⇒" (1) If $\mu_A(xy) \vee 0.5 < t = \mu_A(x) \wedge \mu_A(y)$, then $\mu_A(x) \geq t > 0.5$, $\mu_A(y) \geq t > 0.5$ and $\mu_A(xy) < t$. Thus $[x_{0.5} y_{0.5} \in \vee qA] \geq [x_{0.5} \in \wedge qA] \wedge [y_{0.5} \in \wedge qA] = 1$. So $\mu_A(xy) \geq 0.5$ or $\mu_A(xy) + 0.5 > 1$. Hence $\mu_A(xy) \geq 0.5 \geq \mu_A(x) \wedge \mu_A(y) \wedge 0.5$.

(3) If $\nu_A(xy) \wedge 0.5 > t = 1 - s = \nu_A(x) \vee \nu_A(y)$, then $s \leq 1 - \nu_A(x)$, $s \leq 1 - \nu_A(y)$ and $s > 0.5$. Thus $[x_{0.5} y_{0.5} \in \vee qA] \geq [x_{0.5} \in \wedge qA] \wedge [y_{0.5} \in \wedge qA] \geq \frac{1}{2}$. So $0.5 \leq 1 - \nu_A(xy)$ or $\nu_A(xy) < 0.5$. Hence $\nu_A(xy) \leq 0.5 \leq \nu_A(x) \vee \nu_A(y) \vee 0.5$.

(2) and (4) can be proved similarly.

"⇐" For any $x, y \in G$ and $s, t \in (0, 1]$, let $a = [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$.

Case 1. $a = 1$. Then $\mu_A(x) \geq s$, $\mu_A(x) > 1 - s$, $\mu_A(y) \geq t$ and $\mu_A(y) > 1 - t$. Thus $\mu_A(x) \wedge \mu_A(y) > 0.5$. Next we show $[x_s y_t \in \vee qA] = 1$. Otherwise, we have $[x_s y_t \in \vee qA] \leq \frac{1}{2}$. Then $\mu_A(xy) < s \wedge t$ and $\mu_A(xy) \leq 1 - s \wedge t$. Thus $\mu_A(xy) < 0.5 < \mu_A(x) \wedge \mu_A(y)$. So $\mu_A(xy) < \mu_A(x) \wedge \mu_A(y) \wedge 0.5$ and $\mu_A(xy) \vee 0.5 < \mu_A(x) \wedge \mu_A(y)$, which is a contradiction to Eq. (15). Hence, $[x_s y_t \in \vee qA] = 1$.

Case 2. $a = \frac{1}{2}$. Then $1 - \nu_A(x) \geq s > \nu_A(x)$ and $1 - \nu_A(y) \geq t > \nu_A(y)$. Thus $\nu_A(x) \vee \nu_A(y) < 0.5$. If $[x_s y_t \in \vee qA] = 0$, then $\nu_A(xy) \geq s \wedge t > 1 - \nu_A(xy)$. Thus $\nu_A(xy) > 0.5$. So $\nu_A(xy) \wedge 0.5 = 0.5 \geq \nu_A(x) \vee \nu_A(y)$ and $\nu_A(xy) > \nu_A(x) \vee \nu_A(y) \vee 0.5$, which is a contradiction to Eq. (17). Hence, $[x_s y_t \in \vee qA] \geq \frac{1}{2}$.

Therefore, $[x_s y_t \in \vee qA] \geq [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$.

Similarly, we have $[x_s^{-1} \in \vee qA] \geq [x_s \in \wedge qA]$.

Hence, A is a $(\in \wedge q, \in \vee q)$-intuitionistic fuzzy subgroup of G. □

5. $(s, t]$-intuitionistic fuzzy subgroups

From the discussing in Section 4 we know that

(a) A is a $(\in, \in)$-intuitionistic fuzzy subgroup of G if and only if for any $x, y \in G$

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y), \qquad \mu_A(x^{-1}) \geq \mu_A(x)$$

and

$$\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y), \qquad \nu_A(x^{-1}) \leq \nu_A(x);$$

(b) A is a $(\in, \in \vee q)$-intuitionistic fuzzy subgroup of G if and only if for any $x, y \in G$

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \wedge 0.5, \qquad \mu_A(x^{-1}) \geq \mu_A(x) \wedge 0.5$$

and

$$\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y) \vee 0.5, \qquad \nu_A(x^{-1}) \leq \nu_A(x) \vee 0.5;$$

(c) A is a $(\in \wedge q, \in)$-intuitionistic fuzzy subgroup of G if and only if for any $x, y \in G$

$$\mu_A(xy) \vee 0.5 \geq \mu_A(x) \wedge \mu_A(y), \qquad \mu_A(x^{-1}) \vee 0.5 \geq \mu_A(x)$$

and

$$\nu_A(xy) \wedge 0.5 \leq \nu_A(x) \vee \nu_A(y), \qquad \nu_A(x^{-1}) \wedge 0.5 \leq \nu_A(x).$$

We can generalize the above three kinds of intuitionistic fuzzy subgroups to $(s, t]$-intuitionistic fuzzy subgroup.

Definition 5.1. Let $s, t \in [0, 1]$ and $s < t$. If

$$(1)\ \mu_A(xy) \vee s \geq \mu_A(x) \wedge \mu_A(y) \wedge t, \qquad \mu_A(x^{-1}) \vee s \geq \mu_A(x) \wedge t; \tag{19}$$

$$(2)\ \nu_A(xy) \wedge (1-s) \leq \nu_A(x) \vee \nu_A(y) \vee (1-t), \qquad \nu_A(x^{-1}) \wedge (1-s) \leq \nu_A(x) \vee (1-t), \tag{20}$$

then A is called a $(s, t]$-intuitionistic fuzzy subgroup of G.

Obviously, when $s = 0$ and $t = 1$, then A is a $(0, 1]$-intuitionistic fuzzy subgroup of G if and only if A is a $(\in, \in)$-intuitionistic fuzzy subgroup of G; when $s = 0$ and $t = 0.5$, then A is a $(0, 0.5]$-intuitionistic fuzzy subgroup of G if and only if A is a $(\in, \in \vee q)$-intuitionistic fuzzy subgroup of G; when $s = 0.5$ and $t = 1$, then A is a $(0.5, 1]$-intuitionistic fuzzy subgroup of G if and only if A is a $(\in \wedge q, \in)$-intuitionistic fuzzy subgroup of G.

By Theorem 4.7 we know that

(i) A is a $(\in, \in)$-intuitionistic fuzzy subgroup of G if and only if for any $a \in [0, 1]$, A_a is a 3-valued fuzzy subgroup of G;

(ii) A is a $(\in, \in \vee q)$-intuitionistic fuzzy subgroup of G if and only if for any $a \in (0, 0.5]$, A_a is a 3-valued fuzzy subgroup of G;

(iii) A is a $(\in \vee q, \in)$-intuitionistic fuzzy subgroup of G if and only if for any $a \in (0.5, 1]$, A_a is a 3-valued fuzzy subgroup of G.

Then we have the following theorem

Theorem 5.1. *A is a $(s, t]$-intuitionistic fuzzy subgroup of G if and only if A_a is a 3-valued fuzzy subgroup of G for any $a \in (s, t]$.*

Proof. "⇒" Let $a \in (s, t]$.

If $A_a(x) \wedge A_a(y) = 1$, then $\mu_A(x) \geq a > s$, $\mu_A(y) \geq a > s$. By $\mu_A(xy) \vee s \geq \mu_A(x) \wedge \mu_A(y) \wedge t > a$, we know that $\mu_A(xy) \geq a$. Then $A_a(xy) = 1$.

If $A_a(x) \wedge A_a(y) = \frac{1}{2}$, then $1 - \nu_A(x) \geq a$ and $1 - \nu_A(y) \geq a$. Thus $\nu_A(x) \vee \nu_A(y) \leq 1 - a < 1 - s$. By $\nu_A(xy) \wedge (1-s) \leq \nu_A(x) \vee \nu_A(y) \vee (1-t) \leq 1 - a$, we know that $\nu_A(xy) \leq 1 - a$. Then $A_a(xy) \geq \frac{1}{2}$.

Hence, we have $A_a(xy) \geq A_a(x) \wedge A_a(y)$ for any $x, y \in G$.

Similarly, we have $A_a(x^{-1}) \geq A_a(x)$ for any $x \in G$.

Therefore, A_a is a 3-valued fuzzy subgroup of G.

"⇐" If $\mu_A(xy) \vee s < a = \mu_A(x) \wedge \mu_A(y) \wedge t$, then we have $a \in (s, t]$, $\mu_A(x) \geq a$ and $\mu_A(y) \geq a$. Thus $A_a(xy) \geq A_a(x) \wedge A_a(y) = 1$. So $\mu_A(xy) \geq a$, which contradicts to $\mu_A(xy) < a$. Hence, $\mu_A(xy) \vee s \geq \mu_A(x) \wedge \mu_A(y) \wedge t$.

If $\nu_A(xy) \wedge (1-s) > a = \nu_A(x) \vee \nu_A(y) \vee (1-t)$, then $(1 - \nu_A(xy)) \vee s < 1 - a = (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \wedge t$. Thus $b = 1 - a \in (s, t]$, $1 - \nu_A(x) \geq b$ and $1 - \nu_A(y) \geq b$. So $A_b(xy) \geq A_b(x) \wedge A_b(y) \geq \frac{1}{2}$, which implies that $b \leq 1 - \nu_A(xy)$, i.e., $\nu_A(xy) \leq a$. This is a contradiction to $\nu_A(xy) > a$. Hence, we have $\nu_A(xy) \wedge (1-s) \leq \nu_A(x) \vee \nu_A(y) \vee (1-t)$.

Similarly, we have $\mu_A(x^{-1}) \vee s \geq \mu_A(x) \wedge t$ and $\nu_A(x^{-1}) \wedge (1-s) \leq \nu_A(x) \vee (1-t)$.

Therefore, A is a $(s, t]$- intuitionistic fuzzy subgroup of G. □

Next, we will use the neighborhood relations between a fuzzy point x_a and an intuitionistic fuzzy set A to characterize the $(s, t]$-intuitionistic fuzzy subgroup. First we give the following definition.

Definition 5.2. Let x_a be a fuzzy point, $s \in (0, 1)$ and $A = (X, \mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X. We set

(1)

$$[x_a q_s A] = \begin{cases} 1 & a + \mu_A(x) > 2s; \\ \dfrac{1}{2} & \mu_A(x) \leq 2s - a < 1 - \nu_A(x); \\ 0 & \nu_A(x) + 2s \geq a + 1. \end{cases}$$

(2)

$$[x_a \in \vee q_s A] = [x_a \in A] \vee [x_a q_s A];$$

(3)

$$[x_a \in \wedge q_s A] = [x_a \in A] \wedge [x_a q_s A].$$

Remark 5.1. When $s = 0.5$, $[x_a q_s A] = [x_a q A]$.

Then we have the following theorems.

Theorem 5.2. *Let $s, t \in [0, 1]$ and $0 < s < t$. Then A is a $(s, t]$-intuitionistic fuzzy subgroup of G if and only if*

(1) $\forall a, b \in (0, t]$, $\forall x, y \in G$

$$([x_a \in \wedge q_s A] \wedge [y_b \in \wedge q_s A] \to [x_a y_b \in A]) = 1,$$

i.e., $[x_a y_b \in A] \geq [x_a \in \wedge q_s A] \wedge [y_b \in \wedge q_s A]$.

(2) $\forall a \in (0, t], \forall x \in G$

$$([x_a \in \wedge q_s A] \to [x_a^{-1} \in A]) = 1,$$

i.e., $[x_a^{-1} \in A] \geq [x_a \in \wedge q_s A]$.

Proof. Let $a, b \in (0, t]$ and $c = [x_a \in \wedge q_s A] \wedge [y_b \in \wedge q_s A]$.

Case 1. $c = 1$. Then $\mu_A(x) \geq a$, $a + \mu_A(x) > 2s$, $\mu_A(y) \geq b$ and $b + \mu_A(y) > 2s$. Thus $\mu_A(x) > s$ and $\mu_A(y) > s$. By $\mu_A(xy) \vee s \geq \mu_A(x) \wedge \mu_A(y) \wedge t$, we know that $\mu_A(xy) > s$. So $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \wedge t \geq a \wedge b \wedge t = a \wedge b$. Hence, we have $[x_a y_b \in A] = 1$.

Case 2. $c = \frac{1}{2}$. Then $1 - \nu_A(x) \geq a$, $1 - \nu_A(y) \geq b$, $2s - a < 1 - \nu_A(x)$ and $2s - b < 1 - \nu_A(y)$. Thus $1 - \nu_A(x) > s$ and $1 - \nu_A(y) > s$. By $\nu_A(xy) \wedge (1 - s) \leq \nu_A(x) \vee \nu_A(y) \vee (1 - t)$, we know that $(1 - \nu_A(xy)) \vee s \geq (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \wedge t$. So $1 - \nu_A(xy) > s$ and $1 - \nu_A(xy) \geq a \wedge b \wedge t = a \wedge b$. Hence $[x_a y_b \in A] \geq \frac{1}{2}$.

Therefore, $[x_a y_b \in A] \geq [x_a \in \wedge q_s A] \wedge [y_b \in \wedge q_s A]$.

Similarly, we have $[x_a^{-1} \in A] \geq [x_a \in \wedge q_s A]$.

"$\Leftarrow$" (1) If $\mu_A(xy) \vee s < a = \mu_A(x) \wedge \mu_A(y) \wedge t$, then $s < a \leq t$, $\mu_A(x) \geq a$ and $\mu_A(y) \geq a$. Thus $a + \mu_A(x) > 2s$ and $a + \mu_A(y) > 2s$. So $[x_a \in \wedge q_s A] = [y_a \in \wedge q_s A] = 1$. Hence $[x_a y_a \in A] = 1$, i.e., $\mu_A(xy) \geq a$, which is a contradiction to $\mu_A(xy) < a$.

Therefore, we have $\mu_A(xy) \vee s \geq \mu_A(x) \wedge \mu_A(y) \wedge t$.

(2) If $\nu_A(xy) \wedge (1 - s) > 1 - a = \nu_A(x) \vee \nu_A(y) \vee (1 - t)$, then $(1 - \nu_A(xy)) \vee s < a = (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \wedge t$. Thus $s < a \leq t$, $1 - \nu_A(x) \geq a$ and $1 - \nu_A(y) \geq a$. So $1 - \nu_A(x) > 2s - a$ and $1 - \nu_A(y) > 2s - a$, which implies that $[x_a y_a \in A] \geq [x_a \in \wedge q_s A] \wedge [y_b \in \wedge q_s A] \geq \frac{1}{2}$. Then $1 - \nu_A(xy) \geq a$, which contradicts to $1 - \nu_A(xy) < a$.

Hence, $\nu_A(xy) \wedge (1 - s) \leq \nu_A(x) \vee \nu_A(y) \vee (1 - t)$.

Similarly, we have $\mu_A(x^{-1}) \vee s \geq \mu_A(x) \wedge t$ and $\nu_A(x^{-1}) \wedge (1 - s) \leq \nu_A(x) \vee (1 - t)$.

Therefore, A is a $(s, t]$-intuitionistic fuzzy subgroup of G. □

Theorem 5.3. *Let $s, t \in [0, 1]$ and $s < t < 1$. Then A is a $(s, t]$-intuitionistic fuzzy subgroup of G if and only if*

(1) $\forall a, b \in (s, 1], \forall x, y \in G$

$$([x_a \in A] \wedge [y_b \in A] \to [x_a y_b \in \vee q_t A]) = 1,$$

i.e., $[x_a y_b \in \vee q_t A] \geq [x_a \in A] \wedge [y_b \in A]$.

(2) $\forall a \in (s, 1], \forall x \in G$

$$([x_a \in A] \to [x_a^{-1} \in \vee q_t A]) = 1,$$

i.e., $[x_a^{-1} \in \vee q_t A] \geq [x_a \in A]$.

Proof. "$\Rightarrow$" For any $a, b \in (s, 1]$, let $c = [x_a \in A] \wedge [y_b \in A]$.

Case 1. $c = 1$. Then $\mu_A(x) \geq a$ and $\mu_A(y) \geq b$. By $\mu_A(xy) \vee s \geq \mu_A(x) \wedge \mu_A(y) \wedge t$ we know that $\mu_A(xy) \vee s \geq a \wedge b \wedge t$.

If $a \wedge b \leq t$, then $\mu_A(xy) \geq a \wedge b$. Thus $[x_a y_b \in A] = 1$;

If $a \wedge b > t$, then $a \wedge b + \mu_A(xy) > 2t$. Thus $[x_a y_b q_t A] = 1$, so $[x_a y_b \in \vee q_t]A = 1$.

Case 2. $c = \frac{1}{2}$. Then $1 - \nu_A(x) \geq a$ and $1 - \nu_A(y) \geq b$. By $\nu_A(xy) \wedge (1 - s) \leq \nu_A(x) \vee \nu_A(y) \vee (1 - t)$, we know that $(1 - \nu_A(xy)) \wedge s \geq (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \wedge t \geq a \wedge b \wedge t$.

If $a \wedge b \leq t$, then $1 - \nu_A(xy) \geq a \wedge b$. So $[x_a y_b \in A] \geq \frac{1}{2}$;

If $a \wedge b > t$, then $1 - \nu_A(xy) \geq t$. Thus $2t - (a \wedge b) \leq 1 - \nu_A(xy)$. So $[x_a y_b q_t A] \geq \frac{1}{2}$.

Hence, $[x_a y_b \in \vee q_t A] \geq [x_a \in A] \wedge [y_b \in A]$.

(2) can be proved similarly.

"$\Leftarrow$" (1) If $\mu_A(xy) \vee s < a = \mu_A(x) \wedge \mu_A(y) \wedge t$, then $[x_a \in A] = [y_a \in A] = 1$ and $s < a \leq t$. Thus $[x_a y_a \in \vee q_t A] = 1$. By $\mu_A(xy) < a$, we have that $[x_a y_a q_t A] = 1$. Thus $a + \mu_A(xy) > 2t$. So $\mu_A(xy) > 2t - a \geq a$, which is a contradiction to $\mu_A(xy) < a$. Hence, $\mu_A(xy) \vee s \geq \mu_A(x) \wedge \mu_A(y) \wedge t$.

(2) If $\nu_A(xy) \wedge (1-s) > (1-a) = \nu_A(x) \vee \nu_A(y) \vee (1-t)$, then $s < a \leq t$, $\nu_A(xy) > 1-a$, $1-a \geq \nu_A(x)$ and $1-a \geq \nu_A(y)$. Thus $[x_a \in A] \geq \frac{1}{2}$ and $[y_a \in A] \geq \frac{1}{2}$. So $[x_a y_a \in \vee q_t A] \geq \frac{1}{2}$. Hence $1 - \nu_A(xy) \geq 2t - a \geq a$. This is a contradiction to $\nu_A(xy) > 1 - a$. Therefore, we have $\nu_A(xy) \wedge (1 - s) \leq \nu_A(x) \vee \nu_A(y) \vee (1 - t)$.

(3) Similarly, we can prove that $\mu_A(x^{-1}) \vee s \geq \mu_A(x) \wedge t$ and $\nu_A(x^{-1}) \wedge (1 - s) \leq \nu_A(x) \vee (1 - t)$. □

Remark 5.2. When $s = 0.5$ and $t = 1$, then Theorem 5.2 is coincident with Theorem 4.6(3); when $s = 0$ and $t = 0.5$, then Theorem 5.3 is coincident with Theorem 4.6(2).

6. Conclusions

In this paper, we established the neighborhood relations between a fuzzy point x_a and an intuitionistic fuzzy set A and gave the definition of (α, β)- intuitionistic fuzzy subgroups based on the concept of cut sets on intuitionistic fuzzy sets. We obtained the following results with no conditions attached:

1. Among 16 kinds of (α, β)-intuitionistic fuzzy subgroups, the significant ones are the $(\in, \in)$-intuitionistic fuzzy subgroup, the $(\in, \in \vee q)$-intuitionistic fuzzy subgroup and the $(\in \wedge q, \in)$-intuitionistic fuzzy subgroup.

2. A is a $(\in, \in)$-intuitionistic fuzzy subgroup of G if and only if, for any $a \in (0, 1]$, the cut set A_a of A is a 3-valued fuzzy subgroup of G, and A is a $(\in, \in \vee q)$-intuitionistic fuzzy subgroup (or $(\in, \in \vee q)$-intuitionistic fuzzy subgroup) of G if and only if, for any $a \in (0, 0.5]$(or for any $a \in (0.5, 1]$), the cut set A_a of A is a 3-valued fuzzy subgroup of G.

3. We generalize the $(\in, \in)$-intuitionistic fuzzy subgroup, $(\in, \in \vee q)$- intuitionistic fuzzy subgroup and $(\in \wedge q, \in)$-intuitionistic fuzzy subgroup to intuitionistic fuzzy subgroup with thresholds, i.e., $(s, t]$-intuitionistic fuzzy subgroup. We also show that A is a $(s, t]$-intuitionistic fuzzy subgroup of G if and only if, for any $a \in (s, t]$, the cut set A_a of A is a 3-valued fuzzy subgroup of G.

4. By the neighborhood relations between a fuzzy point x_a and an intuitionistic fuzzy set A, we characterize the $(s, t]$-intuitionistic fuzzy subgroup.

Our works have shown that our method is better than that in [22,23].

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