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The bounded-complete hull of an α-space

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1. Introduction

In the paper [3], the author suggested a general topological approach to domain theory as highly convenient and more general than the established more traditional approach using *dcpos* (directed-complete partial orders) starting from D. Scott's work. This approach was realized by the author in the papers [2, 3] for the cases of f-spaces and A-spaces (complete f_0-space = algebraic bounded-complete domain, complete A_0-space = bounded-complete domain; in the sequel, the term bc-domain will be used to denote bounded-complete domains). In the introduction to [3], the properties of the relation $\prec$ of a *recognizable* approximation was discussed.

I would like to quote from [3]: "It is natural to require that all *recognizable* approximations of a fixed element x should form a directed set and this can be satisfied in the strong (but sufficiently reasonable) form:

(5) if $x_0 \prec x$ and $x_1 \prec x$ then there exists an element $x_2 \in X_0$ [basis] which is the exact upper bound ($x_2 = x_0 \vee x_1$) of these elements in $\langle X, \leqslant \rangle$ and $x_2 \prec x$." (Compare with the definition of an abstract basis in [1]).

One of the arguments for the reasonability of the condition (5) is the following (naive) consideration: "If I know that x_0 and x_1 are approximations of an element x, then the pair (x_0, x_1) can be considered as an *approximation* of x, which contains only that information about x which is carried by the approximations x_0 and x_1. So from the point of view of information about x a couple (x_0, x_1) is the exact upper bound for x_0 and x_1." The problem is that the pair (x_0, x_1) does not belong to the space of approximations X.

It is the aim of the present paper to present a mathematically correct realization of the idea described above and to show that for any α-space (= a basis for a domain, see Proposition 4 below) there exists a *uniquely defined* bc-domain B and a homeomorphic embedding $\lambda : X \to B$ with the properties of universality and minimality (the exact

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definitions see below). This pair (λ, B) will be called the bounded-complete hull (bc-hull) of X.

The survey paper [1] contains all the needed definitions and properties of the domains, which will be used in the proofs.

2. α-spaces and their d-hulls

With any topological space X connect two binary relations $\leqslant$ ($\leqslant_X$) and $\prec$ ($\prec_X$) defined as follows:

$$x \leqslant y \rightleftharpoons \text{for every open } V, \text{ if } x \in V \text{ then } y \in V;$$

$$(\downarrow x \rightleftharpoons \{z \mid z \leqslant x\}, \uparrow x \rightleftharpoons \{y \mid x \leqslant y\})$$

$$x \prec y \rightleftharpoons y \in \operatorname{Int} \uparrow x.$$

($\operatorname{Int} V$ is the interior of a set V = the largest open set contained in V.)

The relation $\leqslant_X$ on a topological space X is a partial order (in general it is a preorder) iff X is a T_0-space. Below all topological spaces are supposed to be the T_0-spaces.

If X is a subspace of Y so call X a *smooth* subspace of Y iff the relation $\prec_X$ is the restriction of the relation $\prec_Y$ on $X (\prec_X = \prec_Y \cap X^2)$. Note that if X is a subspace of Y then $\prec_Y \cap X^2 \subseteq \prec_X$ (and $\leqslant_X = \leqslant_Y \cap Y^2$).

A space X will be called an *α-space* (see [3, 4]), if for any open V and any $x \in V$ there is $y \in V$ such that $y \prec x$.

Corollary 1. *If X is an α-space, $x, y \in X$, and $x \prec y$ then there exists an $z \in X$ such that $x \prec z \prec y$.*

Indeed, the set $U_x \in \{z \mid x \prec z\}$ is open, $y \in U_x$, and for $z \in U_x$ such that $z \prec y$ we also have $x \prec z$.

Now we establish one of the most specific properties.

Proposition 2. *Let X be an α-space and Y and Z be arbitrary topological spaces. The mapping $f : X \times Y \to Z$ is continuous if and only if f is continuous in each argument.*

Proof. We only need to check that if f is continuous in each argument then for any open $U \subseteq Z$ and $\langle x_0, y_0 \rangle \in f^{-1}(U)$ there exist neighborhoods V and W of x_0 and y_0 such that $V \times W \subseteq f^{-1}(U)$.

Since $\lambda x f(x, y_0)$ is continuous, it follows that there exists an open $V_0 \subseteq X$ such that $x_0 \in V_0$ and $V_0 \times \{y_0\} \subseteq f^{-1}(U)$. Let $x_1 \in V_0$ be such that $x_1 \prec x_0$ (X is an α-space!), and let $W \subseteq Y$ be an open set such that $y_0 \in W$, $\{x_1\} \times W \subseteq f^{-1}(U)$ $(\lambda y f(x_1, y) :$

$Y \to Z$ is continuous!). If $V \rightleftharpoons \{x' \mid x' \in X,\ x_1 \prec x'\}$ then V is open, $x_0 \in V$. We show that $V \times W \subseteq f^{-1}(U)$. If $\langle x', y' \rangle \in V \times W$ then $f(x_1, y') \in U$ by the choice of W. We have $f(x_1, y') \leqslant_Z f(x', y')$ because $x_1 \leqslant_X x'$ and $\lambda x f(x, y')$ is a continuous mapping. But then $f(x', y') \in U$ and $V \times W \subseteq f^{-1}(U)$. $\square$

We now give a characterization of α-spaces in terms of the lattices of all open subsets. [2]

Recall (cf. e.g., [1]) that a complete lattice L is called completely distributive if for any index set I and any families $W_i \subseteq L$, $i \in I$, the following equality holds:

$$\bigwedge_{i \in I} \left(\bigvee W_i\right) = \bigvee_{f \in \prod_{i \in I} W_i} \left(\bigwedge_{i \in I} f(i)\right) \tag{$*$}$$

Theorem 3. *A topological (T_0)-space X is an α-space if and only if its lattice $\Omega(X)$ of all open sets is completely distributive.*

Proof. Let $\Omega(X)$ be completely distributive, $U \subseteq X$ be open, and $x \in U$. For any $y \in U \setminus (\downarrow x)$ there exists an open $U_y (\subseteq U)$ such that $y \in U_y$ and $x \notin U_y$.

Let $W_0 \rightleftharpoons \{U_y \mid y \in U \setminus (\downarrow x)\}$. For any open cover S (by open subsets of U) of the set $(\downarrow x) \cap U$ we denote $W_S \rightleftharpoons W_0 \cup S$. Since W_0 covers $U \setminus (\downarrow x)$ and S covers $\downarrow x \setminus \cap U$ then W_S covers U and $\bigvee W_S = U$ (in the lattice $\Omega(X)$ the union has the same set-theoretic sense). Thus, if $I \rightleftharpoons \{S \mid S$ is an open cover of $(\downarrow x) \setminus U\}$ then $\bigwedge_{S \in I}(\bigvee W_S) = U$. Since in this case $\Omega(X)$ is completely distributive, there exists $f \in \prod_{S \in I} W_S$ such that $x \in U_x \rightleftharpoons \bigwedge_{S \in I} f(S)$. For any $S \in I$ we have $x \in f(S) \in W_S$ but $x \notin \bigvee W_0 = \bigcup_{y \in U \setminus (\downarrow x)} (U_y)$ by the choice of U_y. Consequently, $f(S) \in S$.

Thus, for any cover $S \in I$ an open set U_x containing x is contained in some $V \in S$. We show that there exists $z_0 \in (\downarrow x) \cap U$ such that $U_x \subseteq \uparrow z_0$ (and hence $z_0 \prec x$).

Assume that there is no such z_0. Then, for any $z \in (\downarrow x) \cap U$, there exists $u_z \in U_x \setminus (\uparrow z)$. Since $u_z \notin \uparrow z$ and $u_z \not\leqslant z$, there exists an open set $V_z (\subseteq U)$ such that $z \in V_z$ and $u_z \notin V_z$. Let $S_0 \rightleftharpoons \{V_z \mid z \in (\downarrow x) \cap U\}$. Then $S_0 \in I$, $f(S_0) \in S_0$, and $U_x \subseteq V_z$ for some $z \in (\downarrow x) \cap U$. This contradicts $u_z \in U_x \setminus V_z$. Thus, there exists a $z_0 \in (\downarrow x) \cap U$ such that $U_x \subseteq \uparrow z_0$, $z_0 \in U$, and $z_0 \prec x$. Therefore X is an α-space.

Let X be an α-space. Establish the relation $(*)$ in the lattice $\Omega(X)$. It suffices to prove the inclusion $\subseteq$.

Let $x \in U \rightleftharpoons \bigwedge_{i \in I}(\bigvee W_i)$, then U is open; let $x_0 \in U$ and $x_0 \prec x$. For any $i \in I$,

$$U \subseteq \bigvee W_i = \bigcup \{V \mid V \in W_i\}.$$

Hence, there exists $V_i \in W_i$ such that $x_0 \in V_i$. We assume that $f \in \prod_{i \in I} W_i$ is such that $f(i) = V_i$ (i.e., $x_0 \in f(i)$ for all $i \in I$); if $U_{x_0} \rightleftharpoons \{x' \in X,\ x_0 \prec x'\}$ then U_{x_0} is open, $x \in U_{x_0}$, and $U_{x_0} \subseteq V_i = f(i)$ for all $i \in I$. But in this case $U_{x_0} \subseteq \bigwedge_{i \in I} f(i)$

[2] I am grateful to E. Griffor who asked me a question which had led to this characterization.

because $\bigwedge_{i\in I} f(i)$ is the largest open subset of the set $\bigcap_{i\in I} f(i)$. Thus, we have

$$x \in U_{x_0} \subseteq \bigwedge_{i\in I} f(i) \subseteq \bigvee_{f\in\Pi_{i\in I} W_i} \left(\bigwedge_{i\in I} f(i) \right).$$

The inclusion $\subseteq$, and hence the theorem, is proved. □

Remark. One can prove this theorem making use of the following proposition and the well-known corresponding result for domains [1]. However, the direct proof demonstrates that it is very easy to work with the notion of an "α-space" directly.

Note that every domain, considered as a topological space, is an α-space.

Proposition 4. *A space X is an α-space if and only if X is homeomorphic to a basis for some domain.*

Proof. Let X be an α-space, then (it is not hard to see) the system $\langle X, \prec\rangle$ is an (abstract) basis [1]; let $X^* \rightleftharpoons Idl(X,\prec)$ be the ideal completion of $\langle X,\prec\rangle$, then by Proposition 2.2.22 in [1], X^* is a domain and $i: X \to X^*$ is a homeomorphism from X onto the basis $i(X)$ of the domain $X^*(i(x) \rightleftharpoons \Downarrow x = \{y \mid y \prec x\})$.

Vice versa, let D be a domain (considered as a topological space) and $X \subseteq D$ be a basis of D (i.e., for every $d\in D$ the family $X_d \rightleftharpoons \{x \mid x \in X, x \prec_D d\}$ is directed and $d = \sup X_d$).

Show that for every open V_0 in X, every element $x_0 \in V_0$ there exists $x \in V_0$ such that $x \prec_D x_0$: X is a basis of D, so $x_0 = \sup X_{x_0}$ and if V is open in D such that $V_0 = V \cap X$ then (D is a domain!) $V \cap X_{x_0} \neq \emptyset$; but $V \cap X_{x_0} \subseteq V_0$ and if $x \in V \cap X_{x_0} \subseteq V_0$ then $x \prec_D x_0$. Because $x \prec_D x_0$ implies $x \prec_X x_0$, we have that X is an α-space.

Note in addition that X is also a smooth subspace of D: if $x_0 \prec_X x_1$ and if V_0 is an open subset of X such that $x_1 \in V_0 \subseteq \{x \mid x_0 \leqslant_X x_0,\ x \in X\}$, then, as it was proved above, there is $x \in V_0$ such that $x \prec_D x_1$; but $x_0 \leqslant x$, $x \prec_D x_1$ implies $x_0 \prec_D x_1$; so X is a smooth subspace of D. □

The following proposition, expressing known properties of bases of domains (for A-spaces see [3]), is formulated in a form which is convenient for the later considerations here.

Proposition 5. *For any α-space X there are a domain D and a homeomorphic embedding $\lambda : X \to D$ of the space X onto a basis of D. The pair (λ, D) enjoys the following two conditions*:

(1) *For every continuous map $f: X \to D'$ from X into a domain D' there is a continuous map $f^*: D \to D'$ such that $f = f^*\lambda$.*

(2) *If $f: D \to D$ is a continuous map from D into itself such that $f \restriction \lambda(X) = id_{\lambda(X)}$ (i.e., $f\lambda = \lambda$), then $f = id_D$.*

The pair (λ, D) with the properties (1) *and* (2) *is uniquely defined up to a homeomorphism over λ.*

Proof. (1) For simplicity suppose that $X \subseteq D$ (and $\lambda = id_X$). Let f be a continuous map from X into a domain D'; for every element $d \in D$ the set $X_d \rightleftharpoons \{x \mid x \in X, x \prec d\}$ is directed and $d = \sup X_d$; let $f^*(d) \rightleftharpoons \sup\{f(x) \mid x \in X_d\}$; f is continuous, hence monotonic (relative to the orders $\leqslant_X$ and $\leqslant_{D'}$) and consequently the set $\{f(x) \mid x \in X_d\}$ is directed and the map f^* is well defined. Check that $f^* \upharpoonright X = f$: from the definition of f^* one can see that $f^*(x) \leqslant_{D'} f(x)$ for all $x \in X$. Suppose that there is $x_0 \in X$ such that $f(x_0) \nleqslant_{D'} f^*(x_0)$; then there is an open V in D' such that $f(x_0) \in V$, but $f^*(x_0) \notin V$. The set $f^{-1}(V)$ is open in X and contains x_0; then there is $x_1 \in f^{-1}(V)$ such that $x_1 \prec_D x_0$, then $x_1 \in X_{x_0}$ and $f(x_1) \leqslant f^*(x_0) = \sup\{f(x) \mid x \in X_{x_0}\}$. But $x_1 \in f^{-1}(V)$, so $f(x_1) \in V$ and $f^*(x_0)$ $(\geqslant f(x_1)) \in V$; a contradiction. Check that f^* is continuous: let V' be open in D', $V \rightleftharpoons f^{*-1}(V')$ and $d \in V$. Then $f^*(d) \in V'$; $f^*(d) = \sup\uparrow \{f(x) \in X_d\}$ and $\{f(x) \mid x \in X_d\} \cap V' \neq \emptyset$; let $x_0 \in X_d$ be such that $f(x_0)$ $(= f^*(x_0)) \in V'$; then $x_0 \in V$ and $x_0 \prec d$. From the obvious equality $\uparrow V = V$ it follows that the open set $\{d' \mid x_0 \prec d'\}$ is contained in V and contains d. So, V is open and f^* is continuous. Note in addition that f^* is the unique extension of f on D. Really, if $g: D \to D'$ is a continuous map such that $g \upharpoonright X = f$, then for every $d \in D$, $d = \sup\uparrow X_d$ we have $g(d) = \sup\{g(x) \mid x \in X_d\} = \sup\{f(x) \mid x \in X_d\} = f^*(d)$.

(2) From this uniqueness, property (2) follows, because id_D extends id_X. □

A pair (λ, D) which satisfied the conditions (1) and (2) of the Proposition 5 will be called a *d-hull* of X and we will use the notation $H_d(X)$ for D and δ_X for λ.

There is a description of the bases of algebraic domains analogous to Proposition 4.

An element x of a topological space X is called *compact* if the set $\uparrow x = \{y \mid y \in X, x \leqslant y\}$ is open in X.

A (topological) space X is called a *φ-space* if for any open V in X and any $x \in V$, there exists a compact element $y \in V$ such that $y \leqslant x$ (note that for a compact y the relations $y \leqslant x$ and $y \prec x$ are equivalent).

The following proposition practically has the same proof as Proposition 4.

Proposition 6. *A space X is a φ-space if and only if X is homeomorphic to a basis of some algebraic domain.*

Corollary 7. *The d-hull $H_d(X)$ of an α-space X is an algebraic domain iff X is a φ-space.*

3. The bounded-complete hull: The case of φ-spaces

Consider now the question of embedding an α-space into a bc-domain. Define the notion of a bc-hull of an α-space X in analogy with the d-hull as follows:

Let X be an α-space, let B be a bc-domain, and let $\lambda : X \to B$ be a homeomorphic embedding of X into B. The pair (λ, B) is called a bc-hull of X if the following two conditions hold:

(1) *Universality*: For every continuous map $f : X \to B'$ from the α-space X into a bc-domain B' there exists a continuous map $f^* : B \to B'$ such that $f^*\lambda = f$.

(2) *Minimality*: If $f : B \to B$ is a continuous map of B into itself such that $f\lambda = \lambda$ then $f = id_B$.

Remark that the uniqueness of a bc-hull of X follows from the conditions (1) and (2) in the following exact sense:

Proposition 8. *If (λ, B) and (λ', B') are* bc-*hulls of X, then there exists a unique homeomorphism $\varphi : B \to B'$ of the spaces B and B' such that $\varphi\lambda = \lambda'$.*

Proof. Due to universality of the bc-hulls of X, there are continuous maps $\varphi : B \to B'$ and $\varphi' : B' \to B$ such that $\lambda' = \varphi\lambda$ and $\lambda = \varphi'\lambda'$. But then $\varphi'\varphi$ is a continuous map of B into itself such that $\varphi'\varphi\lambda = \varphi'\lambda' = \lambda$ and by minimality for (λ, B) we have $\varphi'\varphi = id_B$. In the same way it is possible to deduce that $\varphi\varphi' = id_{B'}$. So, φ (φ') is a homeomorphism from B onto B' (from B' onto B). □

Note. The example below shows that it is impossible to require uniqueness in the condition (1) of the definition of a bc-hull, though for a d-hull the uniqueness of f^* in condition (1) holds, as it was noticed in the proof of Proposition 4.

For the bc-hull of X (if it exists) we will use the notation $(\beta_X, H_{\mathrm{bc}}(X))$.

Now let us turn to the question of the existence of bc-hulls. Start from the case of φ-spaces.

Let X be a φ-space, $K = K(X)$ the set of all compact elements of X.

Let K^* be the family of all nonempty subsets of K representable in the form $k_F \rightleftharpoons \bigcap_{d\in F}(\uparrow d \cap K)$, where F is a finite subset of K (notice that $K = k_\emptyset \in K^*$). Let $\leqslant^*$ be the partial order on K^* opposite to the inclusion relation $k \leqslant^* k' \rightleftharpoons k' \subseteq k$; (then K is the least element of $\langle K^*, \leqslant^* \rangle$).

Remark that $\langle K^*, \leqslant^* \rangle$ is a partial upper semilattice: if $k_{F_0}, k_{F_1} \in K^*$ and if there is a $k \in K^*$ such that $k_{F_0} \leqslant^* k$ and $k_{F_1} \leqslant^* k$ ($\emptyset \neq k \subseteq k_{F_0}, k \subseteq k_{F_1}$), then $k_{F_0} \cap k_{F_1} = k_{F_0 \cup F_1} \in K^* (k_{F_0 \cup F_1} \supseteq k \neq \emptyset)$ and is obviously the least upper bound for k_{F_0} and k_{F_1} in $\langle K^*, \leqslant^* \rangle$.

Let $B \rightleftharpoons Idl(K^*, \leqslant^*)$ be the ideal completion of $\langle K^*, \leqslant^* \rangle$; then B is an algebraic bc-domain. Define a continuous map $\lambda : X \to B$ as the (unique) continuous extension of the monotonic (= continuous) map $\lambda_0 : K \to K^* \subseteq Idl(K^*, \leqslant^*)$ defined by $\lambda_0(d) \rightleftharpoons (\uparrow d \cap K) (\in K^*)$.

Theorem 9. *The pair (λ, B) is a* bc-*hull of X.*

Proof. Let us check universality (property (1)): Let $f : X \to B'$ be any continuous map from X into a bc-domain B'. Define a monotone (= continuous) map $f_0 : K^* \to B'$ as follows:

$$f_0^*(k) \rightleftharpoons \inf_{B'}\{f(x) \mid x \in k\} \quad \text{for any } k \in K^* .$$

B' is a bc-domain, so inf exists in B' for any nonempty subset of B'. From the definition of f_0^* clear that f_0^* is monotonic: $k_0 \leqslant^* k_1 \Leftrightarrow k_1 \subseteq k_0 \Rightarrow \{f(x) \mid x \in k_1\} \subseteq \{f(x) \mid x \in k_0\} \Rightarrow f_0^*(k_0) = \inf\{f(x) \mid x \in k_0\} \leqslant_{B'} f_0^*(k_1) = \inf\{f(x) \mid x \in k_1\}$. If $k = (\uparrow x_0 \cap K)$ for $x_0 \in K$ then $f_0^*(k) = \inf\{f(x) \mid x \in K,\ x_0 \leqslant x\} = f(x_0)$. The continuous map f_0^* of the basis K^* of the algebraic bc-domain B has a unique continuous extension $f^*: B \to B'$ and for this map we have for $f^*\lambda(x_0) = f_0^*(\uparrow x_0 \cap K) = f(x_0)$ for $x_0 \in K$; so $f^*\lambda$ is a continuous extension of the continuous map $f \upharpoonright K$ of the basis K of the space X into B'. But f is also an extension of $f \upharpoonright K$; by the uniqueness of an extension from a basis we have $f^*\lambda = f$. So the property of universality for the pair (λ, B) is proved.

Note an additional property of the function f^* which will be used later:

If $f': B \to B'$ *is a continuous map such that* $f = f'\lambda$, then $f' \leqslant f^*$. Check that $f'(k) \leqslant_{B'} f^*(k) (= f_0^*(k))$ for every $k \in K$; from that the property just mentioned will follow: Let $k \in K^*$, $x \in k$; then $k \supseteq (\uparrow x \cap K) = \lambda(x)$, $k \leqslant^* \lambda(x)$ and $f'(k) \leqslant_{B'} f'\lambda(x) = f(x)$, because f' is monotonic; so $f'(k)$ is a lower boundary for the set $\{f(x) \mid x \in k\}$ and $f'(k) \leqslant_{B'} \inf\{f(x) \mid x \in k\} = f_0^*(k) = f^*(k)$. □

Let us turn to proving the minimality (property (2)):

To start let us prove a lemma needed for later use also.

A subspace $X \subseteq B$ of a bc-domain B is called $\vee$-*dense* if any element b from B has a presentation in the form $b = \sup \uparrow D$ for a set D such that every element of D has the form $x_0 \vee_B \cdots \vee_B x_n$ for a finite subset $\{x_0, \ldots, x_n\}$ of X.

Lemma 10. *If X is a $\vee$-dense subspace of a* bc-*domain B, and if f is a continuous map of B into itself such that $f \upharpoonright X = id_X$, then $f \geqslant id_B$.*

Any element b from B has the form $b = \sup D_b$, $D_b = \{\bigvee_{x \in F} x \mid F \in \mathscr{F}_b\}$ for some family $\mathscr{F}_b$ of finite subsets of X bounded in X. Then $f(b) = f(\sup D_b) = \sup f(D_b)$; $f(D_b) = \{f(\bigvee_{x \in F} x) \mid F \in \mathscr{F}_b\}$; but $f(\bigvee_{x \in F} x) \geqslant \bigvee_{x \in F} f(x) = \bigvee_{x \in F} x$; so $f(b) = \sup f(D_b) \geqslant \sup D_b = b$. □

Proof of Theorem 9 (*Continued*). It is not hard to see that the condition of the lemma is satisfied for $\lambda(X) \subseteq B$.

Let $f: B \to B$ be a continuous map of B into itself such that $f\lambda = \lambda(f \upharpoonright \lambda(X)) = id_{\lambda(X)}$. Then $f \geqslant id_B$ by the lemma. For proving $f = id_X$ check that $k = \inf_B\{\uparrow x \cap K \mid x \in k\}$ for every $k \in K^*$.

If $x \in k$, then $\uparrow x \cap K \subseteq k$, $k \leqslant^* \uparrow x \cap K$ and K and so k is a lower bound for the set $\{\uparrow x \cap K \mid x \in k\}$ and $k \leqslant^* \inf\{\uparrow x \cap K \mid x \in k\}$. Suppose that $k \neq \inf\{\uparrow x \cap K \mid x \in k\}$; then there is $k' \in K^*$ such that $k' \leqslant^* \inf\{\uparrow x \cap K \mid x \in k\}$, but $k' \ {}^* \ k$, $k \not\subseteq k'$. Let $x_0 \in k \backslash k'$, then $\uparrow x_0 \cap K \not\subseteq k'$, $k' \ {}^* \uparrow x_0 \cap K$. But this is impossible because k' must be a lower bound for $\{\uparrow x \cap K \mid x \in k\} \ni \uparrow x_0 \cap K$.

Let $\lambda^*: B \to B$ be the continuous map constructed as in the proof of the theorem (proving the universality). On elements from K^* we have $\lambda^*(k) = \lambda_0^*(k) =$

$\inf\{\lambda(x) \mid x \in k\} = \inf\{\uparrow x \cap K \mid x \in k\} = k$. Then $\lambda^* = id_B$ and by the property of the construction * we have $f \leqslant \lambda^* = id_B$; so $f = id_B$ and the theorem is proved. □

Notice two corollaries of the (proof of the) theorem.

Corollary A. *For a φ-space X, the embedding $\beta_X : X \to H_{\mathrm{bc}}(X)$ is a homeomorphism onto a smooth subspace of $H_{\mathrm{bc}}(X)$.*

Proof. This follows from the construction of $H_{\mathrm{bc}}(X)$ and from the following remark that is not hard to verify: If a φ-space X is a subspace of an α-space Y, then X is smooth iff every compact element of X is a compact element of Y. □

Corollary B. *If X is a coherent algebraic domain then $H_{\mathrm{bc}}(X)$ is a natural subspace of the Smith powerdomain $P^{\mathrm{S}}(X)$ of X.*

Proof. This easily follows from the construction of $H_{\mathrm{bc}}(X)$, the definition of coherence, and from Theorem 6.2.14 [1]. □

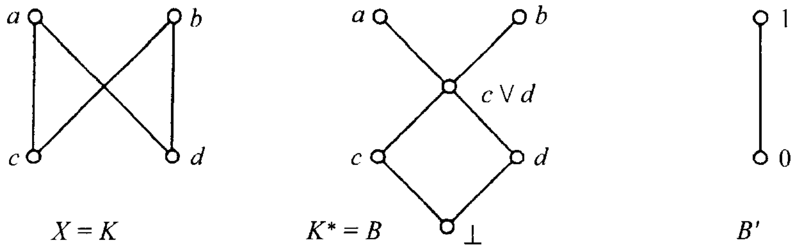
We prove one more property of the construction which is not a direct corollary of the (proof of the) theorem.

Proposition 11. *Let X be a φ-space, $f : X \to B'$ a homeomorphic embedding of X onto a smooth subspace of B'. Then the largest continuous extensions $f^* : H_{\mathrm{bc}}(X) \to B'$ of f such that $f = f^*\beta_X$ is injective.*

In the proof we shall use the concrete bc-hull of X constructed in the proof of the theorem; so we keep the same notations ($H_{\mathrm{bc}}(X) = B$, $\beta_X = \lambda, \ldots$).

Proof. Suppose that f^* is not injective, then $f^*(b_0) = f^*(b_1)$ for some $b_0 \neq b_1 \in B$. Find an element $k \in K^*$ such that $k_0 \leqslant_B b_0$, $k_0 \;_B\; b_1$ (if $k_0 \leqslant_B b_1$, $k_0 \;_B\; b_0$, then exchange 0 and 1 in the indices). The elements b_0 and b_1 have a presentation $b_i = \sup\{k_F \mid k_F \in K^*,\ F \in \mathscr{F}_i\}$, $i = 0, 1$, for some families $\mathscr{F}_0$ and $\mathscr{F}_1$ of finite subsets of K bounded in K closed under finite unions. We can suppose also that $k_0 = k_{F_0}$ for some $F_0 \in \mathscr{F}_0$. For every $F_1 \in \mathscr{F}_1$ we have $k_{F_0} \;^*\; k_{F_1}$, $k_{F_1} \nsubseteq k_{F_0} = \bigcap_{x \in F_0}(\uparrow x \cap K)$. So for every $F_1 \in \mathscr{F}_1$ there is an $x \in F_0$ such that $k_{F_1} \nsubseteq \uparrow x \cap K$. The set F_0 is finite, the family $\mathscr{F}_1$ is directed (under inclusion); so there is $x_0 \in F_0$ such that $k_{F_1} \nsubseteq \uparrow x_0 \cap K$ for every $F_1 \in \mathscr{F}_1$. Now, $f^*(\uparrow x_0 \cap K) \leqslant_{B'} f^*(b_0)$, because $\uparrow x_0 \cap K \leqslant^* k_{F_0} \leqslant_B b_0$ and $f^*(b_0) = f^*(b_1) = f^*(\sup\{k_F \mid F \in \mathscr{F}_1\}) = \sup\{f^*(k_F) \mid F \in \mathscr{F}_1\}$. We have $f^*(\uparrow x_0 \cap K) = f^*\lambda(x_0) = f(x_0)$. Now x_0 is a compact element of X, so $f(x_0)$ is a compact element of B' because f is a homeomorphism onto a smooth subspace of B! Then $f(x_0) \leqslant_{B'} f^*(k_{F_1})$ for some $F_1 \in \mathscr{F}_1$; $f^*(k_{F_1}) = f_0^*(k_{F_1}) = \inf_{B'}\{f(x) \mid x \in k_{F_1}\}$ and $f(x_0) \leqslant_{B'} f(x)$ for every $x \in k_{F_1}$. Because f is a homeomorphism we have $x_0 \leqslant x$ for every $x \in K_{F_1}$, $\uparrow x_0 \supseteq K_{F_1}$, but this contradicts to the choice of x_0. □

Example 12. We use the simple diagrams instead of many words:



$X = K$ $K^* = B$ B'

If $f: X \to B'$ is defined by $f(a) = f(b) = 1$, $f(c) = f(d) = 0$, then $f^*(c \vee d) = 1$, $f^*(\bot) = 0$. But for the monotonic (=continuous) map $f': B \to B'$ extending f and such that $f'(c \vee d) = 0$, $f'(\bot) = 0$, we have $f'\lambda = f$ and $f' \neq f^*$.

In concluding this section where we considered the case of φ-spaces let us formulate a proposition concerning the b_0-spaces with a constructivizable basis (see [4]).

If X is a b-space with a constructivizable basis then $H_{\text{bc}}(X)$ is a bc-*domain with a constructivizable basis.*

4. The bounded-complete hull: The general case of α-spaces

In this section the general case of an arbitrary α-space X will be considered. Note one well-known fact for domains.

Proposition 13. *Any α-space is a projection of some φ-space.*

Proof. Let X be an α-space we can suppose (Proposition 4) that X is a basis of some domain X_0. If $Y_0 \rightleftharpoons Idl(X, \leqslant_X)$ is the ideal completion of the partial ordered set $\langle X, \leqslant_X \rangle$ then the pair of the maps

$$e_0 : x_0 \mapsto X_{x_0} (\rightleftharpoons \{x \mid x \in X,\ x \prec x_0\}), \quad x_0 \in X_0,$$

$$p_0 : I \mapsto \sup I, \quad I \in Idl(X, \leqslant_X)$$

is an embedding-projection pair [1]. If $Y \rightleftharpoons p_0^{-1}(X)$, $e \rightleftharpoons e_0 \restriction X$, $p \rightleftharpoons p_0 \restriction Y$ then (e, p) is an embedding-projection pair, and Y is a φ-space because Y has the same compact elements $(\downarrow x, x \in X)$ as Y_0. □

Let us establish one more general fact needed for the construction.

Proposition 14. *If an α-space X is a smooth subspace of a* bc-*domain B then there exists a subspace $B' \subseteq B$ such that B' is a* bc-*domain, $X \subseteq B'$ and X is $\vee$-dense in B'.*

Proof. Let

$$X_0 \rightleftharpoons \left\{ x_F \rightleftharpoons \bigvee_{x\in F}{}_B\, x \;\middle|\; F \subseteq X \text{ is a finite bounded subset in } X \right\}.$$

Note that $\bot_B = x_\emptyset \in X_0$. Check that X_0 is an α-space (as a subspace of B). Let W be open in X_0 and $x_0 \vee_B \cdots \vee_B x_n \in W$, $x_i \in X$, $i \leqslant n$; let V be open in B such that $V \cap X_0 = W$. There are open sets $V_0, \ldots, V_n$ of B such that $x_i \in V_i$, $i \leqslant n$, and if $y_i \in V_i$, $y_i \leqslant_B x_i$, $i \leqslant n$, then $y_0 \vee_B \cdots \vee_B y_n \in V$ (because $x_0 \vee_B \cdots \vee_B x_n \in V$ and because the operation $\vee_B$ restricted to $\downarrow_B (x_0 \vee_B \cdots \vee_B x_n)$ is continuous). Let $W_i \rightleftharpoons V_i \cap X$, $i \leqslant n$; $x_i \in W_i$, $i \leqslant n$. Because X is an α-space there are $y_i \in W_i$, $i \in n$, such that $y_i \prec x_i$, $i \leqslant n$; but then $y_0 \vee_B \cdots \vee_B y_n \prec_B x_0 \vee_B \cdots \vee_B x_n$ and $y_0 \vee_B \cdots \vee_B y_n \in W = V \cap X_0$. $\langle X_0, \leqslant_{X_0} \rangle$ is a partial upper semilattice; then $H_d(X_0)$ is a bc-domain and the embedding of X_0 into B can be extended to a homeomorphic embedding of $H_d(X_0)$ into B ([3, Theorem 1, Section 3]). The image of this embedding B' obviously satisfies the conclusion of the proposition. □

Combining these two propositions we will have the following:

Proposition 15. *For every α-space X there exists a homeomorphic embedding $\lambda' : X \to B'$ of X into a* bc-*domain B' such that $\lambda'(X)$ is $\vee$-dense in B' and the pair (λ', B') is universal for X.*

Proof. Let $e : X \to Y$, $p : Y \to X$ be an embedding-projection pair for some φ-space Y. By Theorem 9, there exists a bc-hull $(\beta_Y, H_{\mathrm{bc}}(Y))$ for Y. The embedding $\beta_Y e : X \to H_{\mathrm{bc}}(Y)$ is a homeomorphism onto a smooth subspace. Really, it is not hard to check that e embeds X into Y homeomorphically onto a smooth subspace of Y, because (e, p) is an embedding-projection pair (see, e.g., [1, Proposition 3.1.17]). The map β_Y embeds Y homeomorphically into $H_{\mathrm{bc}}(Y)$ onto a smooth subspace (property (1) after the proof of the Theorem 9). From that it follows that $\beta_Y e$ is a homeomorphism from X onto a smooth subspace of $H_{\mathrm{bc}}(Y)$. By Proposition 14 there is a subspace $B' \subseteq H_{\mathrm{bc}}(Y)$ such that B' is a bc-domain and $\beta_Y e(X)$ is $\vee$-dense in B'. Let us show that $(\beta_Y e, H_{\mathrm{bc}}(Y))$ is universal relative to X. Let $f : X \to \overline{B}$ be a continuous map from X into a bc-domain $\overline{B}$; then $fp : Y \to \overline{B}$ is a continuous map and by the universality of $(\beta_Y, H_{\mathrm{bc}}(Y))$ for Y there is a continuous map $(fp)^* : H_{\mathrm{bc}}(Y) \to \overline{B}$ such that $fp = (fp)^* \beta_Y$, then $f = fpe = (fp)^* \beta_Y e$ and the universality of the pair $(\beta_Y e, H_{\mathrm{bc}}(Y))$ for X holds. Now one needs only to notice that for every subspace $Z \subseteq H_{\mathrm{bc}}(Y)$ such that $\beta_Y e(X) \subseteq Z$ the pair $(\beta_Y e, Z)$ is universal for X; in particular, the pair $(\beta_Y e, B')$ is universal for X. □

Theorem 16. *For any α-space X there exists a* bc-*hull.*

Proof. By Proposition 15, there is a pair (λ', B') such that $\lambda' : X \to B'$ is a homeomorphic embedding from X into a bc-domain B', $\lambda'(X)$ is $\vee$-dense in B' and the pair (λ', B') is universal for X. Consider the family $\mathscr{F} \subseteq C(B', B')$ of the all continuous maps f from the bc-domain B' into itself such that $f \restriction \lambda'(X) = id_{\lambda'(X)}$ (or $f\lambda' = \lambda'$).

Because the space $C(B',B')$ of all continuous maps of B' into itself is a *dcpo* (directed-complete partial order) as a partial ordered set $\langle C(B',B'), \leqslant_{C(B',B')}\rangle$, then by Zorn's lemma in $\langle \mathscr{F}, \leqslant\rangle$ there is a maximal element f_0. By Lemma 10, $f \geqslant id_{B'}$ for any $f \in \mathscr{F}$; so $f_0 \geqslant id_{B'}$, $f_0^2 \geqslant f_0$ and $f_0^2\lambda' = f_0(f_0\lambda') = f_0\lambda' = \lambda'$, so $f_0^2 \in \mathscr{F}$. From the maximality of f_0, we have $f_0^2 = f_0$; f_0 is a retraction (closure operator) on B'. Let $B \rightleftharpoons f_0(B') \subseteq B'$; B is a bc-domain (by [3, Proposition 3, Section 2]), B is an A-space and as a continuous image of the complete A_0-space (= bc-domain) B', B is a complete A_0-space (= bc-domain)). From $f_0\lambda' = \lambda'$ it follows that $\lambda'(X) \subseteq B$. Because (λ', B') is universal for X, then (λ', B) is universal for X. Let us check that (λ', B) is minimal. Let $g: B \to B$ be such a continuous map that $g\lambda' = \lambda'$. By Lemma 10 (it is easy to check that $\lambda'(X)$ is $\vee$-dense in B) we have $g \geqslant id_B$. If $g > id_B$, then $f \rightleftharpoons gf_0 > f_0$ and $f\lambda' = gf_0\lambda' = g\lambda' = \lambda'$, $g \in \mathscr{F}$ and $g > f_0$. This is impossible because f_0 is a maximal element of $\mathscr{F}$. □

The construction of a bc-hull in the general case is highly nonconstructive (using a Zorn's lemma); so two natural questions are open:

1. Is Proposition 11 true for arbitrary α-spaces X?
2. Is $\beta_X : X \to H_{bc}(X)$ embedding X onto a smooth subspace of $H_{bc}(X)$?

Remark. Proposition 10 in [4] (more exactly, the second part of the proposition) is not proved, because the intended proof of it contains a gap. I am grateful to Dr. A. Jung for the indication of this gap.

Noted added in proof. After I submitted this paper Prof. K. Keimel pointed out to me that M. Erné introduced in his paper "Scott convergence and Scott topology in partially ordered sets, II" (in: B. Banaschewski and R.-E. Hoffman, eds., *Continuous lattices*, Lecture Notes Mathmatics, Vol. 871 (Springer, Berlin, 1981) 61–96) the notion of C-space which is equivalent to the notion of α-space and so Theorem 2.11a of Erné's paper is equivalent to Theorem 3 of this paper. (See also: M. Erné, The ABC of Order and Topology, in: H. Herrlich and H.-E. Porst eds., *Category Theory at Work* (Heldermann Verlag, Berlin, 1991) 57–83).

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