JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 44, 701-709 (1973)

A Generalization of the Gronwall-Bellman Lemma and Its Applications

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1. INTRODUCTION

In [5] we have characterised the solutions of the inequality

$$F(t^{a_1}x_1, \dots, t^{a_n}x_n) \leqslant t^{a_0}F(x_1, \dots, x_n) \qquad (t \geqslant 1, x_i > 0, i = 1, \dots, n) \tag{1}$$

assuming that F is a continuously differentiable positive function and $a_0, \dots, a_n$ are constants. We have found that (1) is valid if and only if

$$a_1 x_1 \frac{\partial F}{\partial x_1} + \cdots + a_n x_n \frac{\partial F}{\partial x_n} \leqslant a_0 F(x_1, \dots, x_n) \tag{2}$$

holds for all $x_i > 0$, $i = 1, \dots, n$.

The generalization of this result lead us to the following problem.

Let u, v be continuously differentiable functions defined on $[\xi, \xi + a]$ such that

$$u'(t) - f(t, u(t)) \leqslant v'(t) - f(t, v(t)) \qquad t \in [\xi, \xi + a] \tag{3}$$

and

$$u(\xi) = v(\xi) \tag{4}$$

where $f: [\xi, \xi + a] \times R \to R$ is a given continuous function, R is the set of reals. Under what conditions do (3) *and* (4) *imply*

$$u(t) \leqslant v(t) \qquad t \in [\xi, \xi + a]? \tag{5}$$

The continuity of f is surely not enough since if (3) and (4) imply (5) then the solution of the initial-value problem $y'(t) - f(t, y(t)) = 0$, $y(\xi) = \eta$ is unique on $[\xi, \xi + a]$.

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Instead of (3) and (4) we may consider the corresponding integral inequality

$$u(t) - \int_{\xi}^{t} f(s, u(s))\, ds \leqslant v(t) - \int_{\xi}^{t} f(s, v(s))\, ds \qquad t \in [\xi, \xi + a], \tag{6}$$

or even the inequality

$$u - Au \leqslant v - Bv$$

where u, v are elements of a Banach space X; A, B are operators mapping X into itself and $\leqslant$ is a partial ordering on X. Under what conditiohs does this inequality imply $u \leqslant v$?

Our first theorem, which may be regarded as an abstract generalization of the Gronwall–Bellman lemma, gives an answer to the above question (Section 2). In Section 3 we apply this theorem to get estimates for solutions of initial-value problems. Finally in Section 4 we give a necessary and sufficient condition for a function F to satisfy the inequality

$$F(k_1(t, x_1),..., k_n(t, x_n)) \leqslant k_0(t, F(x_1 ,..., x_n))$$

where the functions k_i and the variable t are subjected to certain conditions. In the special cases $k_i(t, x) = t^{a_i}x$ and $a_i t + x$ we prove the same result under weaker assumptions and generalize an inequality of [3].

2. A Generalization of the Gronwall–Bellman Lemma

Let X denote a real Banach space and let C, a subset of X, be a cone i.e. a closed convexe set such that $x \in C$, $\alpha \geqslant 0$ imply $\alpha x \in C$ and from x, $-x \in C$ it follows that $x = 0$. By help of C a partial ordering $\leqslant$ can be defined in X: for $x, y \in X$

$$x \leqslant y \qquad \text{if} \quad y - x \in C. \tag{7}$$

This partial ordering has the usual properties of the ordinary inequalities (see, e.g., [4]).

The following theorem plays an important role in our investigations.

Theorem 1. *Assume that X is a real Banach space, C is a cone in X, $\leqslant$ is the partial ordering defined by (7). Assume further that A, B are two operators (not necessarily linear) mapping X into itself such that*

(i) *$x, y \in X$, $x \leqslant y$ imply $Ax \leqslant By$ and*

(ii) *the equations*

$$\varphi = g + A\varphi \qquad \psi = h + B\psi$$

have unique solutions φ, ψ *whatever be the elements* $g, h \in X$, *and these solutions can be obtained as the limits (in norm convergence) of the sequences of the corresponding successive approximations.*

Then the inequality

$$u - Au \leqslant v - Bv \qquad u, v \in X$$

implies that

$$u \leqslant v.$$

Remark 1. The condition (i) is satisfied if $Ax \leqslant Bx$ for all $x \in X$ and A (or B) is monotone in the sense that $x, y \in X$, $x \leqslant y$ imply $Ax \leqslant Ay$.

Remark 2. The condition (ii) is fulfilled if both A and B are contractions.

Remark 3. If A and B are linear bounded operators defined on the whole X and for some natural number n A^n and B^n are contractions, then (ii) is satisfied again. Namely in this case the sequence of successive approximations of the equation $\varphi = g + A\varphi$ can be written as

$$\varphi_n = (E + A + A^2 + \cdots + A^n)g$$

($n = 1, 2, \ldots$, E is the identity operator) which converges (necessarily to the unique solution of the equation), since the spectral radius of A is

$$r(A) = \inf_k (\| A^k \|)^{1/k} \leqslant (\| A^n \|)^{1/n} < 1$$

and the same is true for the other equation.

Proof of Theorem 1. Denote by g and h the element $u - Au$ and $v - Bv$ respectively then

$$g \leqslant h \tag{8}$$

and by (ii)

$$u = \lim \varphi_n \qquad v = \lim \psi_n \tag{9}$$

where $\varphi_0 = g$, $\varphi_{n+1} = g + A\varphi_n$ ($n = 0, 1, \ldots$); $\psi_0 = h$, $\psi_{n+1} = h + B\psi_n$ ($n = 0, 1, \ldots$). We prove by induction that

$$\varphi_n \leqslant \psi_n \qquad (n = 0, 1, \ldots). \tag{10}$$

For $n = 0$ this is valid by (8). Assume (10) is true for $n = k$ then by (i) and (8)

$$\varphi_{k+1} = g + A\varphi_k \leqslant h + B\psi_k = \psi_{k+1}.$$

Letting $n \to \infty$ in (10) we obtain $u \leqslant v$ which completes the proof.

3. Estimates for Solutions of Operator-Equations

Theorem 2. *Assume that A and B_1, B_2 (in place of B) satisfy the conditions of Theorem* 1, *except that instead of* (i) *we require the validity of the inequalities*

$$B_1 x \leqslant Ax \leqslant B_2 x \qquad x \in X$$

and

$$Ax \leqslant Ay \qquad \text{if} \quad x \leqslant y; \quad x, y \in X.$$

Then the solutions v_1, v_2 of the equations

$$v_1 - B_1 v_1 = 0 \qquad v_2 - B_2 v_2 = 0$$

approximate the solution u of

$$u - Au = 0$$

in the sense that

$$v_1 \leqslant u \leqslant v_2 .$$

Theorem 3. *Assume that $A = B$ satisfy the conditions of Theorem* 1. *Then*

$$u - Au \leqslant v - Av \qquad u, v \in X$$

implies the inequality

$$u \leqslant v$$

that is the inverse operator $(E - A)^{-1}$ is monotone increasing.

The *proof* of these theorems follows immediately from Theorem 1.

Choosing in Theorem 3 the element v as the solution of $v - Av = 0$ and specializing A we can get many results obtained earlier. Instead of listing these we refer the reader to [1] where also detailed references can be found. Here we want to specialize Theorems 2, 3 only for the case of integral and differential operators.

Let f, $g_1, g_2 : [\xi, \xi + a] \times R \to R$ be continuous real-valued functions satisfying Lipschitz condition in their second variable. Assume further that f is an increasing function in its second variable.

Corollary 1. *If*

$$g_1(x, y) \leqslant f(x, y) \leqslant g_2(x, y) \qquad x \in [\xi, \xi + a], \quad y \in R \tag{11}$$

then the solution y of the initial-value problem

$$y' = f(x, y) \qquad y(\xi) = \eta$$

is approximated by the solutions y_1, y_2 *of the initial-value problems*

$$y_1 = g_1(x, y_1) \quad y_1(\xi) = \eta, \qquad y_2 = g_2(x, y_2) \quad y_2(\xi) = \eta,$$

that is

$$y_1(x) \leqslant y(x) \leqslant y_2(x) \qquad x \in [\xi, \xi + a]. \tag{12}$$

COROLLARY 2. *Under the above mentioned conditions* (*for* f) *the inequalities*

$$\begin{aligned} u'(t) - f(t, u(t)) &\leqslant v'(t) - f(t, v(t)) \qquad t \in [\xi, \xi + a] \\ u(\xi) &= v(\xi) \end{aligned} \tag{13}$$

imply

$$u(t) \leqslant v(t) \qquad t \in [\xi, \xi + a]. \tag{14}$$

The proofs are obvious if we apply Theorem 2 and 3 respectively for $X = C[\xi, \xi + a]$, the Banach space of all real-valued functions defined and continuous on $[\xi, \xi + a]$, and for the operators A, B_1, B_2 defined by

$$(A\varphi)(x) = \eta + \int_\xi^x f(t, \varphi(t))\, dt$$

$$(B_i\varphi)(x) = \eta + \int_\xi^x g_i(t, \varphi(t)) \qquad (i = 1, 2).$$

We remark that if f, g_1, g_2 are defined only on $[\xi, \xi + a] \times [\eta, \eta + b]$ then the validity of inequalities (12) and (14) can be guaranteed only on the interval $[\xi, \xi + \alpha]$, where $\alpha = \min\{a, b/M\}$ and M is a common bound for the absolute values of f, g_1, g_2. Instead of a Lipschitz condition we may use weaker assumptions as well, namely we only have to provide the uniform convergence of the sequence of successive approximations. For this see [2].

4. SUBHOMOGENEOUS FUNCTIONS

Let I, J be open intervals, k_i: $J \times I \to I$ $(i = 1,..., n)$, k_0: $J \times R \to R$ given functions. Assume that

(i) there exists a $t_0 \in J$ such that

$$k_i(t_0, x) = x \qquad x \in I (\text{if } i = 1,..., n); \qquad x \in R (\text{if } i = 0),$$

(ii) the functions k_0, k_i $(i = 1,..., n)$ are differentiable with respect to their first variable on $J \times R$ and $J \times I$ respectively,

(iii) $\qquad k_i'(t, x) = k_i'(t_0, k_i(t, x))\, h(t) \qquad (i = 0,..., n)$

holds for all possible values of t and x, where h is a continuous non-negative function on J and the prime denotes the partial derivative with respect to the first variable,

(iv) $k_0'(t_0, x)$ is a continuous increasing function satisfying Lipschitz condition:

$$| k_0'(t_0, x_1) - k_0'(t_0, x_2)| \leqslant \alpha | x_1 - x_2 | \qquad x_1, x_2 \in R$$

with constant α.

DEFINITION. A function $F: I^n \to R$ is called a *positive subhomogeneous* function with respect to the functions $k_0, \ldots, k_n$ satisfying (i)–(iv) if

$$F(k_1(t, x_1), \ldots, k_n(t, x_n)) \leqslant k_0(t, F(x_1, \ldots, x_n)) \tag{15}$$

holds for all $x = (x_1, \ldots, x_n) \in I^n$, $t \in J_x \cap [t_0, \infty)$, where

$$J_x = \{t \mid t \in J, k_i(t, x_i) \in J (i = 1, \ldots, n)\}.$$

F is called *negative subhomogeneous* with respect to $k_0, \ldots, k_n$ if (15) holds for all $x \in I^n$ and $t \in J_x \cap (-\infty, t_0]$.

The notion of *positive*, respectively, *negative superhomogeneous function* can be defined analogously changing the sign $\leqslant$ to $\geqslant$ in (15).

Of course these definitions have sense even if the functions $k_0, \ldots, k_n$ satisfy only condition (i), but our theorems shall be true only under the assumptions (i)–(iv).

THEOREM 4. *Let $F: I^n \to R$ be a continuously differentiable function on I^n. F is positive subhomogeneous with respect to $k_0, \ldots, k_n$ (satisfying (i)–(iv)!) that is*

$$F(k_1(t, x_1), \ldots, k_n(t, x_n)) \leqslant k_0(t, F(x_1, \ldots, x_n)) \tag{15}$$

holds for all $x \in I^n$, $t \in J_x \cap [t_0, \infty)$ if and only if

$$k_1'(t_0, x_1) \frac{\partial F(x)}{\partial x_1} + \cdots + k_n'(t_0, x_n) \frac{\partial F(x)}{\partial x_n} \leqslant k_0'(t_0, F(x)), \qquad x \in I^n. \tag{16}$$

Proof. *Necessity.* Let $x \in I^n$ be a fixed vector and denote by $u(t)$ and $v(t)$ the left and right side of (15) respectively. Then $u(t_0) = v(t_0)$, thus (15) may be written as

$$(u(t) - u(t_0))/(t - t_0) \leqslant (v(t) - v(t_0))/(t - t_0), \qquad t \in J_x \cap (t_0, \infty).$$

Letting $t \to t_0 + 0$ we have

$$u'(t_0) \leqslant v'(t_0)$$

which is identical to (16).

Sufficiency. Put $k_i(t, x_i)$ instead of x_i in (16) and multiply the obtained inequality by $h(t)$, the function occured in (iii). Using the property (iii) we get

$$\sum_{i=1}^{n} \frac{\partial F(k(t, x))}{\partial x_i} k_i'(t, x_i) \leqslant k_0'(t, F(k(t, x)) h(t)$$

where $x \in I^n$, $t \in J_x$ and $k(t, x) = (k_1(t, x_1),..., k_n(t, x_n))$. Hence

$$u'(t) - v'(t) \leqslant k_0'(t_0, F(k(t, x))) h(t) - k_0'(t, F(x))$$

since $v'(t) = k_0'(t, F(x))$. By (iii) (used for $i = 0$)

$$u'(t) - k_0'(t_0, u(t)) h(t) \leqslant v'(t) - k_0'(t_0, v(t))h(t), \qquad t \in J_x. \tag{17}$$

Applying Corollary 2 we get

$$u(t) \leqslant v(t), \qquad t \in J_x \cap [t_0, \infty) \tag{18}$$

which was to be proved.

We remark that Theorem 4 remains in force if we write $\geqslant$ instead of $\leqslant$ both in (15) and (16). Changing only the condition $t \in J_x \cap [t_0, \infty)$ into $t \in J_x \cap (-\infty, t_0]$ the inequality sign in (16) will change. This implies that if (15) is satisfied for all $x \in I^n$, $t \in J_x$ then (16) holds with equality sign thus (15) can hold also with equality sign.

In the special cases $k_i(t, x) = t^{a_i}x$ $(i = 0,..., n)$, $J = R^+ = (0, \infty)$ and $k_i(t, x) = a_i t + x$ $(i = 0,..., n)$, $J = R$ we can obtain stronger result then Theorem 4 (see also [5] Theorems 1, 2).

Let $F: I^n \to R$ be a (totally) differentiable function on I^n. The inequality

$$F(t^{a_1}x_1, ..., t^{a_n}x_n) \leqslant t^{a_0}F(x_1, ..., x_n), \qquad x \in I^n, \quad t \in R_x^+ \cap [1, \infty)$$

is equivalent to

$$a_1x_1(\partial F(x)/\partial x_1) + \cdots + a_nx_n(\partial F(x)/\partial x_n) \leqslant a_0F(x), \qquad x \in I^n.$$

Similarly, the inequality

$$F(a_1t + x_1, ..., a_nt + x_n) \leqslant a_0t + F(x_1, ..., x_n), \qquad x \in I^n, \quad t \in R_x \cap [0, \infty)$$

is equivalent to

$$a_1(\partial F(x)/\partial x_1) + \cdots + a_n(\partial F(x)/(\partial x_n) \leqslant a_0, \qquad x \in I^n.$$

The *proof* is the same as that of Theorem 4 except the implication

(17) → (18). The continuity of the partial derivatives were used only in this step. In the first case the inequality corresponding to (17) has the form

$$u'(t) - (a_0/t)\, u(t) \leqslant v'(t) - (a_0/t)\, v(t).$$

After a multiplication by t^{-a_0} this can be written as

$$(d/dt)\,(t^{-a_0}u(t)) \leqslant (d/dt)\,(t^{-a_0}v(t))$$

from which

$$u(t) \leqslant v(t), \qquad t \in R_x^+ \cap [1, \infty)$$

since $u(1) = v(1)$.

In the second case (17) has the form

$$u'(t) - a_0 \leqslant v'(t) - a_0$$

which obviously implies $u(t) \leqslant v(t)$ for $t \in R_x \cap [0, \infty)$ since $u(0) = v(0)$.

Let $n \geqslant 2$ be a fixed natural number and denote by W_n the set of all vectors $p = (p_1, ..., p_n)$ having the properties $p_i \geqslant 0$ $(i = 1, ..., n)$, $\sum_{i=1}^{n} p_i = 1$. If $x = (x_1, ..., x_n)$ then tx and $t + x$ denote the vectors $(tx_1, ..., tx_n)$ and $(t + x_1, ..., t + x_n)$ respectively.

Applying the above results to the function,

$$F(x) = F_p(x) = \Phi\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i \Phi(x_i)$$

where $\Phi: I \to R$ is a differentiable function on the open interval I, $p \in W_n$, $x \in I^n$ we get further interesting inequalities.

Corollary 3. *The inequality*

$$F_p(tx) \leqslant F_p(x)$$

is true for all $x \in I^n$, $p \in W_n$ and $t \geqslant 1$ with $tx \in I^n$ if and only if the function Ψ defined by $\Psi(x) = x\Phi'(x)$ is a convex function on I.

Corollary 4. *In order that the inequality*

$$F_p(t + x) \leqslant F_p(x) \tag{19}$$

holds for all $x \in I^n$, $p \in W_n$ and $t \geqslant 0$ with $t + x \in I^n$ it is necessary and sufficient that Φ', the derivative of Φ, be a convex function on I.

This is a generalization of Theorem 4 of [3]. There it was proved that (19) is true if Φ is a concave and Φ' is a convex function on $I(=R^+)$.

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