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A Variational-like Inequality for Multifunctions with Applications

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We study a variational-like inequality problem for a multifunction. A certain relevant complementarity problem is defined. We establish some existence theorems for the above two problems. An application to convex mathematical programming is also shown. © 1987 Academic Press, Inc.

1. INTRODUCTION

Given a set S in R^n and a multifunction (i.e., a point-to-set map) V from R^n into itself, the generalized variational inequality (see, e.g., [1]) is to find $\bar{x} \in S$, $\bar{y} \in V(\bar{x})$ such that

$$\langle \bar{y}, x - \bar{x} \rangle \geqslant 0 \qquad \text{for all} \quad x \in S. \tag{1}$$

We introduce an extension of (1) as follows: Let S and C be subsets of R^n and R^p, respectively. Given two maps $M: S \times C \to R^n$ and $\eta: S \times S \to R^n$, and a point-to-set map $V: S \to C$, find $\bar{x} \in S$, $\bar{y} \in V(\bar{x})$ such that

$$\langle M(\bar{x}, \bar{y}), \eta(x, \bar{x}) \rangle \geqslant 0 \qquad \text{for all} \quad x \in S. \tag{2}$$

We call it the generalized variational-like inequality problem. If $\eta(x, \bar{x}) = x - \bar{x}$, then (2) reduces to the problem of finding $\bar{x} \in S$, $\bar{y} \in V(\bar{x})$ such that

$$\langle M(\bar{x}, \bar{y}), x - \bar{x} \rangle \geqslant 0 \qquad \text{for all} \quad x \in S, \tag{2'}$$

which is a special case of (1).

Let K be a closed convex cone in R^n. The polar cone K^* of K is the set $\{\xi \in R^n: \langle \xi, x \rangle \geqslant 0 \text{ for each } x \in K\}$. Let C be a closed convex subset of R^p. Given a map $M: K \times C - R^n$ and a point-to-set map $V: K \to C$, the

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generalized complementarity problem related to (2′) is to find $x \in R^n$, $y \in V(x)$ such that

$$x \in K, \qquad M(x, y) \in K^*, \qquad \langle M(x, y), x \rangle = 0. \tag{3}$$

If $M(x, y) = y$, then (3) becomes the generalized complementarity problem studied by Saigal [2]. Problem (3) is the mathematical form for a variety of problems in mathematical programming, game theory, economics, mechanics, etc., hence its importance.

Our first main result, Theorem 2, is an existence theorem for (2). As a consequence of Theorem 2 we obtain, in Section 2, several other existence theorems for (2) and (2′). Section 3 is devoted to establishing some fairly general existence results for (3). Finally, in Section 4, we apply some existence results of Section 2 to the existence of solutions to two specific problems: (a) a nonlinear programming problem (P), and (b) a saddle point problem (SPP), both associated with a function $L(x, y)$ defined for $(x, y) \in X \times D$, where X and D are sets in R^n and R^p, respectively.

(P): $\mathrm{Min}_{(x,y) \in U} L(x, y)$, where

$$U = \{(x, y) : x \in X,\ y \in D,\ L(x, y) = \max_{v \in D} L(x, v)\}.$$

(SPP): Find $x^* \in X$, $y^* \in D$ such that

$$L(x^*, y) \leqslant L(x^*, y^*) \leqslant L(x, y^*)$$

for all $x \in X$ and $y \in D$.

2. A Variational-Like Inequality

Given a point-to-set map V from R^n into itself, V is said to be upper semicontinuous if $\{x^n\}$ converging to x, and $\{y^n\}$, with $y^n \in V(x^n)$, converging to y, implies $y \in V(x)$. For a set C in R^p, we denote by $P(C)$ the collection of all compact convex subsets of C.

Theorem 1. *Let S be a compact convex set in R^n, and C a closed convex set in R^p. Let $V: S \to P(C)$ be upper semicontinuous and $\varphi: S \times C \times S \to R$ continuous. Suppose that*

(i) *$\varphi(x, y, x) \geqslant 0$ for each $x \in S$,*

(ii) *for each fixed $(x, y) \in S \times C$, $\varphi(x, y, u)$ is quasiconvex in $u \in S$.*

Then there exist $\bar{x} \in S$, $\bar{y} \in V(\bar{x})$ such that

$$\varphi(\bar{x}, \bar{y}, x) \geqslant 0 \qquad \textit{for all} \quad x \in S$$

Proof. For each $(x, y) \in R^n \times R^p$, let

$$\pi(x, y) = \{s \in S: \varphi(x, y, s) = \min_{u \in S} \varphi(x, y, u)\}.$$

Since φ is quasiconvex in u, $\pi(x, y)$ is a convex set. It is easy to show that π is an upper semicontinuous map. By [3, Theorem 3, p. 110], the set $V(S) = U_{x \in S} V(x)$ is a compact subset of C. Hence the convex hull of $V(S)$, denoted by H, is a compact convex set. Consequently, the point-to-set map F: $S \times H \to S \times H$, defined by $F(x, y) = (\pi(x, y), V(x))$, is nonempty, convex valued and upper semicontinuous. Now invoking Kakutani fixed-point theorem [4], we get $(\bar{x}, \bar{y}) \in F(\bar{x}, \bar{y})$. Hence, $\bar{y} \in V(\bar{x})$ and $\bar{x} \in \pi(\bar{x}, \bar{y})$, or for each $x \in S$, $\varphi(\bar{x}, \bar{y}, x) \geqslant \varphi(\bar{x}, \bar{y}, \bar{x}) \geqslant 0$. This completes the proof of the theorem.

Theorem 1 gives immediately the first existence theorem for (2).

THEOREM 2. *Let S, C, V be as in Theorem* 1, *and let the maps M: $S \times C \to R^n$ and η: $S \times S \to R^n$ be continuous. Suppose that*

(i) $\eta(x, x) = 0$ *for each* $x \in S$,

(ii) *for each fixed* $(x, y) \in S \times C$, *the function*

$$\langle M(x, y), \eta(u, x) \rangle \quad \text{is quasiconvex in } u \in S.$$

Then there exist $\bar{x} \in S$, $\bar{y} \in V(\bar{x})$ *such that*

$$\langle M(\bar{x}, \bar{y}), \eta(x, \bar{x}) \rangle \geqslant 0 \quad \text{for all} \quad x \in S.$$

Proof. The proof follows directly from Theorem 1 by setting $\varphi(x, y, u) = \langle M(x, y), \eta(u, x) \rangle$.

COROLLARY 1. *Let S, C, V, M be as in Theorem* 2. *Then there exist* $\bar{x} \in S$, $\bar{y} \in V(\bar{x})$ *which satisfy* (2').

Proof. Define $\eta(u, x) = u - x$ for all u, $x \in S$, and apply Theorem 2.

The following result is an extension of Theorem 2 to noncompact sets.

THEOREM 3. *Let S and C be closed convex sets in R^n and R^p, respectively, and let V, M, η be as in Theorem* 2. *Suppose that*

(i) $\eta(x, x) = 0$ *for each* $x \in S$,

(ii) *for each fixed* $(x, y) \in S \times C$, *the function*

$$\langle M(x, y), \eta(u, x) \rangle \quad \text{is convex in} \quad u \in S.$$

If there is a $\bar{u} \in S$ *and a constant* $r > \|\bar{u}\|$ *such that*

$$\max_{y \in V(x)} \langle M(x, y), \eta(\bar{u}, x) \rangle \leqslant 0 \tag{4}$$

for all $x \in S$ *with* $\|x\| = r$, *then there exists a solution to* (2).

Proof. Let $S_r = \{x \in S\colon \|x\| \leqslant r\}$. Since S_r is compact and convex, by Theorem 2 we have $\bar{x} \in S_r$, $\bar{y} \in V(\bar{x})$ such that

$$\langle M(\bar{x}, \bar{y}), \eta(x, \bar{x}) \rangle \geqslant 0 \qquad \text{for all} \quad x \in S_r. \tag{5}$$

We distinguish two cases.

Case 1. $\|\bar{x}\| = r$. Since $\bar{u} \in S_r$, it follows from (4) and (5) that $\langle M(\bar{x}, \bar{y}), \eta(\bar{u}, \bar{x}) \rangle = 0$. Now, let $x \in S$ and choose $0 < \lambda < 1$ small enough so that $w = \lambda x + (1 - \lambda)\bar{u}$ lies in S_r. Then, by the convexity of $\langle M(\bar{x}, \bar{y}), \eta(u, \bar{x}) \rangle$,

$$\begin{aligned} 0 &\leqslant \langle M(\bar{x}, \bar{y}), \eta(w, \bar{x}) \rangle \\ &\leqslant \lambda \langle M(\bar{x}, \bar{y}), \eta(x, \bar{x}) \rangle + (1 - \lambda) \langle M(\bar{x}, \bar{y}), \eta(\bar{u}, \bar{x}) \rangle \\ &= \lambda (M(\bar{x}, \bar{y}), \eta(x, \bar{x}) \rangle. \end{aligned}$$

and consequently, $(\bar{x}, \bar{y})$ satisfies (2).

Case 2. $\|\bar{x}\| < r$. Given $x \in S$, again we can choose $0 < \lambda < 1$ small enough so that $w' = \lambda x + (1 - \lambda)\bar{x}$ lies in S_r. Then, by proceeding as in Case 1, it can be shown that $(\bar{x}, \bar{y})$ satisfies (2).

The following corollary, obtained as a consequence of Theorem 3, generalizes Theorem 2.3 of More' [5] to multifunctions.

COROLLARY 2. *Let S, C, V, M be as in Theorem* 3. *If there is a* $\bar{u} \in S$ *and a constant* $r > \|\bar{u}\|$ *such that*

$$\max_{y \in V(x)} \langle M(x, y), \bar{u} - x \rangle \leqslant 0 \tag{6}$$

for each $x \in S$ *with* $\|x\| = r$, *then there exists a solution to* (2′).

We now introduce the following generalization of monotone functions. Let $\eta\colon S \times S \to R^n$ be such that $\eta(x, x) = 0$ for each $x \in S$. Then, a multifunction $V\colon S \to R^n$ is said to be η-monotone if

$$\langle y, \eta(u, x) \rangle + \langle v, \eta(x, u) \rangle \leqslant 0$$

whenever $y \in V(x)$, $v \in V(u)$. Note that this definition reduces to the usual definition of monotone functions [6] if $\eta(u, x) = u - x$.

The following gives an existence result for (2) under η-monotonicity of the map M.

THEOREM 4. *Let S, C, V, M, η be as in Theorem 3, and let the map $x \mapsto \{M(x, y): y \in V(x)\}$ be η-monotone on S. If there exist $\bar{u} \in S$, $\bar{v} \in V(\bar{u})$ such that*

$$\lim_{\substack{\|x\| \to \infty \\ x \in S}} \langle M(\bar{u}, \bar{v}), \eta(x, \bar{u}) \rangle > 0, \tag{7}$$

then (2) *has a solution.*

Proof. Hypothesis (7) implies that there is an $r > \|\bar{u}\|$ such that $\langle M(\bar{u}, \bar{v}), \eta(x, \bar{u}) \rangle > 0$ for all $x \in S$ with $\|x\| = r$. Hence, for any such x, it follows from the η-monotonicity of M that

$$\langle M(x, y), \eta(\bar{u}, x) \rangle \leqslant -\langle M(\bar{u}, \bar{v}), \eta(x, \bar{u}) \rangle < 0$$

for all $y \in V(x)$. This shows that condition (4) of Theorem 3 is satisfied. Therefore, (2) has a solution.

THEOREM 5. *Let S be a pointed closed cone in R^n, and let C, V, M be as in Theorem 3. Let the map $x \mapsto \{M(x, y): y \in V(x)\}$ be monotone on S. If there is a $\bar{u} \in S$ and a $\bar{v} \in V(\bar{u})$ such that $M(\bar{u}, \bar{v}) \in \operatorname{int} S^*$, then* (2′) *has a solution.*

Proof. Since $M(\bar{u}, \bar{v}) \in \operatorname{int} S^*$, the set $D = \{x \in S: \langle M(\bar{u}, \bar{v}), x - \bar{u} \rangle \leqslant 0\}$ is compact, and consequently, for all $x \in S \backslash D$, $\langle M(\bar{u}, \bar{v}), x - \bar{u} \rangle > 0$. This implies that there is a $r > \|\bar{u}\|$ such that $\langle M(\bar{u}, \bar{v}), x - \bar{u} \rangle > 0$ for all $x \in S$ with $\|x\| = r$. Then it follows from the monotonicity of M that

$$\langle M(x, y), x - \bar{u} \rangle \geqslant \langle M(\bar{u}, \bar{v}), x - \bar{u} \rangle > 0$$

for all $y \in V(x)$ whenever $x \in \{x \in S: \|x\| = r\}$. Hence we can conclude that condition (6) of Corollary 2 is satisfied, and the proof is complete.

3. GENERALIZED COMPLEMENTARITY PROBLEM

It is known (see, e.g., [7]) that if $S = K$, then the solution sets of (2′) and (3) are the same. Corollary 2 now stated in the terminology of the complementarity problem (3), gives

THEOREM 6. *Let $M: K \times C \to R^n$ be continuous, and $V: K \to P(C)$ upper semicontinuous. If there is a $\bar{u} \in K$ and a constant $r > \|\bar{u}\|$ such that*

$$\max_{y \in V(x)} \langle M(x, y), \bar{u} - x \rangle \leqslant 0$$

for each $x \in K$ with $\|x\| = r$, then (3) has a solution.

Consider a special form of $M(x, y)$ given by $M(x, y) = F(x) + y$, where $F: K \to R^n$. Then (3) assumes the form: Find $x \in R^n$, $y \in V(x)$ such that

$$x \in K, \qquad F(x) + y \in K^*, \qquad \langle F(x) + y, x \rangle = 0. \tag{3'}$$

It is noted that the Kuhn–Tucker stationary point problem for a number of nondifferentiable mathematical programming problems such as those studied in [8, 9] can be projected into the form of (3′).

Our next result, Theorem 7, is on the existence of a solution to (3′).

THEOREM 7. *Let K be pointed, and $d \in \operatorname{int} K^*$. Let C be a compact convex set in R^p. Let $F: K \to R^n$ be continuous, and $V: K \to P(C)$ upper semicontinuous. If $G(x) = F(x) - F(0)$ is positively homogeneous of some degree β and the system*

$$\begin{aligned} &G(x) + td \in K^*, \qquad 0 \neq x \in K, \\ &\langle G(x) + td, x \rangle = 0, \qquad t \geqslant 0 \end{aligned} \tag{8}$$

is inconsistent, then (3′) has a solution.

Proof. For any real $\alpha > 0$, the set $K_\alpha = \{x \in K: d^T x \leqslant \alpha\}$ is nonempty, compact and convex, and $F(x) + y$ is continuous on $K_\alpha \times C$. By Corollary 1 there exist $x^\alpha \in K_\alpha$, $y^\alpha \in V(x^\alpha)$ such that

$$\langle F(x^\alpha) + y^\alpha, x^\alpha \rangle = \min_{x \in K_\alpha} \langle F(x^\alpha) + y^\alpha, x \rangle,$$

and applying the duality theory of linear programming, we get a scalar ξ^α such that

$$F(x^\alpha) + y^\alpha + \xi^\alpha d \in K^*, \tag{9}$$

$$\langle F(x^\alpha) + y^\alpha + \xi^\alpha d, x^\alpha \rangle = 0, \tag{10}$$

$$\xi^\alpha \geqslant 0, \qquad (\alpha - d^T x^\alpha)\, \xi^\alpha = 0. \tag{11}$$

If $\xi^\alpha = 0$ for some α, then (x^α, y^α) constitutes a solution to (3′). We establish by contradiction that there is at least one α with $\xi^\alpha = 0$. Assume, thus, that $\xi^\alpha > 0$ for each $0 < \alpha < \infty$. By (10) we have $d^T x^\alpha = \alpha$. Let $u^\alpha = x^\alpha/\alpha$, and then $d^T u^\alpha = 1$ and $u^\alpha \in K$. Now, the set of points (u^α, y^α) is in the compact set $\{u \in K: d^T u = 1\} \times C$, and therefore, there is a convergent sequence of (u^α, y^α) with $\alpha \to \infty$ which converges to some (u, y) in the compact set. Let this sequence be one with $\alpha \in \{\alpha_i\}_{i \in \Gamma}$, and be denoted by $\{(u^i, y^i)\}_{i \in \Gamma}$, where Γ is an index set. Since G is positively homogeneous of degree β, we have from (10) and (9), respectively,

$$0 \geqslant -(\alpha_i)^{-\beta}\xi^i = \langle G(u^i), u^i \rangle + (\alpha_i)^{-\beta} \langle F(0) + y^i, u^i \rangle,$$

$$G(u^i) + (\alpha_i)^{-\beta}(F(0) + y^i) + (\alpha_i)^{-\beta}\xi^i d \in K^*$$

for all $i \in \Gamma$. By taking the limit, we obtain $0 \geqslant \langle G(u), u \rangle = -t$ (say) and $G(u) + td \in K^*$. Since $d^T u = 1$, we also have $\langle G(u) + td, u \rangle = 0$. This implies that (u, t) is a solution to the system (8), which contradicts the assumption of the theorem. Thus the theorem follows.

Theorem 7 extends a result of Karamardian [10, Theorem 3.1]. In [10], the hypothesis concerning the regularity of the function involved is similar to our hypothesis requiring the inconsistency of the system (8).

4. Applications

In [11], Hanson introduced a class of differentiable functions which contains as a subclass the class of differentiable convex functions. As a particular case, we consider the following class of functions: Let $\psi: S \to R^n$ be differentiable. Then ψ is η-convex if there exists a continuous map $\eta: S \times S \to R^n$ such that

(i) $\eta(x, x) = 0$ for each $x \in S$,

(ii) $\psi(x) - \psi(u) \geqslant \langle \nabla\psi(u), \eta(x, u) \rangle$ for all $x, u \in S$.

Note that the examples given in [11, p. 547] are η-convex functions. It is known that if ψ is convex on S, then $\nabla\psi$ is monotone on S. In the same vein, we have here that $\nabla\psi$ is η-monotone whenever ψ is η-convex.

Now, we associate with (SPP) the following variational-like inequality problem: Find $x^* \in X$, $y^* \in Y(x^*)$ such that

$$\langle \nabla_x L(x^*, y^*), \eta(x, x^*) \rangle \geqslant 0 \qquad \text{for all} \quad x \in X, \tag{12}$$

where

$$Y(x^*) = \{ y \in D: L(x^*, y) = \max_{v \in D} L(x^*, v) \}.$$

It is easily seen that if $L(x, y)$ is η-convex in $x \in X$ for every fixed $y \in D$, and if (x^*, y^*) is a solution of (12), then (x^*, y^*) is also a solution of (SPP). Further, it follows from a lemma of Mangasarian and Ponstein [12, Lemma 3.4] that any solution (x^*, y^*) of (SPP) is an optimal solution of

(P). Hence, the question of the existence of solutions to (P) and (SPP) can be investigated via (12). Consequently, we have

THEOREM 8. *Let X be a closed convex set in R^n, and D a compact convex set in R^p. Suppose that $L(x, y)$ is η-convex in $x \in X$ for every fixed $y \in D$, and concave in $y \in D$ for every fixed $x \in X$. Suppose also that for each fixed $(x, y) \in X \times D$, $\langle \nabla_x L(x, y), \eta(u, x) \rangle$ is convex in $u \in X$. If there exist $\bar{u} \in X$, $\bar{v} \in Y(\bar{u})$ such that*

$$\lim_{\substack{\|x\| \to \infty \\ x \in X}} \langle \nabla_x L(\bar{u}, \bar{v}), \eta(x, \bar{u}) \rangle > 0,$$

then (SPP) *has a solution, and hence, so does* (P).

Proof. Since D is compact and convex, and $L(x, y)$ is concave in $y \in D$, $Y(x)$ is compact and convex. It can be easily checked that the map Y is upper semicontinuous From the η-convexity of L, we have

$$L(x, y) - L(u, v) \geqslant L(x, v) - L(u, v) \geqslant \langle \nabla_x L(u, v), \eta(x, u) \rangle,$$

$$L(u, v) - L(x, y) \geqslant L(u, y) - L(x, y) \geqslant \langle \nabla_x L(x, y), \eta(u, x) \rangle,$$

for every $y \in Y(x)$ and $v \in Y(u)$, from which it follows that the map $x \mapsto \{\nabla_x L(x, y) \colon y \in Y(x)\}$ is η-monotone on X. Hence, by Theorem 4, (12) has a solution, and consequently, (SPP) and (P) have solutions.

By strengthening the convexity requirements on L, we obtain the next existence result.

THEOREM 9. *Let X, D be as in Theorem* 8, *and let $L(x, y)$ be convex–concave on $X \times D$. If there is a $\bar{u} \in X$ and a $\bar{v} \in Y(\bar{u})$ such that*

$$\lim_{\substack{\|x\| \to \infty \\ x \in X}} \langle \nabla_x L(\bar{u}, \bar{v}), x - \bar{u} \rangle > 0,$$

then there exists a solution to (SPP), *and hence an optimal solution to* (P).

Proof. This is the special case of Theorem 8 in which $\eta(x, u) = x - u$ for all $x, u \in X$.

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