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# Logical systems for structured specifications ☆

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**Abstract**

We study proof systems for reasoning about logical consequences and refinement of structured specifications, based on similar systems proposed earlier in the literature (Inform. and Comput. 76 (1988) 165; in: F.L. Bauer, W. Brauer, H. Schwichtenberg (Eds.), Logic and Algebra of Specification, NATO ASI Series F: Computer and Systems Sciences, vol. 94, Springer, Berlin, 1991, p. 411). Following Goguen and Burstall, the notion of an underlying logical system over which we build specifications is formalized as an institution and extended to a more general notion, called $(\mathcal{D}, \mathcal{T})$-institution. We show that under simple assumptions (essentially: amalgamation and interpolation) the proposed proof systems are sound and complete. The completeness proofs are inspired by proofs due to Cengarle (Ph.D. Thesis, Institut für Informatik, Ludwig-Maximilians-Universität Müenchen, 1994) for specifications in first-order logic and the logical systems for reasoning about them. We then propose a methodology for reusing proof systems built over institutions rich enough to satisfy the properties required for the completeness results for specifications built over poorer institutions where these properties need not hold.


*Keywords:* Algebraic specifications; Formal languages; Formal semantics; Interpolation property; Logical systems; Specification languages

## 1. Introduction

During the process of software specification and development, we often have to use various logical systems to capture different aspects of software systems and programming paradigms. Each part of a software system may be described by different logical systems that best suit considered problems. The first task is to present a formal concept of a logical system which covers the population of logical systems used in practice. This problem was considered by Goguen and Burstall [14]:

> … because of the proliferation of logics of programming and logic-based programming languages, plus the great expense of implementing tools like theorem provers

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and compilers, it is useful to know when sentences in one logic can be translated into sentences into another logic in such a way that soundness is preserved. … Institutions provide a foundations for approaching these and many other problems in computer science.

Following the above ideas, we formalize the notion of a logical system as an institution. We attempt to work independently from the institution chosen, providing ideas and results that work in an arbitrary institution.

In this paper we consider formal systems for reasoning about logical consequences and refinement of structural specifications built over an arbitrary logical system formalized as institution (see [8, 27, 32] for similar systems). Most of the results, presented in this paper, are based on the results presented in [4, 5]. In the first part of the paper we extend the notion of underlying logical system, formalized as institution, to $(\mathscr{D},\mathscr{T})$-institution, where the classes of morphisms $\mathscr{D}$ and $\mathscr{T}$ are classes of morphisms allowed to be used in the restriction and, respectively, translation of specifications. Next, we show that formal systems for reasoning about logical consequences and refinement of structured specifications are sound and complete for any $(\mathscr{D},\mathscr{T})$-institution satisfying basic closure, amalgamation and interpolation properties. This generalizes to an arbitrary $(\mathscr{D},\mathscr{T})$-institution the results of Cengarle [8] on completeness of similar systems for specifications in first-order logic. At the end of this part we demonstrate that the interpolation property is crucial for completeness.

The underlying logic which is most appropriate in a given context, is not always strong enough to satisfy the conditions that ensure completeness of logical systems mentioned above. In the second part of the paper we use institution representations (see [20, 30]) to embed institutions that may be too weak to ensure completeness of logical systems for reasoning about structured specifications built over them into richer institutions for which completeness holds. We also formulate conditions (essentially: $\rho$-expansion and weak-$\mathscr{D}$-amalgamation, see Sections 5 and 8) under which a complete and sound proof system for reasoning about logical consequences and refinement of structural specifications in a richer institution can be reused for a sound proof system for reasoning about logical consequences and refinement of structural specifications in the represented institution. To obtain this result, inspired by similar results on the theory level presented in [17], we use the notion of the *institution representation* to define the *specification representation* and prove similar results as in [17] but for the model part of representations. In the concluding section we extend our results to a more general case of *maps of institutions* (see [20]).

Problems presented in Sections 6 and 8 were also studied in [1] (also for the case of structured specifications). The results presented there are similar to results presented in Sections 7 and 8 but for the case of *flat specifications*. Similar results as presented in Sections 7 and 8 were also presented in [9, 30] for the case of specifications without structure. Our results extend them to structured specifications.

Concluding, we demonstrate in a few examples how to use the proposed reusing methodology in practice and argue that both assumptions under which the reusing methodology works are really crucial. We also show “how” the results presented in this paper are more general than these presented in [17] and compare them with similar results presented in other papers.

## 2. Definitions

While developing a specification system independently of the underlying logical system, it is necessary to formalize an abstract mathematical concept of what a logical system is. Our choice of an abstract formalization depends on what we mean by a logical system. Following [14] in the model-theoretic tradition of logic:

> One of the most essential elements of a logical system is its relationship of *satisfaction* between its *syntax* (i.e. its sentences) and its *semantics* (i.e. models)...

Based on this principle, the notion of a logical system is formalized as a mathematical object called *institution* in [14].

An institution consists of a collection of signatures, together with a set of $\Sigma$-sentences and a collection of $\Sigma$-models for each signature $\Sigma$, and a satisfaction relation between $\Sigma$-models and $\Sigma$-sentences. The only requirement is that when we change signatures (by signature morphisms), the induced translations of sentences and models preserve the satisfaction relation. That last requirement, called also *satisfaction condition* (see Definition 2.1 below), means that: “*Truth is invariant under change of notation*”.

**Definition 2.1** (*Institution* [14]). An institution $I$ consists of

- a category $\mathbf{Sign}_I$ of signatures;
- a functor $\mathbf{Sen}_I : \mathbf{Sign}_I \to \mathbf{Set}$, giving a set $\mathbf{Sen}_I(\Sigma)$ of $\Sigma$-sentences for each signature $\Sigma \in |\mathbf{Sign}_I|$;
- a functor $\mathbf{Mod}_I : \mathbf{Sign}_I^{op} \to \mathbf{DCat}$, [1] giving a category $\mathbf{Mod}_I(\Sigma)$ of $\Sigma$-models for each signature $\Sigma \in |\mathbf{Sign}_I|$;
- for each $\Sigma \in |\mathbf{Sign}_I|$, a satisfaction relation $\models^I_\Sigma \subseteq |\mathbf{Mod}_I(\Sigma)| \times \mathbf{Sen}_I(\Sigma)$ such that for any signature morphism $\sigma : \Sigma \to \Sigma'$, $\Sigma$-sentence $\varphi \in \mathbf{Sen}_I(\Sigma)$ and $\Sigma'$-model $M' \in |\mathbf{Mod}_I(\Sigma')|$:

$$M' \models^I_{\Sigma'} \mathbf{Sen}_I(\sigma)\ (\varphi) \quad \text{iff} \quad \mathbf{Mod}_I(\sigma)(M') \models^I_\Sigma \varphi \quad \text{(Satisfaction condition)}$$

Examples of various logical systems viewed as institutions can be found in [14] then, we recall a few examples used later in the paper. The first two were presented also in [30]:

**Example 2.2** (*The institution* **EQ** *of equational logic*). Signatures are the usual many-sorted algebraic signatures; sentences are (universally quantified) equations with

[1] **DCat** is the category of all discrete categories. For simplicity, we disregard in this paper morphisms between models. Hence, classes of models, rather than model categories are considered.

translations along a signature morphism essentially by replacing the operation names as indicated by the signature morphism; models are many-sorted algebras with reducts along a signature morphism defined in the usual way; and satisfaction relations are given as the usual satisfaction of an equation in an algebra.

**Example 2.3** (*The institution* **FOEQ** *of first-order logic with equality*). Signatures are first-order many-sorted signatures (with sort names, operation names and predicate names); sentences are the usual closed formulae of first-order logic built over atomic formulae given either as equalities or atomic predicate formulae; models are the usual first-order structures; satisfaction of a formula in a structure is defined in the standard way.

**Example 2.4** (*The institution* **PEQ** *of partial equational logic*). Signatures are (as in **EQ**) many-sorted algebraic signatures; sentences are (universally quantified) equations and definedness formulae with translations along a signature morphism defined similarly as in institution **EQ**; models are partial many-sorted algebras with reducts along a signature morphism defined in the usual way; and satisfaction relations are defined as the satisfaction of an equation[2] and a definedness formula in a partial many-sorted algebra.

In the next two definitions we define what it means that an institution has a certain minimal logical structure.

**Definition 2.5.** We say that an institution $I$ *has conjunction* if for every signature $\Sigma \in |\mathbf{Sign}_I|$ and finite set of $\Sigma$-sentences $\{\varphi_i\}_{i\in\mathscr{I}} \subseteq \mathbf{Sen}_I(\Sigma)$ there exists a $\Sigma$-sentence, which we denote by $\bigwedge_{i\in\mathscr{I}} \varphi_i$, such that for every $\Sigma$-model $M \in |\mathbf{Mod}_I(\Sigma)|$:

$$M \models^I_\Sigma \bigwedge_{i\in\mathscr{I}} \varphi_i \quad \text{iff for every } i \in \mathscr{I}\ M \models^I_\Sigma \varphi_i.$$

We can similarly define what it means that an institution "has infinite conjunction":

**Definition 2.6.** We say that an institution $I$ *has infinite conjunction* if for every signature $\Sigma \in |\mathbf{Sign}_I|$ and set of $\Sigma$-sentences $\{\varphi_i\}_{i\in\mathscr{I}} \subseteq \mathbf{Sen}_I(\Sigma)$, where $\mathscr{I}$ is a (possibly infinite) set of indices, there exists a $\Sigma$-sentence, which we denote by $\bigwedge_{i\in\mathscr{I}} \varphi_i$, such that for every $\Sigma$-model $M \in |\mathbf{Mod}_I(\Sigma)|$:

$$M \models^I_\Sigma \bigwedge_{i\in\mathscr{I}} \varphi_i \quad \text{iff for every } i \in \mathscr{I}\ M \models^I_\Sigma \varphi_i.$$

Obviously, if an institution has infinite conjunction, then it has conjunction as well.

[2] The satisfaction of an equation is strong, i.e. the equation $t_1 = t_2$ holds if $t_1$ and $t_2$ are either both undefined or both defined and equal.

**Definition 2.7.** We say that an institution $I$ *has negation* if for every signature $\Sigma \in |\mathbf{Sign}_I|$ and $\Sigma$-sentence $\varphi \in \mathbf{Sen}_I(\Sigma)$ there exists a $\Sigma$-sentence, which we denote by $\neg\varphi$, such that for every $\Sigma$-model $M \in |\mathbf{Mod}_I(\Sigma)|$:

$$M \models^I_\Sigma \neg\varphi \quad \text{iff it is not true that } M \models^I_\Sigma \varphi.$$

**Definition 2.8.** We say that an institution $I$ *has implication* if for every signature $\Sigma \in |\mathbf{Sign}_I|$ and $\Sigma$-sentences $\varphi_1, \varphi_2 \in \mathbf{Sen}_I(\Sigma)$ there exists a $\Sigma$-sentence, which we denote by $\varphi_1 \Rightarrow \varphi_2$, such that for every $\Sigma$-model $M \in |\mathbf{Mod}_I(\Sigma)|$:

$$M \models^I_\Sigma \varphi_1 \Rightarrow \varphi_2 \quad \text{iff when } M \models^I_\Sigma \varphi_1 \quad \text{then } M \models^I_\Sigma \varphi_2.$$

**Fact 2.9.** *If an institution I has conjunction and negation then it also has implication.*

In the rest of the paper the following abbreviations are used:

- for any set of sentences $\Gamma \subseteq \mathbf{Sen}_I(\Sigma)$ and $M \in |\mathbf{Mod}_I(\Sigma)|$ we define $M \models^I_\Sigma \Gamma$ as an abbreviation for "*for every sentence* $\varphi \in \Gamma$: $M \models^I_\Sigma \varphi$", and similarly for every class of models $\mathscr{M} \subseteq |\mathbf{Mod}_I(\Sigma)|$ and sentence $\varphi \in \mathbf{Sen}_I(\Sigma)$ we define $\mathscr{M} \models^I_\Sigma \varphi$ as an abbreviation for "*for every model* $M \in \mathscr{M}$: $M \models^I_\Sigma \varphi$";
- for any sentences $\varphi, \psi \in \mathbf{Sen}_I(\Sigma)$ we define $\varphi \models^I_\Sigma \psi$ as an abbreviation for "*for every model* $M \in |\mathbf{Mod}_I(\Sigma)|$, $M \models^I_\Sigma \psi$ *whenever* $M \models^I_\Sigma \varphi$", similarly $\Gamma \models^I_\Sigma \varphi$, for any set of sentences $\Gamma \subseteq \mathbf{Sen}_I(\Sigma)$, as an abbreviation for "*for every model* $M \in |\mathbf{Mod}_I(\Sigma)|$, $M \models^I_\Sigma \varphi$ *whenever* $M \models^I_\Sigma \Gamma$", and also $\varphi \models^I_\Sigma \Gamma_1$ and $\Gamma \models^I_\Sigma \Gamma_1$ for $\Gamma, \Gamma_1 \subseteq \mathbf{Sen}_I(\Sigma)$ as abbreviations for "*for every sentence* $\psi \in \Gamma_1$, $\varphi \models^I_\Sigma \psi$" and "*for every sentence* $\psi \in \Gamma_1$, $\Gamma \models^I_\Sigma \psi$";
- if an institution $I$ has conjunction then for any sentences $\varphi_1, \varphi_2 \in \mathbf{Sen}_I(\Sigma)$ we define $\varphi_1 \wedge \varphi_2$ as an abbreviation for the sentence $\bigwedge_{i \in \{1,2\}} \varphi_i$;
- the following abbreviations will be used: $\sigma\varphi$ for $\mathbf{Sen}_I(\sigma)(\varphi)$, $M|_\sigma$ for $\mathbf{Mod}_I(\sigma)(M)$ and $\models$ for $\models^I_\Sigma$ when it is clear what they mean;
- for any set of sentences $\Gamma \subseteq \mathbf{Sen}_I(\Sigma)$ we write $\bigwedge \Gamma$ as an abbreviation for $\bigwedge_{i \in \mathscr{I}} \varphi_i$ where $\Gamma = \{\varphi_i \mid i \in \mathscr{I}\}$, similarly we write $\bigwedge \sigma\Gamma$ for $\bigwedge_{i \in \mathscr{I}} \sigma\varphi_i$.

**Fact 2.10** (Deduction). *For any institution I that has conjunction and implication*, $\Sigma \in |\mathbf{Sign}_I|$ *and sentences* $\varphi_1, \varphi_2, \varphi_3 \in \mathbf{Sen}_I(\Sigma)$, *we have*

$$\varphi_1 \wedge \varphi_2 \models \varphi_3 \quad \textit{iff} \quad \varphi_1 \models \varphi_2 \Rightarrow \varphi_3.$$

**Proof.** Directly from the definition. □

The above fact shows that "semantic" deduction is a property of institutions having conjunction and implication. For instance, institution **FOEQ**, presented in Example 2.3, satisfies these conditions.

The notion of an institution as introduced in Definition 2.1 covers the model-theoretic view of a logical system. Although semantic aspect of a logical system is crucial for our purposes (see Section 3), it is also important to be able to prove properties of a logical system. Therefore, a more *proof-theoretic* view of a logical system is important as well (see also [20] and Chap. 4 of [18] for argumentation).

**Definition 2.11** (*Entailment relation*). For any institution $I$ and signature $\Sigma \in |\mathbf{Sign}_I|$, an *entailment relation on the set* $\mathbf{Sen}_I(\Sigma)$ *of sentences* is a relation $\vdash^I_\Sigma \subseteq \mathscr{P}(\mathbf{Sen}_I(\Sigma)) \times \mathbf{Sen}_I(\Sigma)$ such that

(*Reflexivity*) $\{\varphi\} \vdash^I_\Sigma \varphi$;
(*Transitivity*) if $\Gamma_i \vdash^I_\Sigma \varphi_i$ for $i \in \mathscr{I}$ and $\Gamma \cup \{\varphi_i\}_{i \in \mathscr{I}} \vdash^I_\Sigma \psi$, then $\Gamma \cup \bigcup_{i \in \mathscr{I}} \Gamma_i \vdash^I_\Sigma \psi$; and
(*Weakening*) if $\Gamma \vdash^I_\Sigma \psi$, then $\Gamma \cup \Gamma' \vdash^I_\Sigma \psi$;

where $\mathscr{P}(\mathbf{Sen}_I(\Sigma))$ is the power set of $\mathbf{Sen}_I(\Sigma)$, $\mathscr{I}$ is a set of indices, $\varphi, \psi, \varphi_i \in \mathbf{Sen}_I(\Sigma)$ are sentences and $\Gamma, \Gamma', \Gamma_i \subseteq \mathbf{Sen}_I(\Sigma)$ are sets of sentences, for $i \in \mathscr{I}$.

We also say that the entailment relation $\vdash^I_\Sigma$ is *sound* wrt satisfaction relation $\models_{\Sigma^I}$, if for every $\Gamma \subseteq \mathbf{Sen}_I(\Sigma)$ and $\varphi \in \mathbf{Sen}_I(\Sigma)$

$$\Gamma \vdash^I_\Sigma \varphi \quad \text{implies} \quad \Gamma \models_{\Sigma^I} \varphi.$$

If the converse holds then the entailment relation $\vdash^I_\Sigma$ is called *complete* for the satisfaction relation $\models_{\Sigma^I}$.

A family of entailment relations $\{\vdash^I_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, denoted by $\vdash^I$, is called a *proof system for institution I*, if for every $\Sigma \in |\mathbf{Sign}|$, $\vdash^I_\Sigma$ is sound wrt the satisfaction relation $\models^I_\Sigma$.

**Definition 2.12** (*Entailment system* [17,20]). For a given institution $I$, $\mathscr{E} = (\mathbf{Sign}_I, \mathbf{Sen}_I, \vdash^I)$ is called an *entailment system for I*, if $\vdash^I$ is a proof system for the institution $I$ and is *stable under translation*, i.e. if for every $\Sigma, \Sigma' \in |\mathbf{Sign}_I|$, $\Gamma \in \mathbf{Sen}_I(\Sigma)$, $\psi \in \mathbf{Sen}_I(\Sigma)$ and $(\sigma : \Sigma \to \Sigma') \in \mathbf{Sign}_I$:

$$\text{if} \quad \Gamma \vdash^I_\Sigma \psi \quad \text{then } \sigma\Gamma \vdash^I_{\Sigma'} \sigma\psi.$$

In the specification formalisms such as presented in [8, 12, 27, 26, 32] and also in this paper (see Definition 3.1), signature morphisms are used at least in two ways:
1. to hide some symbols in the signature of the (target) specification and
2. to add and/or rename some symbols in the (source) signature.

According to this observation, in each institution $I$ we distinguish two classes of signature morphisms:
1. a class $\mathscr{D}_I$ of the signature morphisms considered appropriate for hiding symbols and
2. a class $\mathscr{T}_I$ for adding and renaming symbols.

For instance, many specification formalisms, based on the usual signatures, limit the classes $\mathscr{D}_I$, implicitly involved in their definition, to injective or even inclusive signature morphisms only, and the class $\mathscr{T}_I$ to injective morphisms.

The above observations, plus some technical conditions, are formally expressed by the following definition.

**Definition 2.13** (*$(\mathscr{D},\mathscr{T})$-institution*). Let $\mathscr{D}_I, \mathscr{T}_I \subseteq \mathbf{Sign}_I$ be classes of signature morphisms in an institution $I$. We say that the institution $I$ with distinguished $\mathscr{D}_I$ and $\mathscr{T}_I$ is $(\mathscr{D},\mathscr{T})$-*institution* iff:

- classes $\mathscr{D}_I$ and $\mathscr{T}_I$ are closed under composition and include all identities;
- for every $(d : \Sigma \to \Sigma_1) \in \mathscr{D}_I$ and $(t : \Sigma \to \Sigma_2) \in \mathscr{T}_I$ there exist $(t' : \Sigma_1 \to \Sigma') \in \mathscr{T}_I$ and $(d' : \Sigma_2 \to \Sigma') \in \mathscr{D}_I$ such that the following diagram is a pushout in $\mathbf{Sign}_I$:

$$\begin{tikzcd}
 & \Sigma \arrow[dl, "d\in\mathcal{D}_I"'] \arrow[dr, "t\in\mathcal{T}_I"] & \\
\Sigma_1 \arrow[dr, "t'\in\mathcal{T}_I"'] & & \Sigma_2 \arrow[dl, "d'\in\mathcal{D}_I"] \\
 & \Sigma' &
\end{tikzcd}$$

The above definitions put some limitation on the signature part of "usual" institutions. For a given institution $I$ not all choices of $\mathscr{D}_I$ and $\mathscr{T}_I$ are appropriate. For example:

**Example 2.14.** Let us consider any institution $I$ where $\mathbf{Sign}_I$ is the category of algebraic signatures with derived morphisms $\mathbf{AlgSig}^{\mathrm{der}}$ (see [25]) and let both classes of morphisms $\mathscr{D}_I$ and $\mathscr{T}_I$ contain all the morphisms from $\mathbf{AlgSig}^{\mathrm{der}}$. Then the pushout from Definition 2.13 does not exists in general because the category $\mathbf{AlgSig}^{\mathrm{der}}$ does not have all pushouts. On the other hand, when for instance $\mathscr{D}_I$ is the class of inclusions and $\mathscr{T}_I$ of all derived morphisms then the required pushouts exist.

A positive example could be any institution $I$ with (finitely) cocomplete category of signatures $\mathbf{Sign}_I$ and $\mathscr{D}_I = \mathscr{T}_I = \mathbf{Sign}$, e.g. the category of algebraic signatures $\mathbf{AlgSig}$ is such a category.

In the rest of this section we define properties of a logical system formalized as $(\mathscr{D},\mathscr{T})$-institution, which are used in the completeness theorem (see Theorem 3.9 and also [8]). The first property is the interpolation property. The following definition of the $(\mathscr{D},\mathscr{T})$-interpolation property is inspired by the formalization of Craig Interpolation Theorem presented in [29].

**Definition 2.15** (*$(\mathscr{D},\mathscr{T})$-interpolation*). A $(\mathscr{D},\mathscr{T})$-institution $I$ satisfies the $(\mathscr{D},\mathscr{T})$-*interpolation property* iff for any $d, d' \in \mathscr{D}_I$ and $t, t' \in \mathscr{T}_I$ that form a pushout in $\mathbf{Sign}_I$ (as in Definition 2.13) and $\varphi_i \in \mathbf{Sen}_I(\Sigma_i)$ for $i = 1, 2$, if

$$\mathbf{Sen}_I(t')(\varphi_1) \models^I_{\Sigma'} \mathbf{Sen}_I(d')(\varphi_2)$$

then there exists $\varphi \in \mathbf{Sen}_I(\Sigma)$, called $(\mathscr{D},\mathscr{T})$-interpolant of $\varphi_1$ and $\varphi_2$, such that

$$\varphi_1 \models^I_{\Sigma_1} \mathbf{Sen}_I(d)(\varphi) \quad \text{and} \quad \mathbf{Sen}_I(t)(\varphi) \models^I_{\Sigma_2} \varphi_2.$$

In the above definition we can weaken the requirement of existence of $(\mathscr{D},\mathscr{T})$-interpolant to the existence of a set of $(\mathscr{D},\mathscr{T})$-interpolants. Then we obtain:

**Definition 2.16** (*Weak-$(\mathscr{D},\mathscr{T})$-interpolation*). A $(\mathscr{D},\mathscr{T})$-institution $I$ satisfies the *weak-$(\mathscr{D},\mathscr{T})$-interpolation property* iff for any $d,d' \in \mathscr{D}_I$ and $t,t' \in \mathscr{T}_I$ that form a pushout in $\mathbf{Sign}_I$ (as in Definition 2.13) and $\varphi_i \in \mathbf{Sen}_I(\Sigma_i)$ for $i=1,2$, if

$$\mathbf{Sen}_I(t')(\varphi_1) \models^I_{\Sigma'} \mathbf{Sen}_I(d')(\varphi_2)$$

then there exists $\Gamma \subseteq \mathbf{Sen}_I(\Sigma)$ such that

$$\varphi_1 \models^I_{\Sigma_1} \mathbf{Sen}_I(d)(\Gamma) \quad \text{and} \quad \mathbf{Sen}_I(t)(\Gamma) \models^I_{\Sigma_2} \varphi_2.$$

A characterization of above interpolation properties in terms of a *module algebra* can be found in [3] and also in [11].

**Lemma 2.17.** *If the $(\mathscr{D},\mathscr{T})$-institution $I$ has infinite conjunction and satisfies the weak-$(\mathscr{D},\mathscr{T})$-interpolation property then it also satisfies $(\mathscr{D},\mathscr{T})$-interpolation property.*

**Example 2.18.** The $(\mathscr{D},\mathscr{T})$-institution **EQ** where $\mathscr{D}_{\mathbf{EQ}}$ is the class of signature inclusions and $\mathscr{T}_{\mathbf{EQ}}$ is the class of signature injections satisfies the weak-$(\mathscr{D},\mathscr{T})$-interpolation property (see [23]) but not the $(\mathscr{D},\mathscr{T})$-interpolation property, whereas the $(\mathscr{D},\mathscr{T})$-institution **FOEQ** where $\mathscr{D}_{\mathbf{FOEQ}}$ is the class of signature inclusions and $\mathscr{T}_{\mathbf{FOEQ}}$ is the class of signature injections satisfies both the interpolation properties. The above facts follow from the arguments presented in [3].

**Remark 2.19.** It follows from Example 2.18 and Lemma 2.17 that the $(\mathscr{D},\mathscr{T})$-institution **EQ** from the above example does not have infinite conjunction in the sense of Definition 2.6 (which is obvious anyway).

**Definition 2.20** (*Compactness*). The institution $I$ is *compact* iff for any $\Sigma$-sentence $\varphi \in \mathbf{Sen}_I(\Sigma)$ and any set of $\Sigma$-sentences $\Gamma \subseteq \mathbf{Sen}_I(\Sigma)$, if $\Gamma \models \varphi$ then there exists a finite set $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models \varphi$.

The next property is inspired by the well-known amalgamation property.

**Definition 2.21** (*Weak-$(\mathscr{D},\mathscr{T})$-amalgamation*). A $(\mathscr{D},\mathscr{T})$-institution $I$ satisfies the *weak-$(\mathscr{D},\mathscr{T})$-amalgamation* property iff for any $d,d' \in \mathscr{D}_I$ and $t,t' \in \mathscr{T}_I$ that form a pushout in $\mathbf{Sign}_I$ (as in Definition 2.13) and for any $M_1 \in \mathbf{Mod}_I(\Sigma_1)$ and $M_2 \in \mathbf{Mod}_I(\Sigma_2)$, if $M_1|_d = M_2|_t$, then there exists a model $M' \in \mathbf{Mod}_I(\Sigma')$ uch that $M'|_{t'} = M_1$ and $M'|_{d'} = M_2$.

**Assumption 2.22.** Through Sections 3 and 4 we will work with an arbitrary but fixed $(\mathscr{D},\mathscr{T})$-institution $I$ that has conjunction and negation and for which $\mathscr{D}_I \subseteq \mathscr{T}_I$.

## 3. Specifications

From now on we will work with specifications similar to specifications defined in [27].

As in [27] we assume that software systems, described by specifications, are adequately represented by models of institutions. This means that a specification must describe a signature and a class of models over this signature called the *class of models of the specification*. For any specification $SP$ we denote its signature by $\mathbf{Sig}[SP]$ and the collection of its models by $\mathbf{Mod}[SP]$; we have $\mathbf{Sig}[SP] \in |\mathbf{Sign}_I|$ and $\mathbf{Mod}[SP] \subseteq |\mathbf{Mod}(\mathbf{Sig}[SP])|$. If $\mathbf{Sig}[SP] = \Sigma$ we will call $SP$ a $\Sigma$-specification, and we denote the class of $\Sigma$-specifications by $\mathbf{Spec}_\Sigma$.

**Definition 3.1** (*Specifications*). Specifications over a $(\mathscr{D},\mathscr{T})$-institution $I$ and their semantics are defined inductively as follows:

1. Any pair $\langle \Sigma, \Gamma \rangle$, where $\Sigma \in \mathbf{Sign}_I$ and $\Gamma \subseteq \mathbf{Sen}_I(\Sigma)$, is a specification, called also *flat specification* or *presentation*, with the following semantics:
   $\mathbf{Sig}[\langle \Sigma, \Gamma \rangle] = \Sigma$;
   $\mathbf{Mod}[\langle \Sigma, \Gamma \rangle] = \{M \in |\mathbf{Mod}_I(\Sigma)| \mid M \models^I_\Sigma \Gamma\}$.
2. For any signature $\Sigma$ and $\Sigma$-specifications $SP_1$ and $SP_2$, $SP_1 \cup SP_2$ is a specification with the following semantics:
   $\mathbf{Sig}[SP_1 \cup SP_2] = \Sigma$;
   $\mathbf{Mod}[SP_1 \cup SP_2] = \mathbf{Mod}[SP_1] \cap \mathbf{Mod}[SP_2]$.
3. For any morphism $(t : \Sigma \to \Sigma') \in \mathscr{T}_I$ and $\Sigma$-specification $SP$, **translate** $SP$ **by** $t$ is a specification with the following semantics:
   $\mathbf{Sig}[\mathbf{translate}\ SP\ \mathbf{by}\ t] = \Sigma'$;
   $\mathbf{Mod}[\mathbf{translate}\ SP\ \mathbf{by}\ t] = \{M' \in |\mathbf{Mod}_I(\Sigma')| \mid M'|_t \in \mathbf{Mod}[SP]\}$.
4. For any morphism $(d : \Sigma \to \Sigma') \in \mathscr{D}_I$ and $\Sigma'$-specification $SP'$,
   **derive from** $SP'$ **by** $d$ is a specification with the following semantics:
   $\mathbf{Sig}[\mathbf{derive\ from}\ SP'\ \mathbf{by}\ d] = \Sigma$;
   $\mathbf{Mod}[\mathbf{derive\ from}\ SP'\ \mathbf{by}\ d] = \{M'|_d \mid M' \in \mathbf{Mod}[SP']\}$.

The above definition introduces a number of operations on specifications (union, translate, derive) called *specification building operations* or SBOs for short. The above SBOs semantically refer to certain functions on classes of models and provide some flexible mechanism for expressing basic ways of putting specifications together in a structured manner.

**Definition 3.2.** Specifications $SP_1$ and $SP_2$ are *equivalent* (written $SP_1 \cong SP_2$) if

$$\mathbf{Sig}[SP_1] = \mathbf{Sig}[SP_2] \quad \text{and} \quad \mathbf{Mod}[SP_1] = \mathbf{Mod}[SP_2].$$

In the above definition we use equality of signatures. We can also use signature equivalences defined separately for each category of signatures (as a certain class of isomorphisms) without any influence on the results presented in the rest of the paper.

**Definition 3.3** (*Semantic consequence*). A $\Sigma$-sentence $\varphi$ is a *semantic consequence* of a $\Sigma$-specification $SP$ (written $SP \models_\Sigma \varphi$) if $\mathbf{Mod}[SP] \models^I_\Sigma \varphi$.

Each $\Sigma$-sentence $\varphi$ that is a semantic consequence of a $\Sigma$-specification $SP$ is called a *theorem* of $SP$.

The above definition gives us a model-theoretic view of logical consequences of specifications. Although it is the most fundamental concept in this paper, it is also crucial to be able to prove properties of specifications from its definitions. This proof-theoretic view is given by the following definition:

**Definition 3.4.** For a given $(\mathscr{D},\mathscr{T})$-institution $I$ the *family of entailment relations*[3] $\vdash_\Sigma \subseteq \mathbf{Spec}_\Sigma \times \mathbf{Sen}(\Sigma)$ for $\Sigma \in |\mathbf{Sign}_I|$, parametrized by the entailment system $(\mathbf{Sign}_I, \mathbf{Sen}_I, \vdash^I)$ for $I$, is defined by the following set of rules:

$$\text{(CR)}\ \frac{\{SP \vdash_\Sigma \varphi_i\}_{i\in\mathscr{I}} \quad \{\varphi_i\}_{i\in\mathscr{I}} \vdash^I_\Sigma \varphi}{SP \vdash_\Sigma \varphi} \qquad \text{(basic)}\ \frac{\varphi \in \Gamma}{\langle\Sigma\rangle\Gamma \vdash_\Sigma \varphi}$$

$$\text{(sum1)}\ \frac{SP_1 \vdash_\Sigma \varphi}{SP_1 \cup SP_2 \vdash_\Sigma \varphi} \qquad \text{(sum2)}\ \frac{SP_2 \vdash_\Sigma \varphi}{SP_1 \cup SP_2 \vdash_\Sigma \varphi}$$

$$\text{(trans)}\ \frac{SP \vdash_\Sigma \varphi}{\mathbf{translate}\ SP\ \mathbf{by}\ t \vdash_{\Sigma'} t\varphi} \qquad \text{(derive)}\ \frac{SP' \vdash_{\Sigma'} d\varphi}{\mathbf{derive\ from}\ SP'\ \mathbf{by}\ d \vdash_\Sigma \varphi}$$

where $(t : \Sigma \to \Sigma') \in \mathscr{T}_I$ and $(d : \Sigma \to \Sigma') \in \mathscr{D}_I$.

The set of rules presented in the above definition yields a *compositional proof system*: it allows one to perform proofs of theorems of a given specification $SP$ according to the structure of $SP$. The above structured proof system is parameterized (see rule (CR)) by the proof system for the underlying institution. The main difference between the above set of rules and those presented in [27] are rules (trans) and (derive). In [27] morphisms occurring in (trans) and (derive) rules (and in corresponding SBOs) can be any signature morphisms, whereas in the rules presented above morphisms are restricted to fixed classes of morphisms: $\mathscr{T}_I$ for the rule (trans) and $\mathscr{D}_I$ for the rule (derive).

Let us notice that all the SBOs presented in [8] can be expressed by the generic SBOs presented in this section. Moreover the proof rules presented in [8] can be derived from the rules presented above.

One of the aims of this paper is to study mutual relations between the semantic consequence relation and the entailment relation, especially soundness and completeness.

[3] Entailment relations defined in this definition are not the entailment relations in the sense of Definition 2.11.

**Definition 3.5** (*Soundness and completeness*). For any $(\mathscr{D},\mathscr{T})$-institution $I$ and signature $\Sigma \in |\mathbf{Sign}|$, we say that the entailment relation $\vdash_\Sigma \; \subseteq \mathbf{Spec}_\Sigma \times \mathbf{Sen}(\Sigma)$ is *sound* wrt the semantic consequence relation $\models_\Sigma \subseteq \mathbf{Spec}_\Sigma \times \mathbf{Sen}(\Sigma)$, if for any $\Sigma$-specification $SP$ over $(\mathscr{D},\mathscr{T})$-institution $I$ and $\Sigma$-sentence $\varphi$:

$$SP \vdash_\Sigma \varphi \quad \text{implies} \quad SP \models_\Sigma \varphi.$$

We also say that the entailment relation $\vdash_\Sigma$ is *complete*, if

$$SP \models_\Sigma \varphi \quad \text{implies} \quad SP \vdash_\Sigma \varphi.$$

The entailment relation defined by Definition 3.4 is sound wrt the semantic consequence relation defined by Definition 3.3 (provided $\vdash^I$ is so). The proof follows directly from semantics of SBOs presented in Definition 3.1 (see also proof of soundness presented in [27]).

Now, to prove completeness of the entailment relation $\vdash_\Sigma$ we need some more notions. The first is the notion of a normal form of a given specification. A similar definition was presented in [8] (cf. also [3]).

**Definition 3.6** (*Normal form*). We say that the specification $SP$ over $(\mathscr{D},\mathscr{T})$-institution $I$ is in *the normal form* if it has a form

$$\textbf{derive from } \langle \Sigma, \Gamma \rangle \textbf{ by } d,$$

where $(d : \mathbf{Sig}[SP] \to \Sigma) \in \mathscr{D}_I$ and $\Gamma \subseteq \mathbf{Sen}_I(\Sigma)$.

The following definition introduces an operation **nf** that for every specification $SP$ gives the specification $\mathbf{nf}(SP)$ that is in the normal form and is equivalent to $SP$ in the sense of Definition 3.2.

**Definition 3.7** (*nf operation*). **nf** operation on specifications build over $(\mathscr{D},\mathscr{T})$-institution $I$ is defined as follows:

1. If $SP$ is a specification of the form $\langle \Sigma, \Gamma \rangle$, then
   $\mathbf{nf}(SP) = \textbf{derive from } \langle \Sigma, \Gamma \rangle \textbf{ by } id_\Sigma$.
2. If $SP$ is a specification of the form $SP_1 \cup SP_2$, then
   $\mathbf{nf}(SP) = \textbf{derive from } \langle \Sigma', t_1' \Gamma_2 \cup d_2' \Gamma_1 \rangle \textbf{ by } d$, where

$$\begin{tikzcd}
\Sigma \arrow[r, "d_1"] \arrow[d, "d_2"'] \arrow[dr, "d"] & \Sigma_1 \arrow[d, "d_2'"] \\
\Sigma_2 \arrow[r, "t_1'"'] & \Sigma'
\end{tikzcd}$$

$\mathbf{nf}(SP_i) = \mathbf{derive\ from}\ \langle \Sigma_i, \Gamma_{-i}\rangle\ \mathbf{by}\ d_i$ for $i = 1,2$, $d = d_1; d_2' = d_2; t_1'$ and $\Sigma', t_1' \in \mathscr{T}_I$ and $d_2' \in \mathscr{D}_I$ are given by a pushout in $\mathbf{Sign}_I$:

3. If $SP$ is a specification of the form **translate** $SP_1$ **by** $t$, then
$\mathbf{nf}(SP) = \mathbf{derive\ from}\ \langle \Sigma', t'\Gamma_1\rangle\ \mathbf{by}\ d_1'$, where
$\mathbf{nf}(SP_1) = \mathbf{derive\ from}\ \langle \Sigma_1, \Gamma_1\rangle\ \mathbf{by}\ d_1$ and $\Sigma'$, $t' \in \mathscr{T}_I$ and $d_1' \in \mathscr{D}_I$ are given by a pushout in $\mathbf{Sign}_I$:

$$\begin{tikzcd}
\mathbf{Sig}[SP_1] \arrow[r, "d_1"] \arrow[d, "t"'] & \Sigma_1 \arrow[d, "t'"] \\
\mathbf{Sig}[SP] \arrow[r, "d_1'"'] & \Sigma'
\end{tikzcd}$$

4. If $SP$ is a specification of the form **derive from** $SP_1$ **by** $d$, then
$\mathbf{nf}(SP) = \mathbf{derive\ from}\ \langle \Sigma_1, \Gamma_1\rangle\ \mathbf{by}\ (d; d_1)$, where
$\mathbf{nf}(SP_1) = \mathbf{derive\ from}\ \langle \Sigma_1, \Gamma_1\rangle\ \mathbf{by}\ d_1$.

In Definition 3.1 we introduce structured specifications using sets and specific language constructions. It is also possible to introduce structured specifications as diagrams in a suitable category. Then the construction presented in the above definition can be considered as the colimit of a proper diagram.

**Theorem 3.8.** *For any specification SP build over* $(\mathscr{D}, \mathscr{T})$*-institution I satisfying the weak-*$(\mathscr{D}, \mathscr{T})$*-amalgamation property, we have*

$$\mathbf{nf}(SP) \cong SP.$$

**Proof.** By induction on the structure of $SP$. The signature part of the equivalence, $\mathbf{Sig}[\mathbf{nf}(SP)] = \mathbf{Sig}[SP]$, follows directly from Definition 3.7 (recall that $\mathscr{D}_I \subseteq \mathscr{T}_I$, and both are closed under composition).

Proof of the model part, $\mathbf{Mod}[\mathbf{nf}(SP)] = \mathbf{Mod}[SP]$ (notation as in Definition 3.7).

1. If $SP$ is a specification of the form $\langle \Sigma, \Gamma\rangle$: This case is obvious, since the reduct along identity is the identity.
2. If $SP$ is a specification of the form $SP_1 \cup SP_2$ : $\subseteq$—Let $M \in \mathbf{Mod}[\mathbf{nf}(SP)]$. Then there exists a model $M' \in \mathbf{Mod}[\langle \Sigma', t_1'\Gamma_2 \cup d_2'\Gamma_1\rangle]$ such that $M'|_d = M$. It means that:

$$M' \models t_1'\Gamma_2 \quad \text{and} \quad M' \models d_2'\Gamma_1$$

which by the satisfaction condition is equivalent to

$$M'|_{t_1'} \models \Gamma_2 \quad \text{and} \quad M'|_{d_2'} \models \Gamma_1.$$

By Definitions 3.1 and 3.7 we have

$$M'|_{d_2;t_1'} \in \mathbf{Mod}[\mathbf{nf}(SP_2)] \quad \text{and} \quad M'|_{d_1;d_2'} \in \mathbf{Mod}[\mathbf{nf}(SP_1)].$$

By the induction hypothesis and because $M'|_{d_2;t_1'} = M'|_d = M$ and similarly $M'|_{d_1;d_2'} = M'|_d = M$:

$$M \in \mathbf{Mod}[SP_1] \cap \mathbf{Mod}[SP_2] = \mathbf{Mod}[SP].$$

$\supseteq$—Let $M \in \mathbf{Mod}[SP]$. Then $M \in \mathbf{Mod}[SP_1]$ and $M \in \mathbf{Mod}[SP_2]$ which by the induction hypothesis gives: $M \in \mathbf{Mod}[\mathbf{nf}(SP_1)]$ and $M \in \mathbf{Mod}[\mathbf{nf}(SP_2)]$. By the definitions there exist models $M_i \in \mathbf{Mod}[\langle \Sigma_i, \Gamma_i \rangle]$ for $i = 1, 2$ such that:

$$M_i|_{d_i} = M \quad \text{and} \quad M_i \models \Gamma_i \quad \text{for } i = 1, 2.$$

By the weak-$(\mathscr{D}, \mathscr{T})$-amalgamation property there exists a model $M' \in \mathbf{Mod}(\Sigma')$ such that $M'|_{d_2'} = M_1$ and $M'|_{t_1'} = M_2$. Now we have:

$$M'|_{d_2'} \models \Gamma_1 \quad \text{and} \quad M'|_{t_1'} \models \Gamma_2$$

which by the satisfaction condition and Definition 3.1 is equivalent to: $M' \in \mathbf{Mod}[\langle \Sigma', d_2'\Gamma_1 \cup t_1'\Gamma_2 \rangle]$, and so, because $M = M'|_{d_1;d_2'} = M'|_d$, we have: $M \in \mathbf{Mod}[\mathbf{nf}(SP)]$.

3. If $SP$ is a specification of the form **translate** $SP_1$ **by** $t$: $\subseteq$—Let $M \in \mathbf{Mod}[\mathbf{nf}(SP)]$. Then by definitions there exists a model $M' \in |\mathbf{Mod}(\Sigma')|$ such that $M' \models t'\Gamma_1$ and $M'|_{d_1'} = M$. By the satisfaction condition we obtain: $M'|_{t'} \models \Gamma_1$ and then: $M'|_{d_1;t'} \in \mathbf{Mod}[\mathbf{nf}(SP_1)]$. Now, by the induction hypothesis and because $M'|_{d_1;t'} = M'|_{t;d_1'} = M|_t$, we have that $M|_t \in \mathbf{Mod}[SP_1]$ and by Definition 3.1, $M \in \mathbf{Mod}[SP]$.
$\supseteq$—Let $M \in \mathbf{Mod}[SP]$. Then $M|_t \in \mathbf{Mod}[SP_1]$ and by the induction hypothesis $M|_t \in \mathbf{Mod}[\mathbf{nf}(SP_1)]$. There exists a model $M_1 \in |\mathbf{Mod}(\Sigma_1)|$ such that $M_1 \models \Gamma_1$ and $M_1|_{d_1} = M|_t$ (see Definition 3.7). By the weak-$(\mathscr{D}, \mathscr{T})$-amalgamation property there exists a model $M' \in |\mathbf{Mod}(\Sigma')|$ such that: $M'|_{d_1'} = M$ and $M'|_{t'} = M_1$. Now we have $M'|_{t'} \models \Gamma_1$ and so, by the satisfaction condition $M' \models t'\Gamma_1$ and $M'|_{d_1'} \in \mathbf{Mod}[\mathbf{nf}(SP)]$, which is equivalent to $M \in \mathbf{Mod}[\mathbf{nf}(SP)]$.
4. If $SP$ is a specification of the form **derive from** $SP_1$ **by** $d$: $\subseteq$—Let $M \in \mathbf{Mod}[\mathbf{nf}(SP)]$. Then by definitions there exists a model $M_1 \in |\mathbf{Mod}(\Sigma_1)|$ such that $M_1 \models \Gamma_1$ and $M_1|_{d;d_1} = M$. Now, by the induction hypothesis: $M_1|_{d_1} \in \mathbf{Mod}[SP_1]$ which by Definition 3.1 means that $M \in \mathbf{Mod}[SP]$.
$\supseteq$—Let $M \in \mathbf{Mod}[SP]$. Then there exists $M_1 \in \mathbf{Mod}[SP_1]$ such that $M_1|_d = M$. By the induction hypothesis: $M_1 \in \mathbf{Mod}[\mathbf{nf}(SP_1)]$. Now there exists $M_2 \in |\mathbf{Mod}(\Sigma_1)|$ such that $M_2|_{d_1} = M_1$ and because $M_2|_{d;d_1} = M$ we have: $M \in \mathbf{Mod}[\mathbf{nf}(SP)]$. □

The above theorem is very important from our point of view and its proof is crucial for understanding of the rules presented above and then of the proof of their completeness. It allows us to replace any specification by its appropriate normal form, for which some basic properties are more easily visible.

**Theorem 3.9** (Completeness). *Let $I$ be a $(\mathscr{D},\mathscr{T})$-institution that has infinite conjunction and implication. If*

1. *institution $I$ satisfies $(\mathscr{D},\mathscr{T})$-interpolation and weak-$(\mathscr{D},\mathscr{T})$-amalgamation properties, and*
2. *entailment relations $\vdash^I$ used in rule (CR) are complete for $\models^I$,*
   *then for any $\Sigma$-specification SP over $(\mathscr{D},\mathscr{T})$-institution $I$ and any $\Sigma$-sentence $\varphi$,*

$$SP \models_\Sigma \varphi \quad \textit{implies} \quad SP \vdash_\Sigma \varphi.$$

**Proof.** By induction on the structure of $SP$.

1. If $SP$ is a specification of the form $\langle \Sigma, \Gamma\rangle$, then

$$\langle \Sigma, \Gamma\rangle \models_\Sigma \varphi \quad \text{iff} \quad \mathbf{Mod}[\langle \Sigma, \Gamma\rangle] \models^I_\Sigma \varphi \quad \text{iff} \quad \Gamma \models^I_\Sigma \varphi$$

   and this, by assumption 2, is equivalent to $\Gamma \vdash^I_\Sigma \varphi$. Now, if $\varphi \in \Gamma$ then the rule (basic) completes the proof. If $\varphi \notin \Gamma$, then (CR) and (basic) rules must be used to complete the proof.
2. Let $SP$ be a specification of the form $SP_1 \cup SP_2$ and let $\mathbf{nf}(SP_i) = \mathbf{derive\ from}\ \langle \Sigma_i, \Gamma_i\rangle\ \mathbf{by}\ (d_i : \Sigma \to \Sigma_i)$ for $i = 1, 2$.
   Then $\mathbf{nf}(SP) = \mathbf{derive\ from}\ \langle \Sigma', t'_2\Gamma_1 \cup d'_1\Gamma_2\rangle\ \mathbf{by}\ (d : \Sigma \to \Sigma')$, where $d'_1 \in \mathscr{D}_I$, $t'_2 \in \mathscr{T}_I$ and $\Sigma'$ are given by the following pushout in **Sign**:

$$\begin{tikzcd} \Sigma \arrow[r, "d_1"] \arrow[d, "d_2"'] \arrow[dr, "d"] & \Sigma_1 \arrow[d, "t'_2"] \\ \Sigma_2 \arrow[r, "d'_1"'] & \Sigma' \end{tikzcd}$$

   From Theorem 3.8 we have $\mathbf{Mod}[\mathbf{nf}(SP)] \models^I_\Sigma \varphi$. Therefore by the satisfaction condition $t'_2\Gamma_1 \cup d'_1\Gamma_2 \models^I_{\Sigma'} d\varphi$, which is equivalent to $t'_2(\bigwedge \Gamma_1) \models^I_{\Sigma'} (\bigwedge d'_1\Gamma_2) \Rightarrow d\varphi$ in $I$. Since $d = d_2; d'_1$, this is equivalent to $t'_2(\bigwedge \Gamma_1) \models^I_{\Sigma'} d'_1(\bigwedge \Gamma_2 \Rightarrow d_2\varphi)$. By $(\mathscr{D},\mathscr{T})$-interpolation property for $I$, we have that there exists a $\Sigma$-sentence $\varphi_3$ such that
   (1) $\bigwedge \Gamma_1 \models^I_{\Sigma_1} d_1\varphi_3$,
   (2) $d_2\varphi_3 \models^I_{\Sigma_2} \bigwedge \Gamma_2 \Rightarrow d_2\varphi$.
   Condition (1) imply $\mathbf{Mod}[\langle \Sigma_1, \Gamma_1\rangle] \models^I_{\Sigma_1} d_1\varphi_3$, which by the satisfaction condition is equivalent to $\mathbf{Mod}[\mathbf{nf}(SP_1)] \models^I_\Sigma \varphi_3$ and so, by Theorem 3.8, to $\mathbf{Mod}[SP_1] \models^I_\Sigma \varphi_3$. Now, by the induction hypothesis we obtain $SP_1 \vdash_\Sigma \varphi_3$.
   Condition (2) by Theorem 2.10 is equivalent to $\bigwedge \Gamma_2 \models^I_{\Sigma_2} d_2(\varphi_3 \Rightarrow \varphi)$. Next, we obtain $\mathbf{Mod}[\langle \Sigma_2, \Gamma_2\rangle] \models^I_{\Sigma_2} d_2(\varphi_3 \Rightarrow \varphi)$ and by the satisfaction condition and Theorem 3.8, $\mathbf{Mod}[SP_2] \models^I_\Sigma \varphi_3 \Rightarrow \varphi$ which by the induction hypothesis gives: $SP_2 \vdash_\Sigma \varphi_3$

$\Rightarrow \varphi$. The following derivation completes the proof:

$$(\mathrm{CR})\ \dfrac{(\mathrm{sum1})\dfrac{SP_1 \vdash_\Sigma \varphi_3}{SP_1 \cup SP_2 \vdash_\Sigma \varphi_3} \qquad (\mathrm{sum2})\dfrac{SP_2 \vdash_\Sigma \varphi_3 \varphi_3 \Rightarrow \varphi}{SP_1 \cup SP_2 \vdash_\Sigma \varphi_3 \Rightarrow \varphi} \qquad \{\varphi_3 \Rightarrow \varphi, \varphi_3\} \vdash^I_\Sigma \varphi}{SP_1 \cup SP_2 \vdash_\Sigma \varphi}$$

where $\{\varphi_3 \Rightarrow \varphi, \varphi_3\} \vdash^I_\Sigma \varphi$ follows from (4) by Theorem 2.10 and because $\vdash^I_\Sigma$ is complete for $\models^I_\Sigma$ (assumption 2).

3. If $SP$ is a specification of the form **translate** $SP'$ **by** $(t : \Sigma' \to \Sigma)$, then let $\mathbf{nf}(SP') =$ **derive from** $\langle \Sigma_1, \Gamma_1 \rangle$ **by** $(d_1 : \Sigma' \to \Sigma_1)$ and
$\mathbf{nf}(SP) =$ **derive from** $\langle \Sigma_1', t'\Gamma_1 \rangle$ **by** $(d_1' : \Sigma \to \Sigma_1')$, where $t'$, $d_1'$ and $\Sigma_1'$ are given by a pushout diagram in **Sign**:

$$\begin{tikzcd} \Sigma' \arrow[r, "t"] \arrow[d, "d_1"'] & \Sigma \arrow[d, "d_1'"] \\ \Sigma_1 \arrow[r, "t'"'] & \Sigma_1' \end{tikzcd}$$

Now, similarly to case 2, $SP \models_\Sigma \varphi$ iff $\mathbf{Mod}[\mathbf{nf}(SP)] \models^I_\Sigma \varphi$.
By the satisfaction condition, we obtain $\mathbf{Mod}[\langle \Sigma_1', t'\Gamma_1 \rangle] \models^I_{\Sigma_1'} d_1'\varphi$, which is equivalent to $t'(\bigwedge \Gamma_1) \models^I_{\Sigma_1'} d_1'\varphi$. By the $(\mathscr{D}, \mathscr{T})$-interpolation property, there exists a $\Sigma'$-sentence $\varphi'$ such that
(1) $\bigwedge \Gamma_1 \models^I_{\Sigma_1} d_1\varphi'$,
(2) $t\varphi' \models^I_\Sigma \varphi$.
Because $\mathbf{Mod}[\langle \Sigma_1, \Gamma_1 \rangle] \models^I_{\Sigma_1} \Gamma_1$ and (1), we have $\mathbf{Mod}[\langle \Sigma_1, \Gamma_1 \rangle] \models^I_{\Sigma_1} d_1\varphi'$ and by the satisfaction condition and Theorem 3.8, $\mathbf{Mod}[SP'] \models^I_{\Sigma'} \varphi'$, which by the induction hypothesis is equivalent to $SP' \vdash_{\Sigma'} \varphi'$. The following derivation completes this case:

$$(\mathrm{CR})\dfrac{(\mathrm{trans})\dfrac{SP' \vdash_{\Sigma'} \varphi'}{\textbf{translate } SP' \textbf{ by } t \vdash_\Sigma t\varphi'} \qquad t\varphi' \vdash^I_\Sigma \varphi}{\textbf{translate } SP' \textbf{ by } t \vdash_\Sigma \varphi}$$

where $t\varphi' \vdash^I_\Sigma \varphi$ follows from (2) by assumption 2.

4. If $SP$ is a specification of the form **derive from** $SP'$ **by** $d$, where $d : \Sigma \to \Sigma'$, then $SP \models_\Sigma \varphi$ iff $(\mathbf{Mod}[SP'])|_d \models^I_\Sigma \varphi$. By the satisfaction condition, we have $\mathbf{Mod}[SP']$ $\models^I_{\Sigma'} d\varphi$, and by the induction hypothesis $SP' \vdash_{\Sigma'} d\varphi$. Application of the (derive) rule completes the proof. $\square$

If the $(\mathscr{D}, \mathscr{T})$-institution over which we build specifications is compact then we can modify Theorem 3.9 and obtain:

**Corollary 3.10.** *Let $I$ be a $(\mathscr{D},\mathscr{T})$-institution that has conjunction and implication. If*
1. *institution $I$ satisfies the weak-$(\mathscr{D},\mathscr{T})$-interpolation and weak-$(\mathscr{D},\mathscr{T})$-amalgamation properties,*
2. *the entailment relations $\vdash^I$ used in the rule (CR) are complete for $\models^I$, and*
3. *the institution $I$ is compact,*

*then for any $\Sigma$-specification SP over the institution $I$ and any $\Sigma$-sentence $\varphi$,*

$$SP \models_\Sigma \varphi \quad \textit{iff} \quad SP \vdash_\Sigma \varphi.$$

**Proof.** By soundness of $\vdash_\Sigma$ wrt $\models_\Sigma$ and by an obvious modification of the proof of Theorem 3.9: in each case when from $\Gamma \models \varphi$ we deduce $\bigwedge \Gamma \models \varphi$, we first have to choose a finite set $\Gamma_1 \subseteq \Gamma$ such that $\Gamma_1 \models \varphi$ and then work with $\bigwedge \Gamma_1 \models \varphi$. $\square$

Directly from Lemma 2.17 and Theorem 3.9 we have:

**Corollary 3.11.** *Let $I$ be a $(\mathscr{D},\mathscr{T})$-institution that has infinite conjunction and implication. If*
1. *institution $I$ satisfies the weak-$(\mathscr{D},\mathscr{T})$-interpolation and weak-$(\mathscr{D},\mathscr{T})$-amalgamation properties, and*
2. *the entailment relations $\vdash^I$, used in the rule (CR) are complete for $\models^I$,*
   *then for any $\Sigma$-specification SP over the institution $I$ and any $\Sigma$-sentence $\varphi$,*

$$SP \models_\Sigma \varphi \quad \textit{iff} \quad SP \vdash_\Sigma \varphi.$$

**Definition 3.12.** We say that specifications defined by Definition 3.1 are *finite* iff in point 1 of Definition 3.1 we additionally assume that the set $\Gamma$ is finite.

**Fact 3.13.** *The normal form of a finite specification is finite.*

**Proof.** By induction on the structure of *SP*. $\square$

Now for finite specifications we can skip assumption 3 in Corollary 3.10 and obtain:

**Corollary 3.14.** *Let $I$ be a $(\mathscr{D},\mathscr{T})$-institution that has conjunction and implication. If*
1. *institution $I$ satisfies the $(\mathscr{D},\mathscr{T})$-interpolation and weak-$(\mathscr{D},\mathscr{T})$-amalgamation properties, and*
2. *the entailment relations $\vdash^I$ used in the rule (CR) are complete for $\models^I$,*
   *then for any finite $\Sigma$-specification SP over the institution $I$ and any $\Sigma$-sentence $\varphi$*

$$SP \models_\Sigma \varphi \quad \textit{iff} \quad SP \vdash_\Sigma \varphi.$$

**Proof.** By inspection of the proof of Theorem 3.9. It is easy to check that all the sets of sentences used there are finite if *SP* is a finite specification. $\square$

Another consequence of completeness of entailment relation $\vdash_\Sigma$ is presented by the following lemma:

**Lemma 3.15.** *For any* $(\mathscr{D}, \mathscr{T})$*-institution* $I$, *signature* $\Sigma \in |\mathbf{Sign}_I|$, $\Sigma$*-specifications* $SP_1$ *and* $SP_2$, $\varphi \in \mathbf{Sen}_I(\Sigma)$ *and the entailment relation* $\vdash_\Sigma$ *which is complete*, *if* $SP_1 \cong SP_2$ *then*

$$SP_1 \vdash_\Sigma \varphi \quad \textit{iff} \quad SP_2 \vdash_\Sigma \varphi.$$

**Proof.** $\Rightarrow$: By soundness of the entailment relation $\vdash_\Sigma$ we obtain $SP_1 \models_\Sigma \varphi$. From $SP_1 \cong SP_2$ we have $\mathbf{Mod}[SP_1] = \mathbf{Mod}[SP_2]$ and next $SP_2 \models_\Sigma \varphi$ which by completeness of $\vdash_\Sigma$ gives us $SP_2 \vdash_\Sigma \varphi$.

$\Leftarrow$: By symmetry. $\square$

In particular, from the above lemma it follows that if $\vdash_\Sigma$ is complete and we can prove the judgment $\mathbf{nf}(SP) \vdash_\Sigma \varphi$, then there also exists a proof of $SP \vdash_\Sigma \varphi$.

In the next lemma we show that the interpolation property is crucial for completeness of the compositional proof system.

**Lemma 3.16.** *If the* $(\mathscr{D}, \mathscr{T})$*-institution* $I$ *satisfying the weak-*$(\mathscr{D}, \mathscr{T})$*-amalgamation property does not have the weak-*$(\mathscr{D}, \mathscr{T})$*-interpolation property*, *then the logical system for proving logical consequences of specifications over* $I$ *presented in Definition* 3.4 *is not complete.*

**Proof.** Let

$$\begin{tikzcd}
 & \Sigma \arrow[dl, "d \in \mathcal{D}_I"'] \arrow[dr, "t \in \mathcal{T}_I"] & \\
\Sigma_1 \arrow[dr, "t' \in \mathcal{T}_I"'] & & \Sigma_2 \arrow[dl, "d' \in \mathcal{D}_I"] \\
 & \Sigma' &
\end{tikzcd}$$

be a diagram in $\mathbf{Sign}_I$ and

$$t'\varphi_1 \models^I_{\Sigma'} d'\varphi_2 \tag{1}$$

where $\varphi_i \in \mathbf{Sen}_I(\Sigma_i)$ are such that there is no $\Gamma \subseteq \mathbf{Sen}_I(\Sigma)$ such that

$$\varphi_1 \models^I_{\Sigma_1} d\Gamma \quad \text{and} \quad t\Gamma \models^I_{\Sigma_2} \varphi_2. \tag{2}$$

Let us assume that the logical system for proving logical consequences of specifications over $I$ is complete.

Now we show that

$$t'\varphi_1 \models^I_{\Sigma'} d'\varphi_2 \quad \text{implies} \quad \textbf{translate}\ (\textbf{derive from}\ \langle \Sigma_1, \{\varphi_1\}\rangle\ \textbf{by}\ d)\ \textbf{by}\ t \models_{\Sigma_2} \varphi_2.$$

Let $M_2 \in \textbf{Mod}[\textbf{translate}\ (\textbf{derive from}\ \langle \Sigma_1, \{\varphi_1\}\rangle\ \textbf{by}\ d)\ \textbf{by}\ t]$. Then by Definition 3.1

$$M_2|_t \in \textbf{Mod}[\textbf{derive from}\ \langle \Sigma_1, \{\varphi_1\}\rangle\ \textbf{by}\ d]$$

and there exists $M_1 \in \textbf{Mod}(\Sigma_1)$ such that

$$M_1|_d = M_2|_t \quad \text{and} \quad M_1 \models^I_{\Sigma_1} \varphi_1.$$

By the weak-$(\mathscr{D},\mathscr{T})$-amalgamation property there exists $M' \in \textbf{Mod}(\Sigma')$ such that

$$M'|_{t'} = M_1 \quad \text{and} \quad M'|_{d'} = M_2.$$

Because $M_1 \models^I_{\Sigma_1} \varphi_1$ and $M'|_{t'} = M_1$ we obtain $M' \models^I_{\Sigma'} t'\varphi_1$ and by the assumption $M' \models^I_{\Sigma'} d'\varphi_2$. Next, we have $M'|_{d'} \models^I_{\Sigma_2} \varphi_2$ and finally because $M'|_{d'} = M_2$, $M_2 \models^I_{\Sigma_2} \varphi_2$.

Now, from the above implication and (1) and also by the assumption (completeness) we obtain

$$\textbf{translate}\ (\textbf{derive from}\ \langle \Sigma_1, \{\varphi_1\}\rangle\ \textbf{by}\ d)\ \textbf{by}\ t \vdash_{\Sigma_2} \varphi_2.$$

Because of the shape of (trans) rule and since $\vdash^I$ has transitivity and is stable under translation (see Definitions 2.11 and 2.12) there exists $\Psi = \{\psi_i \mid i \in \mathscr{I}\} \subseteq \textbf{Sen}_I(\Sigma)$ such that

$$(\text{CR})\frac{\left\{ (\text{trans})\dfrac{(\text{derive})\dfrac{(\text{basic})\dfrac{\varphi_1 \vdash^I_{\Sigma_1} d\psi_i}{\langle \Sigma_1, \{\varphi_1\}\rangle \vdash_{\Sigma_1} d\psi_i}}{\textbf{derive from}\ \langle \Sigma_1, \{\varphi_1\}\rangle\ \textbf{by}\ d \vdash_{\Sigma} \psi_i}}{\textbf{translate}\ (\textbf{derive from}\ \langle \Sigma_1, \{\varphi_1\}\rangle\ \textbf{by}\ d)\ \textbf{by}\ t \vdash_{\Sigma_2} t\psi_i} \right\}_{i\in\mathscr{I}} \quad t\Psi \vdash^I_{\Sigma_2} \varphi_2}{\textbf{translate}\ (\textbf{derive from}\ \langle \Sigma_1, \{\varphi_1\}\rangle\ \textbf{by}\ d)\ \textbf{by}\ t \vdash_{\Sigma_2} \varphi_2}$$

From the above proof tree we have

$$\{\varphi_1 \vdash^I_{\Sigma_1} d\psi_i\}_{i\in\mathscr{I}} \quad \text{and} \quad t\Psi \vdash^I_{\Sigma_2} \varphi_2$$

and because $\vdash^I$ is sound wrt $\models^I$:

$$\varphi_1 \models^I_{\Sigma_1} d\Psi \quad \text{and} \quad t\Psi \models^I_{\Sigma_2} \varphi_2,$$

which is in contradiction to (2). □

There are at least two kinds of negative examples of specifications, known from the literature, where the $(\mathscr{D},\mathscr{T})$-interpolation (and also Theorem 3.9) does not hold for the underlying $(\mathscr{D},\mathscr{T})$-institution and therefore certain semantic consequences of specifications cannot be proved using the rules of Definition 3.4. The first, presented also in [17], is based on empty carriers.

**Example 3.17.** Let us consider a specification $SP$ over $(\mathscr{D},\mathscr{T})$-institution **EQ**, where $\mathscr{D}_{\mathbf{EQ}}$ is the class of signature inclusions and $\mathscr{T}_{\mathbf{EQ}}$ is the class of all signature morphisms, and

$$SP_0 = \langle \Sigma_0, \emptyset \rangle, \qquad SP_1 = \textbf{derive from } SP_0 \textbf{ by } \imath,$$

$$SP_2 = \langle \Sigma_1, \{\forall_{x:s}.b = c\} \rangle, \qquad SP = SP_1 \cup SP_2,$$

where
- $\Sigma_0 = \textbf{sig sorts } s, s' \textbf{ opns } a : s;\ b, c : s' \textbf{ end}$;
- $\Sigma_1 = \textbf{sig sorts } s, s' \textbf{ opns } b, c : s' \textbf{ end}$;
- $\imath : \Sigma_1 \hookrightarrow \Sigma_0$.

As shown in [28]:

$$SP \models_{\Sigma_1} b = c, \tag{3}$$

whereas the judgment:

$$SP \vdash_{\Sigma_1} b = c \tag{4}$$

cannot be proved in **EQ** because the sentence $b = c$ cannot be derived from the sentence $\forall_{x:s}.b = c$ (the nonemptiness of the carrier of sort $s$, ensured by the hidden constant $a$, cannot be expressed using equations, cf. [15]).

The second example is based on the example presented in [3].

**Example 3.18.** Let the $(\mathscr{D},\mathscr{T})$-institution **EQ** be the same as in Example 3.17 and let us consider specification $SP$ over $(\mathscr{D},\mathscr{T})$-institution **EQ** defined as follows:

$$SP_0 = \langle \Sigma_0, \{f(c) = c\} \rangle,$$

$$SP_1 = \langle \Sigma_1, \{h(x, x, y) = y;\ h(x, f(x), a) = h(x, f(x), b)\} \rangle,$$

$$SP_2 = \textbf{derive from } SP_0 \textbf{ by } \imath,$$

$$SP = (\textbf{translate } SP_2 \textbf{ by } \jmath) \cup SP_1,$$

where
- $\Sigma_0 = \textbf{sig sorts } s \textbf{ opns } c : s;\ f : s \to s \textbf{ end}$;
- $\Sigma_1 = \textbf{sig sorts } s \textbf{ opns } a, b : s;\ f : s \to s;\ h : s \times s \times s \to s \textbf{ end}$;
- $\Sigma_2 = \textbf{sig sorts } s \textbf{ opns } f : s \to s; \textbf{ end}$;
- $\imath : \Sigma_2 \hookrightarrow \Sigma_0$ and $\jmath : \Sigma_2 \hookrightarrow \Sigma_1$.

Now, $\mathbf{Mod}[SP_0]$ is the class of all $\Sigma_0$-algebras, which satisfy $f(c) = c$. $\mathbf{Mod}[SP_1]$ consists of all $\Sigma_1$-algebras for which $h(x, x, y) = y$ and $h(x, f(x), a) = h(x, f(x), b)$. For some $\Sigma_1$-algebras in $\mathbf{Mod}[SP_1]$ the equality $a = b$ is satisfied, but not for all. The class $\mathbf{Mod}[SP_2]$ consists of reducts of algebras from $\mathbf{Mod}[SP_0]$ obtained by removing the constant $c$. Let us notice that we do not touch interpretation of $f$ in $\mathbf{Mod}[SP_2]$. It means that for every $M \in \mathbf{Mod}[SP_2]$ there exists value $v \in |M|_s$ such that $f(v) = v$. $\mathbf{Mod}[SP]$

is intersection of **Mod**[$SP_2$], viewed as a class of $\Sigma_1$-algebras, and **Mod**[$SP_1$]. Because in **Mod**[$SP$] there exists value $v \in |M|_s$ such that $f(v) = v$ we have

$$a = h(v, v, a) = h(v, f(v), a) = h(v, f(v), b) = h(v, v, b) = b.$$

It means that $SP \models_{\Sigma_1} a = b$.

On the other hand, we cannot prove $SP \vdash_{\Sigma_1} a = b$, because in **EQ** we cannot express the existence of value $v$ of sort $s$ such that $f(v) = v$.

In Lemma 3.16, we argued that the entailment relation defined in Definition 3.4 is not complete, if the underlying logical system does not satisfy the weak-($\mathscr{D}, \mathscr{T}$)-interpolation property. In both examples presented above the underlying ($\mathscr{D}, \mathscr{T}$)-institution **EQ** satisfies the weak-($\mathscr{D}, \mathscr{T}$)-interpolation property but does not satisfy the ($\mathscr{D}, \mathscr{T}$)-interpolation property. An interesting question is

> What are the minimal conditions, that have to be satisfied by the underlying ($\mathscr{D}$, $\mathscr{T}$)-institution (apart from the weak-($\mathscr{D}, \mathscr{T}$)-interpolation property) in order to ensure completeness of the entailment relation $\vdash_\Sigma$ wrt the semantic consequence $\models_\Sigma$?

In Corollaries 3.11 and 3.10 gives some (not minimal) answer for the above question. In general the problem presented in the above question is open.

Now, we present a positive example based on Example 5.3.3 from [8], and also presented in [4]. It shows how to construct a nontrivial specification and how to use the logical system defined in this section for reasoning about this specification.

**Example 3.19.** In this example we will work with specifications over the ($\mathscr{D}, \mathscr{T}$)-institution **FOEQ**.

First, we define two specifications: the first $ST$ specifying stacks and the second $NAT$ specifying natural numbers. Then we put them together to obtain specification $NAT\text{-}ST$ of stacks of natural numbers. Let us start with signatures:

$SIG\text{-}ST$ = **sig**
  **sorts** $Elem$; $Stack$
  **opns** $empty : Stack$;
    $push : Elem \times Stack \to Stack$;
    $top : Stack \to Elem$;
    $pop : Stack \to Stack$
  **rels** $is\_empty \subseteq Stack$
**end**

$SIG\text{-}NAT$ = **sig**
  **sorts** $Nat$
  **opns** $zero : Nat$;
    $succ : Nat \to Nat$
  **rels** $is\_zero \subseteq Nat$
**end**

In the next step we define specifications of stacks and natural numbers:

$$\begin{aligned}
ST = \langle \mathit{SIG\text{-}ST}, \quad & \{\forall_{e:Elem}.\forall_{x:Stack}.pop(push(e,x)) = x;\\
& \forall_{e:Elem}.\forall_{x:Stack}.top(push(e,x)) = e;\\
& is_empty(empty);\\
& \forall_{e:Elem}.\forall_{x:Stack}.\neg(is_empty(push(e,x)))\}\rangle\\
NAT = \langle \mathit{SIG\text{-}NAT}, \quad & \{\forall_{m,n:Nat}.succ\ m = succ\ n \Rightarrow m = n;\\
& \forall_{m:Nat}.\neg(succ\ m = zero);\\
& is_zero(zero);\\
& \forall_{m:Nat}.\neg is_zero(succ\ m)\}\rangle
\end{aligned}$$

Now, we put above specifications together to obtain the specification of stacks of natural numbers. Let us consider the following pushout in **Sign**:

$$\begin{tikzcd}
\mathit{SIG\text{-}ELEM} \arrow[r, "d"] \arrow[d, "t"'] & \mathit{SIG\text{-}ST} \arrow[d, "t'"] \\
\mathit{SIG\text{-}NAT} \arrow[r, "d'"'] & \mathit{SIG\text{-}NAT\text{-}ST}
\end{tikzcd}$$

where

- $\mathit{SIG\text{-}ELEM} =$ **sig sorts** $Elem$ **end**;
- $t(Elem) = Nat$;
- $d$ is an inclusion.

From the above we can define

$$\mathit{NAT\text{-}ST} = (\textbf{translate}\ NAT\ \textbf{by}\ d') \cup (\textbf{translate}\ ST\ \textbf{by}\ t')$$

and prove several properties of the $\mathit{NAT\text{-}ST}$ specification, e.g.

$$\mathit{NAT\text{-}ST} \vdash_{\mathit{SIG\text{-}NAT\text{-}ST}} \forall_{x:Stack}.is\_zero(top(push(zero,x))).$$

In the following proof of the above property, we write $\vdash$ as an abbreviation for $\vdash_{\mathit{SIG\text{-}NAT\text{-}ST}}$, $Ax\_of\_ST$ for the set of axions of the specification $ST$ and $Ax\_of\_NAT$ for the set of axions of the specification $NAT$:

$$(\mathrm{CR})\ \frac{\dfrac{(3)}{\mathit{NAT\text{-}ST} \vdash is\_zero(zero)} \quad \dfrac{(2)}{\mathit{NAT\text{-}ST} \vdash \forall_{n:Nat}.\forall_{x:Stack}.top(push(n,x))=n} \quad (1)}{\mathit{NAT\text{-}ST} \vdash \forall_{x:Stack}.is\_zero(top(push(zero,x)))}$$

where (1) is a proof in **FOEQ** of the following judgment:

$$\left\{\begin{array}{l} is\_zero(zero),\\ \forall_{n:Nat}.\forall_{x:Stack}.top(push(n,x)) = n \end{array}\right\} \vdash^{\mathbf{FOEQ}}_{\mathit{SIG\text{-}NAT\text{-}ST}} \forall_{x:Stack}.is\_zero(top(push(zero,x))),$$

(2) is the following proof:

$$\text{(sum2)}\ \dfrac{\text{(trans)}\ \dfrac{\text{(basic)}\ \dfrac{(\forall_{n:Elem}.\forall_{x:Stack}.top(push(n,x)) = n) \in Ax\_of\_ST}{ST \vdash_{SIG\text{-}ST} \forall_{n:Elem}.\forall_{x:Stack}.top(push(n,x)) = n}}{\textbf{translate}\ ST\ \textbf{by}\ t' \vdash \forall_{n:Nat}.\forall_{x:Stack}.top(push(n,x)) = n}}{NAT\text{-}ST \vdash \forall_{n:Nat}.\forall_{x:Stack}.top(push(n,x)) = n}$$

and finally (3) is

$$\text{(sum1)}\ \dfrac{\text{(trans)}\ \dfrac{\text{(basic)}\ \dfrac{(is\_zero(zero)) \in Ax\_of\_NAT}{NAT \vdash_{SIG\text{-}NAT} is\_zero(zero)}}{\textbf{translate}\ NAT\ \textbf{by}\ d' \vdash is\_zero(zero)}}{NAT\text{-}ST \vdash is\_zero(zero)}.$$

At the end of this section we want to mention a noncompositional proof system for proving logical consequences of structural specifications (see also [8, 32]). It can be defined by the following rules:

$$\text{(n-nf)}\ \dfrac{\mathbf{nf}(SP) \vdash_{\Sigma}^{(n)} \varphi}{SP \vdash_{\Sigma}^{(n)} \varphi} \qquad \text{(n-derive)}\ \dfrac{\Gamma' \vdash_{\Sigma'}^{I} d\varphi}{\textbf{derive from}\ \langle \Sigma', \Gamma' \rangle\ \textbf{by}\ d \vdash_{\Sigma}^{(n)} \varphi}$$

where $SP$ is a $\Sigma$-specification, $\varphi$ is $\Sigma$-sentence, $\Gamma'$ is a set of $\Sigma'$-sentences and $(d : \Sigma' \to \Sigma) \in \mathscr{D}_I$.

The above proof system is sound wrt the semantic consequence $\models_{\Sigma}$ and complete if $\vdash_{\Sigma}^{I}$ is complete. As we can see this proof system has nice proof-theoretic properties but has also several disadvantages in practice. The first is the technical complexity of computing a normal form which can be very important for larger specifications. The second is the loss of structure. The structured nature of the specification is ignored, and too many axioms occurring in the normal form may cause the proof to become hard to deal with and the proof search more difficult.

## 4. Refinement

In this section we consider the refinement relation for specifications build over $(\mathscr{D}, \mathscr{T})$-institutions and prove that the logical system for reasoning about the refinement relation, presented in this section, is sound and complete.

In addition to Assumption 2.22 throughout this section we also adopt the following restriction on classes $\mathscr{D}$ and $\mathscr{T}$:

**Assumption 4.1.** We assume that every $(\mathscr{D}, \mathscr{T})$-institution $I$ satisfies $\mathscr{T}_I = Iso_I; \mathscr{D}_I$, where $Iso_I$ is a class of isomorphisms of institution $I$, i.e. every $t \in \mathscr{T}_I$ can be presented as $t = i; d$, where $i \in Iso_I$ and $d \in \mathscr{D}_I$.

**Definition 4.2** (*Semantic refinement*). A $\Sigma$-specification $SP_2$ is a *semantic refinement* of a $\Sigma$-specification $SP_1$ (written $SP_1 \rightsquigarrow_{\Sigma} SP_2$) if $\mathbf{Mod}[SP_1] \supseteq \mathbf{Mod}[SP_2]$.

**Definition 4.3** (*Conservative extension along $d$*). For any specifications $SP_1$ and $SP_2$ over $(\mathscr{D},\mathscr{T})$-institution $I$ and $(d : \mathbf{Sig}[SP_2] \to \mathbf{Sig}[SP_1]) \in \mathscr{D}_I$, $SP_1$ is a *conservative extension of $SP_2$ along $d$* if $\mathbf{Mod}[SP_1]|_d = \mathbf{Mod}[SP_2]$.

**Definition 4.4.** For a given $(\mathscr{D},\mathscr{T})$-institution $I$ the *family of refinement relations* $\{\leadsto_\Sigma \subseteq \mathbf{Spec}_\Sigma \times \mathbf{Spec}_\Sigma\}_{\Sigma\in|\mathbf{Sign}_I|}$, parameterized by the family of entailment relations $\{\vdash_\Sigma\}_{\Sigma\in|\mathbf{Sign}_I|}$ (see Definition 3.4) which we assume is sound wrt the family of semantic consequence relations $\{\models_\Sigma\}_{\Sigma\in|\mathbf{Sign}_I|}$ (see Definition 3.3), is defined by the following set of rules:

$$\text{(Basic)}\ \frac{SP \vdash_\Sigma \Gamma}{\langle \Sigma, \Gamma\rangle \leadsto_\Sigma SP} \qquad \text{(Sum)}\ \frac{SP_1 \leadsto_\Sigma SP \quad SP_2 \leadsto_\Sigma SP}{SP_1 \cup SP_2 \leadsto_\Sigma SP}$$

$$\text{(Trans}_1\text{)}\ \frac{SP \leadsto_\Sigma \textbf{translate } SP' \textbf{ by } r^{-1}}{\textbf{translate } SP \textbf{ by } r \leadsto_{\Sigma'} SP'} \qquad \text{(Trans}_2\text{)}\ \frac{SP' \leadsto_{\Sigma'} \textbf{derive from } SP'' \textbf{ by } d}{\textbf{translate } SP' \textbf{ by } d \leadsto_{\Sigma''} SP''}$$

$$\text{(Derive)}\ \frac{SP \leadsto_{\Sigma''} SP''}{\textbf{derive from } SP \textbf{ by } d \leadsto_{\Sigma'} SP'} \quad \begin{array}{l} SP'' \text{ is a conservative extension of} \\ SP' \text{ along } d \end{array}$$

$$\text{(Trans-equiv)}\ \frac{\textbf{translate } (\textbf{translate } SP \textbf{ by } r) \textbf{ by } d \leadsto_{\Sigma''} SP''}{\textbf{translate } SP \textbf{ by } r;d \leadsto_{\Sigma''} SP''}$$

where $(r : \Sigma \to \Sigma') \in Iso_I$ and $(d : \Sigma' \to \Sigma'') \in \mathscr{D}_I$.

The above definition of the refinement relation is inspired by the definition of analogical relation presented in [8, 32]. The semantic side condition in the rule (Derive) is outside the system presented by the above definition and we do not have any general strategy for proving it. In practical examples (see [8, 32] where $\mathscr{D}_I$ is a class of inclusions) $SP''$ can be definitional extension of $SP'$. A more model-theoretic aspect of this side condition can be found in [12].

Now, we prove that the refinement relation is sound and complete wrt the semantic refinement.

**Theorem 4.5** (Soundness and completeness). *For any $\Sigma$-specifications $SP_1$ and $SP_2$ over $(\mathscr{D},\mathscr{T})$-institution $I$,*

*if* $SP_1 \leadsto_\Sigma SP_2$ *then* $SP_1 \rightsquigarrow_\Sigma SP_2$.

*The converse implication holds if the entailment $\vdash_\Sigma$ is complete.*

**Proof.** Soundness ($\Rightarrow$): First we prove that the rules from Definition 4.4 are sound and hence by the soundness of $\vdash_\Sigma$ we obtain soundness of $\leadsto_\Sigma$.

1. (Basic) Let us assume that $SP \models_\Sigma \Gamma$ and let $M \in \mathbf{Mod}[SP]$. By definitions $M \models_\Sigma \Gamma$, and so by Definition 3.1, $M \in \mathbf{Mod}[\langle\Sigma,\Gamma\rangle]$.
2. (Sum) Let $\mathbf{Mod}[SP_1] \supseteq \mathbf{Mod}[SP]$ and $\mathbf{Mod}[SP_2] \supseteq \mathbf{Mod}[SP]$. This means that $\mathbf{Mod}[SP] \subseteq \mathbf{Mod}[SP_1] \cap \mathbf{Mod}[SP_2] = \mathbf{Mod}[SP_1 \cup SP_2]$.

3. (Trans$_1$) Let $\mathbf{Mod}[SP] \supseteq \mathbf{Mod}[\textbf{translate } SP' \textbf{ by } r^{-1}]$ and $M \in \mathbf{Mod}[SP']$. By definitions, since $(M|_r)|_{r^{-1}} = M$, $M|_r \in \mathbf{Mod}[\textbf{translate } SP' \textbf{ by } r^{-1}] \subseteq \mathbf{Mod}[SP]$ which means that $M \in \mathbf{Mod}[\textbf{translate } SP \textbf{ by } r]$.
4. (Trans$_2$) Let $\mathbf{Mod}[SP'] \supseteq \mathbf{Mod}[\textbf{derive from } SP'' \textbf{ by } d]$ and $M \in \mathbf{Mod}[SP'']$. By definitions we have $M|_d \in \mathbf{Mod}[\textbf{derive from } SP'' \textbf{ by } d] \subseteq \mathbf{Mod}[SP']$ and finally $M \in \mathbf{Mod}[\textbf{translate } SP' \textbf{ by } d]$.
5. (Derive) Let $\mathbf{Mod}[SP] \supseteq \mathbf{Mod}[SP'']$ and $M \in \mathbf{Mod}[SP']$. Because $SP''$ is a conservative extension of $SP'$ along $d$, there exists $M'' \in \mathbf{Mod}[SP'']$ such that $M''|_d = M$. By the assumption, $M'' \in \mathbf{Mod}[SP]$, which means that
$M = M''|_d \in \mathbf{Mod}[\textbf{derive from } SP \textbf{ by } d]$.
6. (Trans-equiv) Let $\mathbf{Mod}[\textbf{translate } (\textbf{translate } SP \textbf{ by } r) \textbf{ by } d] \supseteq \mathbf{Mod}[SP'']$ and $M \in \mathbf{Mod}[SP'']$ which means that $M \in \mathbf{Mod}[\textbf{translate } (\textbf{translate } SP \textbf{ by } r) \textbf{ by } d]$. By definitions we have $(M|_d)|_r \in \mathbf{Mod}[SP]$. Because $(M|_d)|_r = M|_{r;d}$ we obtain $M \in \mathbf{Mod}[\textbf{translate } SP \textbf{ by } r; d]$.

*Completeness* ($\Leftarrow$): By induction on the structure of $SP_1$.

1. $SP_1$ is a specification expression of the form $\langle \Sigma, \Gamma \rangle$.
By the assumption, $\mathbf{Mod}[\langle \Sigma, \Gamma \rangle] \supseteq \mathbf{Mod}[SP_2]$, which is equivalent to $SP_2 \models_\Sigma \Gamma$. Because $\vdash_\Sigma$ is complete we obtain $SP_2 \vdash_\Sigma \Gamma$ and by the (Basic) rule $\langle \Sigma, \Gamma \rangle \rightsquigarrow_\Sigma SP_2$.
2. $SP_1$ is a specification expression of the form $SP'_1 \cup SP'_2$. By the assumption, $\mathbf{Mod}[SP'_1] \cap \mathbf{Mod}[SP'_2] \supseteq \mathbf{Mod}[SP_2]$, which is equivalent to $\mathbf{Mod}[SP'_i] \supseteq \mathbf{Mod}[SP_2]$ for $i = 1, 2$, and then by the induction hypothesis we have $SP'_i \rightsquigarrow_\Sigma SP_2$ for $i = 1, 2$. Finally, by the rule (Sum) we obtain $SP'_1 \cup SP'_2 \rightsquigarrow_\Sigma SP_2$.
3. $SP_1$ is a specification expression of the form **translate** $SP$ **by** $(t : \Sigma \to \Sigma'')$, where $t = r; d$ for $r \in Iso_I$ and $d \in \mathscr{D}_I$. By the assumption, we have $\mathbf{Mod}[SP_1] \supseteq \mathbf{Mod}[SP_2]$. Let

$$M \in \mathbf{Mod}[\textbf{translate } (\textbf{derive from } SP_2 \textbf{ by } d) \textbf{ by } r^{-1}].$$

By definitions we have $M|_{r^{-1}} \in \mathbf{Mod}[\textbf{derive from } SP_2 \textbf{ by } d]$ and also there exists $M_2 \in \mathbf{Mod}[SP_2]$ such that $M_2|_d = M|_{r^{-1}}$, which is equivalent to $M_2|_{r;d} = M$. Because also $M_2 \in \mathbf{Mod}[SP_1]$, it means that $M = M_2|_{r;d} \in \mathbf{Mod}[SP]$ which shows that

$$SP \rightsquigarrow_\Sigma \textbf{ translate } (\textbf{derive from } SP_2 \textbf{ by } d) \textbf{ by } r^{-1}.$$

Now by the induction hypothesis we obtain

$$SP \rightsquigarrow_\Sigma \textbf{ translate } (\textbf{derive from } SP_2 \textbf{ by } d) \textbf{ by } r^{-1}$$

and by applying rules (Trans$_1$) and (Trans$_2$)

$$\textbf{translate } (\textbf{translate } SP \textbf{ by } r) \textbf{ by } d \rightsquigarrow_{\Sigma''} SP_2,$$

which by the (Trans-equiv) gives us **translate** $SP$ **by** $t \rightsquigarrow_{\Sigma''} SP_2$.
4. $SP_1$ is a specification expression of the form **derive from** $SP'$ **by** $(d : \Sigma \to \Sigma')$. Let $SP'' = SP' \cup (\textbf{translate } SP_2 \textbf{ by } d)$. We prove that $SP''$ is a conservative extension of $SP_2$ along $d$.

(a) $\mathbf{Mod}[SP'']|_d \supseteq \mathbf{Mod}[SP_2]$: Let $M \in \mathbf{Mod}[SP_2]$. By the assumption, $M \in \mathbf{Mod}[SP_1]$ and so there exists $M' \in \mathbf{Mod}[SP']$ such that $M'|_d = M$. This shows that $M' \in \mathbf{Mod}[\textbf{translate}\ SP_2\ \textbf{by}\ d]$, and $M \in \mathbf{Mod}[SP'']|_d$.

(b) $\mathbf{Mod}[SP'']|_d \subseteq \mathbf{Mod}[SP_2]$: Let $M \in \mathbf{Mod}[SP'']|_d$. There exists $M_2 \in \mathbf{Mod}[\textbf{translate}\ SP_2\ \textbf{by}\ d]$ such that $M_2|_d = M$. By definitions we have $M_2|_d \in \mathbf{Mod}[SP_2]$ and finally $M \in \mathbf{Mod}[SP_2]$.

Now,

$$\mathbf{Mod}[SP'] \supseteq \mathbf{Mod}[SP'] \cap \mathbf{Mod}[\textbf{translate}\ SP_2\ \textbf{by}\ d]$$
$$= \mathbf{Mod}[SP' \cup (\textbf{translate}\ SP_2\ \textbf{by}\ d) = \mathbf{Mod}[SP'']$$

and so by the induction hypothesis we have $SP' \rightsquigarrow_{\Sigma'} SP''$. Next, because $SP''$ is conservative extension of $SP_2$ along $d$, by the rule (Derive) we obtain $\textbf{derive from}\ SP'\ \textbf{by}\ d \rightsquigarrow_\Sigma SP_2$. $\square$

Let us consider the family of refinement relations defined as in Definition 4.4 except the (Derive) rule where we weaken the semantic side condition "*$SP''$ is a conservative extension of $SP'$ along $d$*" to "$\mathbf{Mod}[SP'']|_d \supseteq \mathbf{Mod}[SP']$", then Theorem 4.5 also holds. In fact, this weaker condition is necessary for the rule (Derive) to be sound, and so in particular, if we consider the (Derive) rule without any side conditions, then the family of refinement relations is still complete wrt the semantic refinement relation, but is not sound in general.

Let us notice that if we consider the proof system defined by Definition 4.4 without the (Trans-equiv) rule then the system is still sound but not complete. For instance, the judgment

$$\textbf{translate}\ SP\ \textbf{by}\ r; d \rightsquigarrow_{\Sigma''}\ \textbf{translate}\ (\textbf{translate}\ SP\ \textbf{by}\ r)\ \textbf{by}\ d,$$

where $(r : \Sigma \to \Sigma') \in Iso_I$ and $(d : \Sigma' \to \Sigma'') \in \mathscr{D}_I$, is true but cannot be proved without the (Trans-equiv) rule.

On the other hand, having completeness we can introduce even more general rule then the (Trans-equiv) rule:

**Lemma 4.6.** *For any $(\mathscr{D}, \mathscr{T})$-institution $I$, signature $\Sigma \in |\mathbf{Sign}_I|$, $\Sigma$-specifications $SP_1$, $SP_1'$, $SP_2$ and $SP_2'$ and the refinement relation $\rightsquigarrow_\Sigma$ which is complete, if $SP_1 \cong SP_1'$ and $SP_2 \cong SP_2'$ then*

$$SP_1 \rightsquigarrow_\Sigma SP_2 \quad \textit{iff} \quad SP_1' \rightsquigarrow_\Sigma SP_2'.$$

**Proof.** By analogy to proof of Lemma 3.15. $\square$

Similarly as for Lemma 3.15, it follows that if $\rightsquigarrow_\Sigma$ is complete and we can prove the judgment $SP_1 \rightsquigarrow_\Sigma \mathbf{nf}(SP_2)$, then there also exists a proof of $SP_1 \rightsquigarrow_\Sigma SP_2$.

The following example shows that if the entailment relation $\vdash_\Sigma$ is not complete, then also the refinement relation $\rightsquigarrow_\Sigma$ is not complete.

**Example 4.7.** Let *SP* be a $\Sigma$-specification and $\varphi$ be a $\Sigma$-sentence such that $\varphi$ is satisfied in every model of specification *SP*, but the judgment $SP \vdash_\Sigma \varphi$ cannot be proved. Examples of such *SP* and $\varphi$ are presented in Examples 3.17 and 3.18. Let us consider following judgment:

$$\langle \Sigma, \{\varphi\}\rangle \rightsquigarrow_\Sigma SP.$$

Because $\mathbf{Mod}[\langle \Sigma, \{\varphi\}\rangle] \supseteq \mathbf{Mod}[SP]$, we have $\langle \Sigma, \{\varphi\}\rangle \rightsquigarrow_\Sigma SP$, which means that the above judgment is true. Now we try to prove it. We have to apply the (Basic) rule, for which we need

$$SP \vdash_\Sigma \varphi,$$

which, by assumption, is not provable.

## 5. Representing specifications

The notion of an institution representation, introduced below, is a special case of a *simple map of institutions* (see [20]). The definition presented below is exactly the same as in [30].

**Definition 5.1** (*Institution representation* [30]). Let $I = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|}\rangle$ and $I' = \langle \mathbf{Sign}', \mathbf{Sen}', \mathbf{Mod}', \langle \models'_{\Sigma}\rangle_{\Sigma \in |\mathbf{Sign}'|}\rangle$ be arbitrary institutions. An *institution representation* $\rho : I \to I'$ consists of:

- a functor $\rho^{\mathbf{Sign}} : \mathbf{Sign} \to \mathbf{Sign}'$; and
- a natural transformation: $\rho^{\mathbf{Sen}} : \mathbf{Sen} \to \rho^{\mathbf{Sign}}; \mathbf{Sen}'$, that is, a family of functions $\rho^{\mathbf{Sen}}_\Sigma : \mathbf{Sen}(\Sigma) \to \mathbf{Sen}'(\rho^{\mathbf{Sign}}(\Sigma))$, natural in $\Sigma \in |\mathbf{Sign}|$:

$$\begin{tikzcd}
\Sigma_2 & \mathbf{Sen}(\Sigma_2) \arrow[r, "\rho^{\mathbf{Sen}}_{\Sigma_2}"] & \mathbf{Sen}'(\rho^{\mathbf{Sign}}(\Sigma_2)) \\
\Sigma_1 \arrow[u, "\sigma"] & \mathbf{Sen}(\Sigma_1) \arrow[u, "\mathbf{Sen}(\sigma)"] \arrow[r, "\rho^{\mathbf{Sen}}_{\Sigma_1}"] & \mathbf{Sen}'(\rho^{\mathbf{Sign}}(\Sigma_1)) \arrow[u, "\mathbf{Sen}'(\rho^{\mathbf{Sign}}(\sigma))"']
\end{tikzcd}$$

- a natural transformation $\rho^{\mathbf{Mod}} : (\rho^{\mathbf{Sign}})^{op};\ \mathbf{Mod}' \to \mathbf{Mod}$, that is, a family of functions $\rho^{\mathbf{Mod}}_\Sigma : \mathbf{Mod}'(\rho^{\mathbf{Sign}}(\Sigma)) \to \mathbf{Mod}(\Sigma)$, natural in $\Sigma \in |\mathbf{Sign}|$:

$$\begin{tikzcd}
\Sigma_2 & \mathbf{Mod}(\Sigma_2) \arrow[d, "\mathbf{Mod}(\sigma)"'] & \mathbf{Mod}'(\rho^{\mathbf{Sign}}(\Sigma_2)) \arrow[l, "\rho^{\mathbf{Mod}}_{\Sigma_2}"'] \arrow[d, "\mathbf{Mod}'(\rho^{\mathbf{Sign}}(\sigma))"] \\
\Sigma_1 \arrow[u, "\sigma"] & \mathbf{Mod}(\Sigma_1) & \mathbf{Mod}'(\rho^{\mathbf{Sign}}(\Sigma_1)) \arrow[l, "\rho^{\mathbf{Mod}}_{\Sigma_1}"]
\end{tikzcd}$$

such that for any signature $\Sigma \in |\mathbf{Sign}|$ the translations $\rho^{\mathbf{Sen}}_{\Sigma} : \mathbf{Sen}(\Sigma) \to \mathbf{Sen}'(\rho^{\mathbf{Sign}}(\Sigma))$ of sentences and $\rho^{\mathbf{Mod}}_{\Sigma} : \mathbf{Mod}'(\rho^{\mathbf{Sign}}(\Sigma)) \to \mathbf{Mod}(\Sigma)$ of models preserve the satisfaction relation, that is, for any $\varphi \in \mathbf{Sen}(\Sigma)$ and $M' \in |\mathbf{Mod}'(\rho^{\mathbf{Sign}}(\Sigma))|$:

$$M' \models'_{\rho^{\mathbf{Sign}}(\Sigma)} \rho^{\mathbf{Sen}}_{\Sigma}(\varphi) \quad \text{iff} \quad \rho^{\mathbf{Mod}}_{\Sigma}(M') \models_{\Sigma} \varphi \quad \text{(Representation condition)}$$

An institution representation $\rho : I \to I'$ shows how institution $I$ is encoded in institution $I'$. It means that all parts of $I$ are represented, but only some parts of $I'$ are used for representing various parts of $I$.

The above definition of institution representation can be easily extended to $(\mathscr{D}, \mathscr{T})$-institution representation.

**Definition 5.2** (*$(\mathscr{D}, \mathscr{T})$-institution representation*). A $(\mathscr{D}, \mathscr{T})$-institution representation $\rho : I \to I'$ is a usual institution representation $\rho : I \to I'$ which additionally satisfies

$$\rho^{\mathbf{Sign}}(\mathscr{D}_I) \subseteq \mathscr{D}_{I'} \quad \text{and} \quad \rho^{\mathbf{Sign}}(\mathscr{T}_I) \subseteq \mathscr{T}_{I'}.$$

The following example was also presented in [30]:

**Example 5.3.** The institution representation $\rho_{\mathbf{EQ} \to \mathbf{FOEQ}} : \mathbf{EQ} \to \mathbf{FOEQ}$ is given by the embedding of the category of algebraic signatures into the category of first-order signatures which equips algebraic signatures with the empty set of predicate names. The translation of sentences is an inclusion of (universally quantified) equations as first-order logic sentences, and the translation of models is the identity.

In the next example the model part of the institution representation is an embedding.

**Example 5.4.** The institution representation $\rho_{\mathbf{PEQ} \to \mathbf{EQ}} : \mathbf{PEQ} \to \mathbf{EQ}$ is given by the identity on the category of algebraic signatures. Translation of an equality from **PEQ** is the corresponding equality in **EQ**. Translation of the definedness formulae $D(t)$ is the equality $t = t$. Translation of models is the embedding of the category of total many-sorted algebras into the category of partial many-sorted algebras.

The institution representation presented in the above example does not quite fit our expectations (see the explanations after Definition 5.1). To improve this situation we put some extra condition on the model part of institution representations.

**Definition 5.5** (*$\rho$-Expansion*). An institution representation $\rho : I \to I'$ has the *$\rho$-expansion* property, if for any signature $\Sigma \in |\mathbf{Sign}|$, any $\Sigma$-model $M$ has a *$\rho$-expansion* to a $\rho^{\mathbf{Sign}}(\Sigma)$-model, that is, there exists a $\rho^{\mathbf{Sign}}(\Sigma)$-model $M'$ such that $\rho_{\Sigma}^{\mathbf{Mod}}(M') = M$.

**Example 5.6.** The institution representation $\rho_{\mathbf{EQ}\to\mathbf{FOEQ}} : \mathbf{EQ} \to \mathbf{FOEQ}$ has the $\rho$-expansion property, whereas the institution representation $\rho_{\mathbf{PEQ}\to\mathbf{EQ}} : \mathbf{PEQ} \to \mathbf{EQ}$ does not have it.

**Definition 5.7** (*Weak-$\mathscr{D}$-amalgamation*). Let $\rho : I \to I'$ be an institution representation and $\mathscr{D}$ be a class of signature morphisms in $I$. We say that the institution representation $\rho$ has the *weak-$\mathscr{D}$-amalgamation* property iff for every signatures $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $(d : \Sigma_2 \to \Sigma_1) \in \mathscr{D}$, $M_1 \in |\mathbf{Mod}(\Sigma_1)|$ and $M_2 \in |\mathbf{Mod}'(\rho^{\mathbf{Sign}}(\Sigma_2))|$, as in the following diagram

$$\begin{tikzcd}
\mathbf{Mod}(\Sigma_1) \arrow[d, "-|_d"'] & \mathbf{Mod}'(\rho^{\mathbf{Sign}}(\Sigma_1)) \arrow[l, "\rho^{\mathbf{Mod}}_{\Sigma_1}"'] \arrow[d, "-|_{\rho^{\mathbf{Sign}}(d)}"] \\
\mathbf{Mod}(\Sigma_2) & \mathbf{Mod}'(\rho^{\mathbf{Sign}}(\Sigma_2)) \arrow[l, "\rho^{\mathbf{Mod}}_{\Sigma_2}"]
\end{tikzcd}$$

if $\rho_{\Sigma_2}^{\mathbf{Mod}}(M_2) = M_1|_d$ then there exists $M \in |\mathbf{Mod}'(\rho^{\mathbf{Sign}}(\Sigma_1))|$ such that $\rho_{\Sigma_1}^{\mathbf{Mod}}(M) = M_1$ and $M|_{\rho^{\mathbf{Sign}}(d)} = M_2$.

**Example 5.8.** The institution representation $\rho_{\mathbf{EQ}\to\mathbf{FOEQ}} : \mathbf{EQ} \to \mathbf{FOEQ}$ presented in Example 5.3 has the weak-$\mathscr{D}$-amalgamation property for the class $\mathscr{D}$ being the class of all inclusions in the category of algebraic signatures.

**Example 5.9.** The institution representation $\rho_{\mathbf{PEQ}\to\mathbf{EQ}} : \mathbf{PEQ} \to \mathbf{EQ}$ defined in Example 5.4, for $\mathscr{D}$ being class of all inclusions in the category of algebraic signatures, does not have the weak-$\mathscr{D}$-amalgamation property. Counterexample follows.

In the rest of this example we will write $\rho$ as an abbreviation for the institution representation $\rho_{\mathbf{PEQ}\to\mathbf{EQ}}$.

Let:

- $\Sigma =$ **sig sorts** $s$ **opns** $op : s \to s$ **end** and
- $\Sigma' =$ **sig sorts** $s$ **opns** $op : s \to s$; $pop : s \to s$ **end**

be signatures in **PEQ** and $\imath : \Sigma \hookrightarrow \Sigma'$ an inclusion, then in the model part of representation $\rho$, we have the following diagram:

$$\begin{tikzcd}
\mathbf{Mod}_{\mathbf{PEQ}}(\Sigma') \arrow[d, "\mathbf{Mod}_{\mathbf{PEQ}}(\imath)"'] & \mathbf{Mod}_{\mathbf{EQ}}(\rho^{\mathbf{Sign}}(\Sigma')) \arrow[l, "\rho^{\mathbf{Mod}}_{\Sigma'}"'] \arrow[d, "\mathbf{Mod}_{\mathbf{EQ}}(\rho^{\mathbf{Sign}}(\imath))"] \\
\mathbf{Mod}_{\mathbf{PEQ}}(\Sigma) & \mathbf{Mod}_{\mathbf{EQ}}(\rho^{\mathbf{Sign}}(\Sigma)) \arrow[l, "\rho^{\mathbf{Mod}}_{\Sigma}"]
\end{tikzcd}$$

Let us take $M \in |\mathbf{Mod}_{\mathbf{PEQ}}(\Sigma)|$ such that it interprets operation $op : s \to s$ as a total operation, and $M' \in |\mathbf{Mod}_{\mathbf{PEQ}}(\Sigma')|$ interpreting operation $op : s \to s$ in the same way as $M$ and operation $pop : s \to s$ as a partial operation. The forgetful functor $\mathbf{Mod}_{\mathbf{PEQ}}(\imath)$ just forgets interpretation of $pop : s \to s$. From the definition of $\rho$ (cf. Example 5.4), we know that $\rho^{\mathbf{Sign}}(\Sigma)$ and $\rho^{\mathbf{Sign}}(\Sigma')$ are just $\Sigma$ and $\Sigma'$, but considered as signatures in **EQ**. Now, if $\bar{M} \in |\mathbf{Mod}_{\mathbf{EQ}}(\rho^{\mathbf{Sign}}(\Sigma))|$ interprets $op : s \to s$ in the same way as $M$, then

$$\rho^{\mathbf{Mod}}_{\Sigma}(\bar{M}) = M = \mathbf{Mod}_{\mathbf{PEQ}}(\imath)(M').$$

On the other hand, from the definition of $\rho$ we know that

$$\text{for any } \bar{M}' \in |\mathbf{Mod}_{\mathbf{EQ}}(\rho^{\mathbf{Sign}}(\Sigma'))|, \quad \rho^{\mathbf{Mod}}_{\Sigma'}(\bar{M}') \text{ is total}$$

hence there is no $\bar{M}' \in |\mathbf{Mod}_{\mathbf{EQ}}(\rho^{\mathbf{Sign}}(\Sigma'))|$ such that $\rho^{\mathbf{Mod}}_{\Sigma'}(\bar{M}') = M'$. It means that $\rho$ does not have the weak-$\mathscr{D}$-amalgamation property for $\mathscr{D}$ being the class of all inclusions in the category of algebraic signatures.

The institution representation $\rho_{\mathbf{PEQ}\to\mathbf{EQ}}$ does not have either of the two properties: $\rho$-expansion and weak-$\mathscr{D}$-amalgamation. In general, the two properties are orthogonal, i.e., there are examples of institution representations which have $\rho$-expansion but do not have the weak-$\mathscr{D}$-amalgamation property (see Examples 5.11 and 6.6) and also which have the weak-$\mathscr{D}$-amalgamation property but do not have $\rho$-expansion (see the example below).

**Example 5.10.** Let $I$ and $I'$ be institutions without sentences, defined as follows:

- categories of signatures consists of two distinct objects: $\Sigma_A, \Sigma_B$ in $\mathbf{Sign}_I$ and $\Sigma_{A'}, \Sigma_{B'}$ in $\mathbf{Sign}_{I'}$ and one arrow in each category $d : \Sigma_A \to \Sigma_B$ and $d' : \Sigma_{A'} \to \Sigma_{B'}$ (plus identities);
- model functors:

$$\begin{aligned}
&\mathbf{Mod}_I(\Sigma_A) = \{M_A^1, M_A^2\}, && \mathbf{Mod}_{I'}(\Sigma_{A'}) = \{M_{A'}\},\\
&\mathbf{Mod}_I(\Sigma_B) = \{M_B\}, && \mathbf{Mod}_{I'}(\Sigma_{B'}) = \{M_{B'}\},\\
&\mathbf{Mod}_I(d)(M_B) = M_A^1, && \mathbf{Mod}_{I'}(d')(M_{B'}) = M_{A'},
\end{aligned}$$

where $\mathbf{Mod}_I(\Sigma_A)$, $\mathbf{Mod}_I(\Sigma_B)$, $\mathbf{Mod}_{I'}(\Sigma_{A'})$ and $\mathbf{Mod}_{I'}(\Sigma_{B'})$ are discrete categories;

- satisfaction relations $\models^{I}_{\Sigma_A}$, $\models^{I}_{\Sigma_B}$, $\models^{I'}_{\Sigma_{A'}}$ and $\models^{I'}_{\Sigma_{B'}}$ are empty.

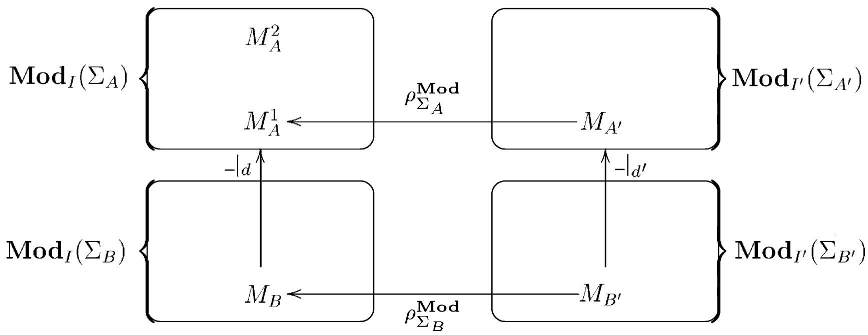
  The satisfaction condition holds obviously for both institutions $I$ and $I'$.

  Let us define institution representation $\rho : I \to I'$:

$$\rho^{\mathbf{Sign}}(\Sigma_A) = \Sigma_{A'}, \qquad \rho^{\mathbf{Mod}}_{\Sigma_A}(M_{A'}) = M^1_A,$$
$$\rho^{\mathbf{Sign}}(\Sigma_B) = \Sigma_{B'}, \qquad \rho^{\mathbf{Mod}}_{\Sigma_B}(M_{B'}) = M_B,$$
$$\rho^{\mathbf{Sign}}(d) = d'.$$

We omit $\rho^{\mathbf{Sen}}_{\Sigma_A}$ and $\rho^{\mathbf{Sen}}_{\Sigma_B}$ in the above definition of $\rho$ because their domains are empty. The representation condition holds obviously.

$\rho$ satisfies the weak-$\mathscr{D}$-amalgamation for $\mathscr{D} = \mathbf{Sign}_I$, but does not satisfy the $\rho$-expansion property. The correspondence between models can be illustrated by the following diagram:




**Example 5.11.** Let $I$ and $I'$ be institutions without sentences, with the same categories of signatures, $\mathbf{Sign}_I$ and $\mathbf{Sign}_{I'}$, as in Example 5.10, and:

- model functors:

$$\mathbf{Mod}_I(\Sigma_A) = \{M_A\}, \qquad \mathbf{Mod}_{I'}(\Sigma_{A'}) = \{M^1_{A'}, M^2_{A'}\},$$
$$\mathbf{Mod}_I(\Sigma_B) = \{M_B\}, \qquad \mathbf{Mod}_{I'}(\Sigma_{B'}) = \{M_{B'}\},$$
$$\mathbf{Mod}_I(d)(M_B) = M_A, \qquad \mathbf{Mod}_{I'}(d')(M_{B'}) = M^1_{A'},$$

  where $\mathbf{Mod}_I(\Sigma_A)$, $\mathbf{Mod}_I(\Sigma_B)$, $\mathbf{Mod}_{I'}(\Sigma_{A'})$ and $\mathbf{Mod}_{I'}(\Sigma_{B'})$ are discrete categories;
- satisfaction relations $\models^{I}_{\Sigma_A}$, $\models^{I}_{\Sigma_B}$, $\models^{I'}_{\Sigma_{A'}}$ and $\models^{I'}_{\Sigma_{B'}}$ are empty.

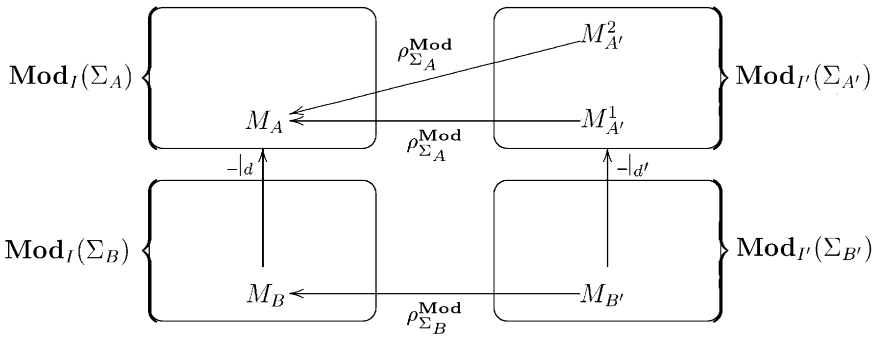
  Similarly as in Example 5.10, the satisfaction condition holds obviously for both institutions $I$ and $I'$.

  Now, we define the institution representation $\rho : I \to I'$ as follows:
- the functor $\rho^{\mathbf{Sign}}$ and the natural transformation $\rho^{\mathbf{Sen}}$ are defined as in Example 5.10;
- the natural transformation $\rho^{\mathbf{Mod}}$ is given as follows:

$$\rho^{\mathbf{Mod}}_{\Sigma_A}(M^1_{A'}) = M_A = \rho^{\mathbf{Mod}}_{\Sigma_A}(M^2_{A'}) \quad \text{and} \quad \rho^{\mathbf{Mod}}_{\Sigma_B}(M_{B'}) = M_B.$$

The representation condition holds obviously.

$\rho$ satisfies the $\rho$-expansion property, but does not satisfy the weak-$\mathscr{D}$-amalgamation property for $\mathscr{D} = \mathbf{Sign}_I$. The correspondence between models can be illustrated by the following diagram:




In the following definition we use the notion of institution representation to translate specifications along a given institution representation.

**Definition 5.12** (*Specification representation*). For any $(\mathscr{D},\mathscr{T})$-institution representation $\rho: I \to I'$, the *specification representation* $\hat{\rho}$ is a family of functions $\{\hat{\rho}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ between classes of specifications over $(\mathscr{D},\mathscr{T})$-institutions $I$ and $I'$ defined as follows:
1. if $SP$ is a $\Sigma$-specification of the form $\langle\Sigma,\Gamma\rangle$, then
   $\hat{\rho}_\Sigma(SP) = \langle\rho^{\mathbf{Sign}}(\Sigma), \rho^{\mathbf{Sen}}_\Sigma(\Gamma)\rangle$;
2. if $SP$ is a $\Sigma$-specification of the form $SP_1 \cup SP_2$, then
   $\hat{\rho}_\Sigma(SP) = \hat{\rho}_\Sigma(SP_1) \cup \hat{\rho}_\Sigma(SP_2)$;
3. if $SP$ is a $\Sigma$-specification of the form **translate** $SP_1$ **by** $(t: \Sigma_1 \to \Sigma)$, then
   $\hat{\rho}_\Sigma(SP) =$ **translate** $\hat{\rho}_{\Sigma_1}(SP_1)$ **by** $\rho^{\mathbf{Sign}}(t: \Sigma_1 \to \Sigma)$;
4. if $SP$ is a $\Sigma$-specification of the form **derive from** $SP_1$ **by** $(d: \Sigma \to \Sigma_1)$, then
   $\hat{\rho}_\Sigma(SP) =$ **derive from** $\hat{\rho}_{\Sigma_1}(SP_1)$ **by** $\rho^{\mathbf{Sign}}(d: \Sigma \to \Sigma_1)$,
   where $t \in \mathscr{T}_I$ and $d \in \mathscr{D}_I$.

For a $\Sigma$-specification $SP$ we will write $\hat{\rho}(SP)$ as an abbreviation for $\hat{\rho}_\Sigma(SP)$.

**Remark 5.13.** For any $(\mathscr{D},\mathscr{T})$-institution representation $\rho: I \to I'$ and $\Sigma$-specification $SP$ over $(\mathscr{D},\mathscr{T})$-institution $I$, $\hat{\rho}(SP)$ is a $\rho^{\mathbf{Sign}}(\Sigma)$-specification over $(\mathscr{D},\mathscr{T})$-institution $I'$.

**Theorem 5.14.** *For any $(\mathscr{D},\mathscr{T})$-institution representation $\rho: I \to I'$, $\Sigma \in |\mathbf{Sign}|$ and $\Sigma$-specification $SP$ over $(\mathscr{D},\mathscr{T})$-institution $I$, if $\rho^{\mathbf{Sign}}: \mathbf{Sign} \to \mathbf{Sign}'$ preserves pushouts then*

$$\mathbf{nf}(\hat{\rho}(SP)) = \hat{\rho}(\mathbf{nf}(SP))$$

**Proof.** By induction on the structure of specification $SP$.
1. $SP$ is a specification of the form $\langle\Sigma,\Gamma\rangle$. By Definition 5.12 we have $\mathbf{nf}(\hat{\rho}(\langle\Sigma,\Gamma\rangle))$ is equal to $\mathbf{nf}(\langle\rho^{\mathbf{Sign}}(\Sigma),\ \rho^{\mathbf{Sen}}(\Gamma)\rangle)$ and next by Definition 3.7 it is equal to **derive from**

$\langle \rho^{\mathbf{Sign}}(\Sigma),\ \rho^{\mathbf{Sen}}(\Gamma)\rangle$ **by** $id_{\rho^{\mathbf{Sign}}(\Sigma)}$. Again by Definition 5.12 we obtain that the last is equal to $\hat{\rho}$(**derive from** $\langle \Sigma, \Gamma\rangle$ **by** $id_\Sigma$) and finally to $\hat{\rho}(\mathbf{nf}(\langle \Sigma, \Gamma\rangle))$.

2. $SP$ is a specification of the form $SP_1 \cup SP_2$. Let us assume that $\mathbf{nf}(SP_i) =$ **derive from** $\langle \Sigma_i \Gamma_i\rangle$ **by** $d_i$) then $\mathbf{nf}(SP) =$ **derive from** $\langle \Sigma', t_1'\Gamma_2 \cup d_2'\Gamma_1\rangle$ **by** $d$, where $d = d_1;\ d_2'$ and $\Sigma', t_1' \in \mathscr{T}_I$ and $d_2' \in \mathscr{D}_I$ are given by the following pushout diagram in $\mathbf{Sign}_I$:

$$\begin{tikzcd} \Sigma \arrow[r, "d_1"] \arrow[d, "d_2"'] \arrow[dr, "d"] & \Sigma_1 \arrow[d, "d_2'"] \\ \Sigma_2 \arrow[r, "t_1'"'] & \Sigma' \end{tikzcd}$$

Now, by the induction hypothesis and Definition 5.12 we have

$$\mathbf{nf}(\hat{\rho}(SP_i)) = \hat{\rho}(\mathbf{nf}(SP_i)) = \textbf{derive from } \hat{\rho}(\langle \Sigma_i, \Gamma_i\rangle) \textbf{ by } \rho^{\mathbf{Sign}}(d_i) \quad \text{for } i = 1, 2.$$

Next by Definitions 3.7 and 5.12 and because $\rho$ preserves pushouts we obtain

$$\begin{aligned}\mathbf{nf}(\hat{\rho}(SP)) &= \mathbf{nf}(\hat{\rho}(SP_1) \cup \hat{\rho}(SP_2))\\ &= \textbf{derive from } \hat{\rho}(\langle \Sigma', t_1'\Gamma_2 \cup d_2'\Gamma_1\rangle) \textbf{ by } \rho^{\mathbf{Sign}}(d) = \hat{\rho}(\mathbf{nf}(SP)).\end{aligned}$$

3. $SP$ is a specification of the form **translate** $SP_1$ **by** $(t : \Sigma_1 \to \Sigma)$. Let us assume that $\mathbf{nf}(SP_1) =$ **derive from** $\langle \Sigma_1, \Gamma_1\rangle$ **by** $d_1$ then $\mathbf{nf}(SP) =$ **derive from** $\langle \Sigma', t'\Gamma_1\rangle$ **by** $d_1'$, where $\Sigma'$, $t' \in \mathscr{T}_I$ and $d_1' \in \mathscr{D}_I$ are given by a pushout in $\mathbf{Sign}_I$:

$$\begin{tikzcd} \mathbf{Sig}[SP_1] \arrow[r, "d_1"] \arrow[d, "t"'] & \Sigma_1 \arrow[d, "t'"] \\ \mathbf{Sig}[SP] \arrow[r, "d_1'"'] & \Sigma' \end{tikzcd}$$

By the induction hypothesis and Definition 5.12 we obtain

$$\mathbf{nf}(\hat{\rho}(SP_1)) = \hat{\rho}(\mathbf{nf}(SP_1)) = \textbf{derive from } \hat{\rho}(\langle \Sigma_1, \Gamma_1) \textbf{ by } \rho^{\mathbf{Sign}}(d_1).$$

Finally, by Definitions 3.7 and 5.12 and because $\rho$ preserves pushouts

$$\begin{aligned}\mathbf{nf}(\hat{\rho}(SP)) &= \mathbf{nf}(\textbf{translate } \hat{\rho}(SP_1) \textbf{ by } \rho^{\mathbf{Sign}}(t))\\ &= \textbf{derive from } \hat{\rho}(\langle \Sigma', (\rho^{\mathbf{Sign}}(t'))(\Gamma_1\rangle)) \textbf{ by } \rho^{\mathbf{Sign}}(d_1') = \hat{\rho}(\mathbf{nf}(SP)).\end{aligned}$$

4. $SP$ is a specification of the form **derive from** $SP_1$ **by** $(d:\Sigma\to\Sigma_1)$. By the induction hypothesis we have $\mathbf{nf}(\hat{\rho}(SP_1))=\hat{\rho}(\mathbf{nf}(SP_1))$. Let us assume that $\mathbf{nf}(SP_1)=$ **derive from** $\langle\Sigma_1,\Gamma_1\rangle$ **by** $d_1$ then

$$\mathbf{nf}(\hat{\rho}(SP_1)) = \textbf{derive from } \hat{\rho}(\langle\Sigma_1,\Gamma_1\rangle) \textbf{ by } \rho^{\mathbf{Sign}}(d_1).$$

Now by Definitions 3.7 and 5.12 we obtain

$$\begin{aligned}&\mathbf{nf}(\textbf{derive from } \hat{\rho}(SP_1) \textbf{ by } \rho^{\mathbf{Sign}}(d))\\ &\quad = \textbf{derive from } \hat{\rho}(\langle\Sigma_1,\Gamma_1\rangle) \textbf{ by } \rho^{\mathbf{Sign}}(d;d_1) = \hat{\rho}(\mathbf{nf}(SP)). \qquad \square\end{aligned}$$

**Corollary 5.15.** *For any $(\mathscr{D},\mathscr{T})$-institution representation $\rho:I\to I'$, $\Sigma\in|\mathbf{Sign}|$ and $\Sigma$-specification $SP$ over $(\mathscr{D},\mathscr{T})$-institution $I$, if $\rho^{\mathbf{Sign}}:\mathbf{Sign}\to\mathbf{Sign}'$ preserves pushouts and $I'$ satisfies the weak-$(\mathscr{D},\mathscr{T})$-amalgamation property then*

$$\hat{\rho}(SP)\cong\hat{\rho}(\mathbf{nf}(SP))$$

**Proof.** By Theorems 3.8 and 5.14 we have $\hat{\rho}(SP)\cong\mathbf{nf}(\hat{\rho}(SP))=\hat{\rho}(\mathbf{nf}(SP))$. $\square$

It follows from the above corollary that to verify the equality of model classes of specifications $\hat{\rho}(SP_1)$ and $\hat{\rho}(SP_2)$, where $\rho$ satisfies assumptions of Corollary 5.15, it is enough to verify the equality of model classes of specifications in the normal form $\hat{\rho}(\mathbf{nf}(SP_1))$ and $\hat{\rho}(\mathbf{nf}(SP_1))$.

## 6. Mod[$SP$] vs. Mod[$\hat{\rho}(SP)$]

In this section we study mutual relationships between models of a given specification $SP$ and the specification $\hat{\rho}(SP)$. We assume that $\rho:I\to I'$ is an arbitrary but fixed $(\mathscr{D},\mathscr{T})$-institution representation.

In the first part we show that inclusion

$$\rho_{\Sigma}^{\mathbf{Mod}}(\mathbf{Mod}[\hat{\rho}(SP)])\subseteq\mathbf{Mod}[SP]$$

holds "for free"—that is, we need just the representation condition to ensure it. The inclusion in the opposite direction is more difficult. As demonstrated in [9, 30], for flat specifications the inclusion in the opposite direction holds if the institution representation $\rho$ has the $\rho$-expansion property. We show that this result can be extended to structured specification provided that $\rho$ additionally satisfies the weak-$\mathscr{D}$-amalgamation property.

**Lemma 6.1.** *For any $(\mathscr{D},\mathscr{T})$-institutions $I$ and $I'$, $(\mathscr{D},\mathscr{T})$-institution representation $\rho:I\to I'$, signature $\Sigma\in|\mathbf{Sign}|$, $\Sigma$-specification $SP$ over the $(\mathscr{D},\mathscr{T})$-institution $I$ and $\rho^{\mathbf{Sign}}(\Sigma)$-model $M'\in\mathbf{Mod}[\hat{\rho}(SP)]$, we have*

$$\rho_{\Sigma}^{\mathbf{Mod}}(M')\in\mathbf{Mod}[SP].$$

**Proof.** By induction on the structure of the specification $SP$. Let us assume that $M' \in \mathbf{Mod}[\hat{\rho}(SP)]$.

1. *If* $SP = \langle \Sigma, \Gamma \rangle$: By assumption and Definition 5.12 we have $M' \in \mathbf{Mod}[\langle \rho^{\mathbf{Sign}}(\Sigma), \rho_\Sigma^{\mathbf{Sen}}(\Gamma)\rangle]$. This is equivalent to $M' \models'_{\rho^{\mathbf{Sign}}(\Sigma)} \rho_\Sigma^{\mathbf{Sen}}(\Gamma)$ and by the representation condition to $\rho_\Sigma^{\mathbf{Mod}}(M') \models_\Sigma \Gamma$, which yields $\rho_\Sigma^{\mathbf{Mod}}(M') \in \mathbf{Mod}[\langle \Sigma, \Gamma \rangle]$.
2. *If* $SP = SP_1 \cup SP_2$: By assumption, $M' \in \mathbf{Mod}[\hat{\rho}(SP_1)] \cap \mathbf{Mod}[\hat{\rho}(SP_2)]$. Now, by the induction hypothesis, we obtain $\rho_\Sigma^{\mathbf{Mod}}(M') \in \mathbf{Mod}[SP_1] \cap \mathbf{Mod}[SP_2] = \mathbf{Mod}[SP]$.
3. *If* $SP = \mathbf{translate}\, SP_1\, \mathbf{by}\ (t : \Sigma_1 \to \Sigma)$: By assumption, $M' \in \mathbf{Mod}[\mathbf{translate}\, \hat{\rho}(SP_1)\, \mathbf{by}\ \rho^{\mathbf{Sign}}(t)]$. By definition, $M'|_{\rho^{\mathbf{Sign}}(t)} \in \mathbf{Mod}[\hat{\rho}(SP_1)]$. Now, by the induction hypothesis

$$\rho_{\Sigma_1}^{\mathbf{Mod}}(M'|_{\rho^{\mathbf{Sign}}(t)}) \in \mathbf{Mod}[SP_1]$$

and because the following diagram commutes

$$\begin{tikzcd}
\mathbf{Mod}(\Sigma) \arrow[d, "-|_t"'] & \mathbf{Mod}'(\rho^{\mathbf{Sign}}(\Sigma)) \arrow[l, "\rho_\Sigma^{\mathbf{Mod}}"'] \arrow[d, "-|_{\rho^{\mathbf{Sign}}(t)}"] \\
\mathbf{Mod}(\Sigma_1) & \mathbf{Mod}'(\rho^{\mathbf{Sign}}(\Sigma_1)) \arrow[l, "\rho_{\Sigma_1}^{\mathbf{Mod}}"]
\end{tikzcd} \tag{5}$$

we have $\rho_{\Sigma_1}^{\mathbf{Mod}}(M'|_{\rho^{\mathbf{Sign}}(t)}) = (\rho_\Sigma^{\mathbf{Mod}}(M'))|_t$, hence $\rho_\Sigma^{\mathbf{Mod}}(M')|_t \in \mathbf{Mod}[SP_1]$ and finally $\rho_\Sigma^{\mathbf{Mod}}(M') \in \mathbf{Mod}[SP]$.
4. *If* $SP = \mathbf{derive\ from}\ SP_1\ \mathbf{by}\ d : \Sigma \to \Sigma_1$: By assumption, $M' \in \mathbf{Mod}[\mathbf{derive\ from}\, \hat{\rho}(SP_1)\ \mathbf{by}\ \rho^{\mathbf{Sign}}(d)]$. Now, there exists $M'' \in \mathbf{Mod}[\hat{\rho}(SP_1)]$ such that $M''|_{\rho^{\mathbf{Sign}}(d)} = M'$. By the induction hypothesis, $\rho_{\Sigma_1}^{\mathbf{Mod}}(M'') \in \mathbf{Mod}[SP_1]$. Then, since $\rho_{\Sigma_1}^{\mathbf{Mod}}(M'')|_d \in \mathbf{Mod}[SP]$ and because the following diagram commutes:

$$\begin{tikzcd}
\mathbf{Mod}(\Sigma_1) \arrow[d, "-|_d"'] & \mathbf{Mod}'(\rho^{\mathbf{Sign}}(\Sigma_1)) \arrow[l, "\rho_{\Sigma_1}^{\mathbf{Mod}}"'] \arrow[d, "-|_{\rho^{\mathbf{Sign}}(d)}"] \\
\mathbf{Mod}(\Sigma) & \mathbf{Mod}'(\rho^{\mathbf{Sign}}(\Sigma)) \arrow[l, "\rho_\Sigma^{\mathbf{Mod}}"]
\end{tikzcd} \tag{6}$$

we have $\rho_{\Sigma_1}^{\mathbf{Mod}}(M'')|_d = \rho_\Sigma^{\mathbf{Mod}}(M''|_{\rho^{\mathbf{Sign}}(d)}) = \rho_\Sigma^{\mathbf{Mod}}(M') \in \mathbf{Mod}[SP]$. $\square$

As a consequence we obtain:

**Corollary 6.2.** *For any* $(\mathscr{D}, \mathscr{T})$*-institutions* $I$ *and* $I'$, $(\mathscr{D}, \mathscr{T})$*-institution representation* $\rho : I \to I'$, *signature* $\Sigma \in |\mathbf{Sign}|$ *and* $\Sigma$*-specification* $SP$ *over* $(\mathscr{D}, \mathscr{T})$*-institution* $I$, *we have*:

$$\rho_\Sigma^{\mathbf{Mod}}(\mathbf{Mod}[\hat{\rho}(SP)]) \subseteq \mathbf{Mod}[SP].$$

Now, we present proof of the inclusion in the opposite direction. Let us start with a simpler result for which we do not need $\rho$-expansion.

**Lemma 6.3.** *For any $(\mathscr{D},\mathscr{T})$-institutions $I$ and $I'$ and $(\mathscr{D},\mathscr{T})$-institution representation $\rho : I \to I'$, if $\rho$ satisfies the weak-$\mathscr{D}$-amalgamation then for every signature $\Sigma \in |\mathbf{Sign}|$, $\Sigma$-specification $SP$ over the $(\mathscr{D},\mathscr{T})$-institution $I$ and $\rho^{\mathbf{Sign}}(\Sigma)$-model $M'$*

$$\rho_\Sigma^{\mathbf{Mod}}(M') \in \mathbf{Mod}[SP] \quad \textit{implies} \quad M' \in \mathbf{Mod}[\hat{\rho}(SP)].$$

**Proof.** By induction on the structure of $SP$.

1. *If $SP = \langle \Sigma, \Gamma \rangle$*: By assumption, $\rho_\Sigma^{\mathbf{Mod}}(M') \in \mathbf{Mod}[\langle \Sigma, \Gamma \rangle]$. It is equivalent to $\rho_\Sigma^{\mathbf{Mod}}(M') \models_\Sigma \Gamma$. By the representation condition, we obtain $M' \models'_{\rho^{\mathbf{Sign}(\Sigma)}} \rho_\Sigma^{\mathbf{Sen}}(\Gamma)$, which is equivalent to $M' \in \mathbf{Mod}[\hat{\rho}(\langle \Sigma, \Gamma \rangle)]$.
2. If $SP = SP_1 \cup SP_2$: By assumption, $\rho_\Sigma^{\mathbf{Mod}}(M') \in \mathbf{Mod}[SP_1] \cap \mathbf{Mod}[SP_2]$. Next, by the induction hypothesis, we obtain $M' \in \mathbf{Mod}[\hat{\rho}(SP_1)] \cap \mathbf{Mod}[\hat{\rho}(SP_2)] = \mathbf{Mod}[\hat{\rho}(SP_1 \cup SP_2)]$.
3. *If $SP =$ **translate** $SP_1$ **by** $(t : \Sigma_1 \to \Sigma)$*: By assumption, $\rho_\Sigma^{\mathbf{Mod}}(M') \in \mathbf{Mod}[\mathbf{translate} SP_1$ **by** $t]$. Next, by Definition 3.1, $\rho_\Sigma^{\mathbf{Mod}}(M')|_t \in \mathbf{Mod}[SP_1]$, which by the commutativity of the diagram (5) (see the proof of Lemma 6.1) is equivalent to $\rho_{\Sigma_1}^{\mathbf{Mod}}(M'|_{\rho^{\mathbf{Sign}}(t)}) \in \mathbf{Mod}[SP_1]$. Now, by the induction hypothesis we obtain
$M' \in \mathbf{Mod}[\mathbf{translate}\, \hat{\rho}(SP_1)$ **by** $\rho^{\mathbf{Sign}}(t)] = \mathbf{Mod}[\hat{\rho}(\mathbf{translate}\, SP_1$ **by** $t)]$.
4. *If $SP =$ **derive from** $SP_1$ **by** $(d : \Sigma \to \Sigma_1)$*: By assumption, $\rho_\Sigma^{\mathbf{Mod}}(M') \in \mathbf{Mod}[$**derive from** $SP_1$ **by** $d]$. There exists $M_1 \in \mathbf{Mod}[SP_1]$ such that $M_1|_d = \rho^{\mathbf{Mod}}(M')$. Now, because $\rho$ has the weak-$\mathscr{D}$-amalgamation property, there exists $M_1' \in |\mathbf{Mod}(\rho^{\mathbf{Sign}}(\Sigma_1))|$ such that $\rho_{\Sigma_1}^{\mathbf{Mod}}(M_1') = M_1$ and $M_1'|_{\rho^{\mathbf{Sign}}(d)} = M'$ (see diagram (6) from the proof of Lemma 6.1). By the induction hypothesis, we obtain $M_1' \in \mathbf{Mod}[\hat{\rho}(SP_1)]$, and finally $M_1'|_{\rho^{\mathbf{Sign}}(d)} \in \mathbf{Mod}[\hat{\rho}(SP)]$. $\square$

In the next step we just add assumption about $\rho$-expansion and obtain expected inclusion.

**Lemma 6.4.** *For any $(\mathscr{D},\mathscr{T})$-institutions $I$ and $I'$ and $(\mathscr{D},\mathscr{T})$-institution representation $\rho : I \to I'$, if $\rho$ has the weak-$\mathscr{D}$-amalgamation property, then for every signature $\Sigma \in |\mathbf{Sign}|$, $\Sigma$-specification $SP$ over the $(\mathscr{D},\mathscr{T})$-institution $I$, if each model $M \in \mathbf{Mod}[SP]$ has a $\rho$-expansion to a $\rho^{\mathbf{Sign}}(\Sigma)$-model, then*

$$\mathbf{Mod}[SP] \subseteq \rho_\Sigma^{\mathbf{Mod}}(\mathbf{Mod}[\hat{\rho}(SP)]).$$

**Proof.** Let $M \in \mathbf{Mod}[SP]$. By the $\rho$-expansion, there exists a $\rho^{\mathbf{Sign}}(\Sigma)$-model $M'$ such that $\rho_\Sigma^{\mathbf{Mod}}(M') = M$. By Lemma 6.3, we have $M' \in \mathbf{Mod}[\hat{\rho}(SP)]$. Finally, since $\rho_\Sigma^{\mathbf{Mod}}(M') = M$, we have $M \in \rho_\Sigma^{\mathbf{Mod}}(\mathbf{Mod}[\hat{\rho}(SP)])$. $\square$

In the following example, we show that the inclusion considered in Lemma 6.4 does not hold for institution representation $\rho_{\mathbf{PEQ}\to\mathbf{EQ}}$ defined in Example 5.4.

**Example 6.5.** In this example we will write $\rho$ for the institution representation $\rho_{\mathbf{PEQ}\to\mathbf{EQ}}$. Let $SP=\langle\Sigma,\emptyset\rangle$ be a $\Sigma$-specification over the $(\mathscr{D},\mathscr{T})$-institution **PEQ**, where $\Sigma\in|\mathbf{Sign_{PEQ}}|$. Let us notice that every $M\in|\mathbf{Mod_{PEQ}}(\Sigma)|$ is a model of $SP$. It means that $\mathbf{Mod}[SP]=|\mathbf{Mod_{PEQ}}(\Sigma)|$. Similarly, $\mathbf{Mod}[\hat{\rho}(SP)]=|\mathbf{Mod_{EQ}}(\rho^{\mathbf{Sign}}(\Sigma))|$. Now, by definition of $\rho$, we have that $\mathbf{Mod}[SP]$ is not included in $\rho_\Sigma^{\mathbf{Mod}}(\mathbf{Mod}[\hat{\rho}(SP)])$.

The next example shows that the weak-$\mathscr{D}$-amalgamation property is really crucial for the inclusion presented in Lemma 6.4 and for Lemma 6.3.

**Example 6.6.** Let $I$ and $I'$ be institutions defined as in Example 5.11, except that

- categories of signatures $\mathbf{Sign}_I$ and $\mathbf{Sign}_{I'}$ have additional objects $\Sigma_C$ and $\Sigma_{C'}$ and additional arrows $d_1:\Sigma_A\to\Sigma_C$ and $d_1':\Sigma_{A'}\to\Sigma_{C'}$, respectively;
- sentence functor is given as follows:

$$\mathbf{Sen}_I(\Sigma_A)=\emptyset=\mathbf{Sen}_I(\Sigma_B),\qquad \mathbf{Sen}_{I'}(\Sigma_{A'})=\emptyset=\mathbf{Sen}_{I'}(\Sigma_{B'}),$$
$$\mathbf{Sen}_I(\Sigma_C)=\{\varphi\},\qquad \mathbf{Sen}_{I'}(\Sigma_{C'})=\{\varphi'\};$$

- model functor is given now as follows:

$$\mathbf{Mod}_I(\Sigma_A)=\{M_A\},\qquad \mathbf{Mod}_{I'}(\Sigma_{A'})=\{M_{A'}^1,M_{A'}^2\},$$
$$\mathbf{Mod}_I(\Sigma_B)=\{M_B\},\qquad \mathbf{Mod}_{I'}(\Sigma_{B'})=\{M_{B'}\},$$
$$\mathbf{Mod}_{I'}(\Sigma_{C'})=\{M_{C'}^1,M_{C'}^2\},\qquad \mathbf{Mod}_I(\Sigma_C)=\{M_C^1,M_C^2\},$$
$$\mathbf{Mod}_{I'}(d')(M_{B'})=M_{A'}^1,\qquad \mathbf{Mod}_I(d)(M_B)=M_A,$$
$$\mathbf{Mod}_I(d_1)(M_C^i)=M_A\quad\text{for } i=1,2,\qquad \mathbf{Mod}_{I'}(d_1')(M_{C'}^i)=M_{A'}^i\quad\text{for } i=1,2;$$

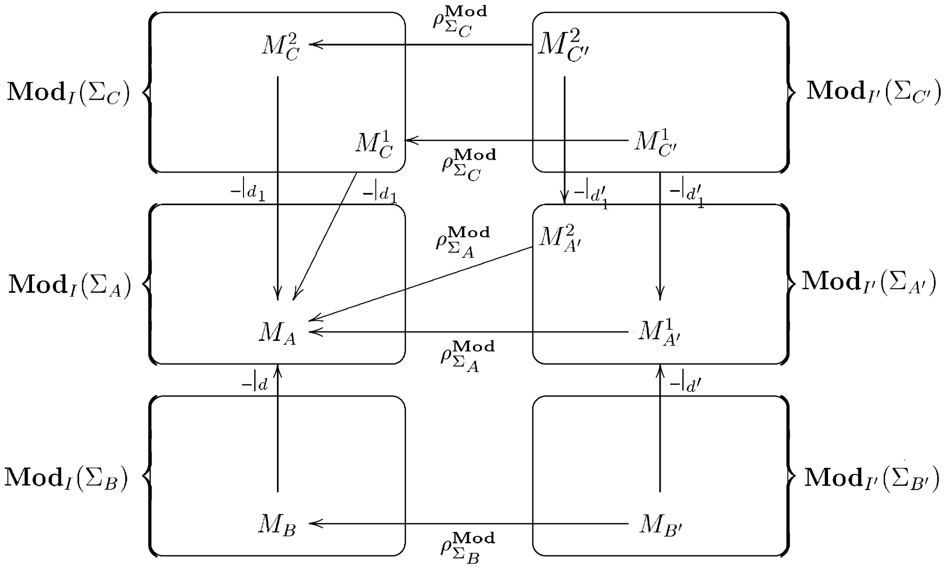
- satisfaction relations $\models^I_{\Sigma_A}$, $\models^I_{\Sigma_B}$, $\models^{I'}_{\Sigma_{A'}}$ and $\models^{I'}_{\Sigma_{B'}}$ are empty, and $\models^I_{\Sigma_C}$ and $\models^{I'}_{\Sigma_{C'}}$ are given as follows:

$$M_C^1\not\models^I_{\Sigma_C}\varphi\qquad M_{C'}^1\not\models^I_{\Sigma_{C'}}\varphi'$$
$$M_C^2\models^I_{\Sigma_C}\varphi\qquad M_{C'}^2\models^I_{\Sigma_{C'}}\varphi'.$$

The satisfaction condition holds trivially for both institutions $I$ and $I'$. The institution representation $\rho:I\to I'$ is defined as follows:

$$\rho^{\mathbf{Sign}}(\Sigma_A)=\Sigma_{A'},\qquad \rho^{\mathbf{Sen}}_{\Sigma_C}(\varphi)=\varphi',\qquad \rho^{\mathbf{Mod}}_{\Sigma_A}(M_{A'}^i)=M_A\qquad\text{for } i=1,2,$$
$$\rho^{\mathbf{Sign}}(\Sigma_B)=\Sigma_{B'},\qquad \rho^{\mathbf{Sign}}(d)=d',\qquad \rho^{\mathbf{Mod}}_{\Sigma_B}(M_{B'})=M_B,$$
$$\rho^{\mathbf{Sign}}(\Sigma_C)=\Sigma_{C'},\qquad \rho^{\mathbf{Sign}}(d_1)=d_1',\qquad \rho^{\mathbf{Mod}}_{\Sigma_A}(M_{C'}^i)=M_C^i\qquad\text{for } i=1,2.$$

The representation condition holds (trivially) as well. $\rho$ satisfies the $\rho$-expansion property, whereas does not satisfy the weak-$\mathscr{D}$-amalgamation for $\mathscr{D}=\mathbf{Sign}_I$ (there is no model $M^?_{B'} \in \mathbf{Mod}_{I'}(\Sigma_{B'})$ with $M^?_{B'}|_{d'} = M^2_{A'}$ and $\rho^{\mathbf{Mod}}_{\Sigma_B}(M^?_{B'}) = M_B$). The correspondence between models can be illustrated by the following diagram:




Now, let $SP = (\textbf{derive from } \langle \Sigma_B, \emptyset \rangle \textbf{ by } d) \cup (\textbf{derive from } \langle \Sigma_C, \{\varphi\} \rangle \textbf{ by } d_1)$. Then $\hat{\rho}\ (SP) = (\textbf{derive from} \langle \Sigma_{B'}, \emptyset \rangle \textbf{ by } d') \cup (\textbf{derive from } \langle \Sigma_{C'}, \{\varphi'\} \rangle \textbf{ by } d'_1)$, and by definitions

$$\mathbf{Mod}[SP] = \{M_A\} \cap \{M_A\} = \{M_A\} \quad \text{and} \quad \mathbf{Mod}[\hat{\rho}(SP)] = \{M^1_{A'}\} \cap \{M^2_{A'}\} = \emptyset.$$

Finally, $\mathbf{Mod}[SP] = \{M_A\} \not\subseteq \emptyset = \rho^{\mathbf{Mod}}_{\Sigma_A}(\mathbf{Mod}[\hat{\rho}(SP)])$ and also $\rho^{\mathbf{Mod}}_{\Sigma_A}(M^1_{A'}) \in \mathbf{Mod}[SP]$ does not imply $M^1_{A'} \in \mathbf{Mod}[\hat{\rho}(SP)]$.

**Corollary 6.7.** *For any $(\mathscr{D}, \mathscr{T})$-institutions $I$ and $I'$ and $(\mathscr{D}, \mathscr{T})$-institution representation $\rho : I \to I'$, if $\rho$ has the weak-$\mathscr{D}$-amalgamation property, then for every signature $\Sigma \in |\mathbf{Sign}|$, $\Sigma$-specification $SP$ over the $(\mathscr{D}, \mathscr{T})$-institution $I$, if each model $M \in \mathbf{Mod}[SP]$ has a $\rho$-expansion to a $\rho^{\mathbf{Sign}}(\Sigma)$-model, then*

$$\mathbf{Mod}[SP] = \rho^{\mathbf{Mod}}_{\Sigma}(\mathbf{Mod}[\hat{\rho}(SP)]) \quad \textit{and} \quad (\rho^{\mathbf{Mod}}_{\Sigma})^{-1}(\mathbf{Mod}[SP]) = \mathbf{Mod}[\hat{\rho}(SP)].$$

## 7. Reusing proof systems

Results presented in this section are consequences of the results presented in the previous section. The first result for the case of flat specifications was presented in [9] for institution maps and also in [10] for institution representations.

**Theorem 7.1.** *For any $(\mathscr{D}, \mathscr{T})$-institutions $I$ and $I'$ and $(\mathscr{D}, \mathscr{T})$-institution representation $\rho : I \to I'$, if $\rho$ has the weak-$\mathscr{D}$-amalgamation property, then for every signature*

$\Sigma \in |\mathbf{Sign}|$, *$\Sigma$-specification $SP$ over $(\mathscr{D},\mathscr{T})$-institution $I$ and $\Sigma$-sentence $\varphi$, if each $\Sigma$-model $M \in \mathbf{Mod}[SP]$ has a $\rho$-expansion to a $\rho^{\mathbf{Sign}}(\Sigma)$-model, then*

$$SP \models_\Sigma \varphi \quad \textit{iff} \quad \hat{\rho}(SP) \models'_{\rho^{\mathbf{Sign}}(\Sigma)} \rho^{\mathbf{Sen}}_\Sigma(\varphi).$$

**Proof.** $\Rightarrow$: Let $M' \in \mathbf{Mod}[\hat{\rho}(SP)]$. Then $\rho^{\mathbf{Mod}}_\Sigma(M') \in \rho^{\mathbf{Mod}}_\Sigma(\mathbf{Mod}[\hat{\rho}(SP)])$. By Lemma 6.1, we obtain $\rho^{\mathbf{Mod}}_\Sigma(M') \in \mathbf{Mod}[SP]$, which implies $\rho^{\mathbf{Mod}}_\Sigma(M') \models_\Sigma \varphi$. By the representation condition, this is equivalent to $M' \models'_{\rho^{\mathbf{Sign}}(\Sigma)} \rho^{\mathbf{Sen}}_\Sigma(\varphi)$.

$\Leftarrow$: Let $M \in \mathbf{Mod}[SP]$. Then by Lemma 6.4 $M \in \rho^{\mathbf{Mod}}_\Sigma(\mathbf{Mod}[\hat{\rho}(SP)])$. This means that there exists $M' \in \mathbf{Mod}[\hat{\rho}(SP)]$ such that $\rho^{\mathbf{Mod}}_\Sigma(M') = M$ and $M' \models'_{\rho^{\mathbf{Sign}}(\Sigma)} \rho^{\mathbf{Sen}}_\Sigma(\varphi)$. By the representation condition, we obtain $\rho^{\mathbf{Mod}}_\Sigma(M') \models_\Sigma \varphi$, that is $M \models_\Sigma \varphi$. $\square$

In the following example we demonstrate that the weak-$\mathscr{D}$-amalgamation property is crucial for Theorem 7.1.

**Example 7.2.** Let $I$ and $I'$ be institutions defined as in Example 6.6 except

- sentence functors, defined now as follows:

$$\begin{aligned} &\mathbf{Sen}_I(\Sigma_A) = \{\mathit{false}\} = \mathbf{Sen}_I(\Sigma_B), && \mathbf{Sen}_{I'}(\Sigma_{A'}) = \{\mathit{false}\} = \mathbf{Sen}_{I'}(\Sigma_{B'}),\\ &\mathbf{Sen}_I(\Sigma_C) = \{\varphi, \mathit{false}\}, && \mathbf{Sen}_{I'}(\Sigma_{C'}) = \{\varphi', \mathit{false}\},\\ &\mathbf{Sen}_I(\sigma)(\mathit{false}) = \mathit{false}, && \mathbf{Sen}_{I'}(\sigma')(\mathit{false}) = \mathit{false}, \end{aligned}$$

  for $\sigma \in \{d, d_1\}$ and $\sigma' \in \{d', d'_1\}$;
- and satisfaction relations, which are the same as in Example 6.6, except that the sentence *false* is not satisfied by any model.

It is easy to check that the satisfaction condition holds for both institutions $I$ and $I'$.

We define the institution representation $\rho : I \to I'$ in the same way as in Example 6.6, except that the sentence part is defined now as follows:

$$\rho^{\mathbf{Sen}}_{\Sigma_A}(\mathit{false}) = \mathit{false}, \quad \rho^{\mathbf{Sen}}_{\Sigma_B}(\mathit{false}) = \mathit{false}, \quad \rho^{\mathbf{Sen}}_{\Sigma_C}(\mathit{false}) = \mathit{false}, \quad \rho^{\mathbf{Sen}}_{\Sigma_C}(\varphi) = \varphi'.$$

The representation condition holds as well. Because the correspondence between models is the same as in Example 6.6, $\rho$ satisfies the $\rho$-expansion property, whereas does not the weak-$\mathscr{D}$-amalgamation for $\mathscr{D} = \mathbf{Sign}_I$.

Now, let $SP$ be the specifications defined in Example 6.6, then we have

$$\hat{\rho}(SP) \models'_{\rho^{\mathbf{Sign}}(\Sigma_A)} \rho^{\mathbf{Sen}}_{\Sigma_A}(\mathit{false}) \quad \text{but} \quad SP \not\models_{\Sigma_A} \mathit{false}.$$

Let us see what the advantages of Theorem 7.1 are. First of all, it ensures soundness of the following scheme of rules:

$$(\rho\text{-join-entailment}) \quad \frac{\hat{\rho}(SP) \vdash'_{\rho^{\mathbf{Sign}}(\Sigma)} \rho^{\mathbf{Sen}}_\Sigma(\varphi)}{SP \vdash_\Sigma \varphi}$$

where $\rho$ and $SP$ satisfy the assumptions of Theorem 7.1. Now, let us assume that we have

1. A sound and complete set of rules for proving logical consequences of specifications over $(\mathscr{D},\mathscr{T})$-institution $I'$.
2. A $(\mathscr{D},\mathscr{T})$-institution representation: $\rho : I \to I'$ satisfying assumptions of Theorem 7.1.

We can construct a sound and complete set of rules for the logical system for reasoning about specifications over $(\mathscr{D},\mathscr{T})$-institution $I$ from rules from point 1 and the ($\rho$-join-entailment) rule schema for $\rho$ from point 2.

In the following example we demonstrate how to use such a proof technique in practice.

**Example 7.3.** In this example we use the ($\rho$-join-entailment) rule schema to prove judgment $SP \vdash_{\Sigma_1} b = c$ from Example 3.17. Let us notice that the institution representation $\rho_{\mathbf{EQ}\to\mathbf{FOEQ}}$ defined in Example 5.3 satisfies assumptions of Theorem 7.1. We will write $\rho$ as an abbreviation for $\rho_{\mathbf{EQ}\to\mathbf{FOEQ}}$.

The following tree makes the proof:

$$
(\mathrm{CR})\ \dfrac{\dfrac{(1)}{\hat{\rho}(SP) \vdash'_{\rho^{\mathbf{Sign}}(\Sigma_1)} \forall_{x:s}.\rho^{\mathbf{Sen}}_{\Sigma_1}(b = c)} \quad \dfrac{(2)}{\hat{\rho}(SP) \vdash'_{\rho^{\mathbf{Sign}}(\Sigma_1)} \exists_{x:s}.\mathit{true}} \quad (3)}{(\rho\text{-join-entailment})\ \dfrac{\hat{\rho}(SP) \vdash'_{\rho^{\mathbf{Sign}}(\Sigma_1)} \rho^{\mathbf{Sen}}_{\Sigma_1}(b = c)}{SP \vdash_{\Sigma_1} b = c}}
$$

where (1) is

$$
(\mathrm{sum2})\ \dfrac{(\mathrm{basic})\ \dfrac{\overline{\forall_{x:s}.\rho^{\mathbf{Sen}}_{\Sigma_1}(b = c) \in \{\rho^{\mathbf{Sen}}_{\Sigma_1}(\forall_{x:s}.b = c)\}}}{\hat{\rho}(\langle \Sigma_1, \forall_{x:s}.b = c\rangle) \vdash'_{\rho^{\mathbf{Sign}}(\Sigma_1)} \forall_{x:s}.\rho^{\mathbf{Sen}}_{\Sigma_1}(b = c)}}{\hat{\rho}(SP_1) \cup \hat{\rho}(SP_2) \vdash'_{\rho^{\mathbf{Sign}}(\Sigma_1)} \forall_{x:s}.\rho^{\mathbf{Sen}}_{\Sigma_1}(b = c)}
$$

(2) is

$$
(\mathrm{sum1})\ \dfrac{(\mathrm{derive})\ \dfrac{(\mathrm{CR})\ \dfrac{\dfrac{\vdots}{\emptyset \vdash^{\mathbf{FOEQ}}_{\rho^{\mathbf{Sign}}(\Sigma_0)} \exists_{x:s}.\mathit{true}}}{\hat{\rho}(\langle\{\{s, s'\}, \{a : s, b, c : s'\}\}, \emptyset\rangle) \vdash'_{\rho^{\mathbf{Sign}}(\Sigma_0)} \exists_{x:s}.\mathit{true}}}{\textbf{derive from } \hat{\rho}(SP_0) \textbf{ by } \rho^{\mathbf{Sign}}(\imath) \vdash'_{\rho^{\mathbf{Sign}}(\Sigma_1)} \exists_{x:s}.\mathit{true}}}{\hat{\rho}(SP_1) \cup \hat{\rho}(SP_2) \vdash'_{\rho^{\mathbf{Sign}}(\Sigma_1)} \exists_{x:s}.\mathit{true}}
$$

and finally, (3) is

$$
\overline{\{\forall_{x:s}.\rho^{\mathbf{Sen}}_{\Sigma_1}(b = c), \exists_{x:s}.\mathit{true}\} \vdash^{\mathbf{FOEQ}}_{\rho^{\mathbf{Sign}}(\Sigma_1)} \rho^{\mathbf{Sen}}_{\Sigma_1}(b = c)}
$$

Similar reasoning as presented in this example can be repeated for Example 3.18.

Having the ($\rho$-join-entailment) rule schema we can show even something slightly more general:

**Example 7.4.** Let $\rho_{\mathbf{EQ}\to\mathbf{FOEQ}}$ be the $(\mathscr{D},\mathscr{T})$-institution representation defined in Example 5.3, $\Sigma\in|\mathbf{Sign_{EQ}}|$, $SP$ be any $\Sigma$-specification over the $(\mathscr{D},\mathscr{T})$-institution **EQ** and $\varphi$ be a $\Sigma$-sentence, then by Theorem 7.1 and completeness of the logical system for structured specifications over the $(\mathscr{D},\mathscr{T})$-institution **FOEQ** (see [8] and also Theorem 3.9) we have

$$SP \models_\Sigma \varphi \quad \text{implies} \quad SP \vdash_\Sigma \varphi$$

in $(\mathscr{D},\mathscr{T})$-institution **EQ**, which means that every theorem of $SP$ can be proved, where $\vdash_\Sigma$ is the entailment relation defined by Definition 3.4 extended by the ($\rho$-join-entailment) rule schema.

The next theorem allows us to repeat the above argument also for the refinement relation.

**Theorem 7.5.** *For any $(\mathscr{D},\mathscr{T})$-institutions $I$ and $I'$ and $(\mathscr{D},\mathscr{T})$-institution representation $\rho: I\to I'$, if $\rho$ has the weak-$\mathscr{D}$-amalgamation property, then for every signature $\Sigma\in|\mathbf{Sign}|$ and $\Sigma$-specifications $SP_1$ and $SP_2$ over $(\mathscr{D},\mathscr{T})$-institution $I$, if any $\Sigma$-model has $\rho$-expansion to a $\rho^{\mathbf{Sign}}(\Sigma)$-model, then*

$$SP_1 \rightsquigarrow_\Sigma SP_2 \quad \textit{iff} \;\; \hat{\rho}(SP_1) \rightsquigarrow_{\rho^{\mathbf{Sign}}(\Sigma)} \hat{\rho}(SP_2).$$

**Proof.** $\Rightarrow$: Assumption $\mathbf{Mod}[SP_2]\subseteq\mathbf{Mod}[SP_1]$. Let $M\in\mathbf{Mod}[\hat{\rho}(SP_2)]$. Then by Lemma 6.1, we have $\rho_\Sigma^{\mathbf{Mod}}(M)\in\mathbf{Mod}[SP_2]$ and next, by the assumption $\rho_\Sigma^{\mathbf{Mod}}(M)\in\mathbf{Mod}[SP_1]$. Now, by Lemma 6.3, we obtain $M\in\mathbf{Mod}[\hat{\rho}(SP_1)]$.

$\Leftarrow$: Assumption $\mathbf{Mod}[\hat{\rho}(SP_2)]\subseteq\mathbf{Mod}[\hat{\rho}(SP_1)]$. By Corollary 6.7 and monotonicity of the image function,

$$\mathbf{Mod}[SP_2] = \rho_\Sigma^{\mathbf{Mod}}(\mathbf{Mod}[\hat{\rho}(SP_2)]) \subseteq \rho_\Sigma^{\mathbf{Mod}}(\mathbf{Mod}[\hat{\rho}(SP_1)]) = \mathbf{Mod}[SP_1]. \qquad \square$$

The weak-$\mathscr{D}$-amalgamation property is also crucial for the above theorem.

**Example 7.6.** Let $SP$ be the specification and $\Sigma_A$ be the signature, both defined in Example 6.6, then

$$SP \rightsquigarrow_{\Sigma_A} \langle\Sigma_A,\emptyset\rangle \quad \text{but} \quad \hat{\rho}(SP) \rightsquigarrow_{\rho^{\mathbf{Sign}}(\Sigma_A)} \hat{\rho}(\langle\Sigma_A,\emptyset\rangle).$$

Now, similarly as for Theorem 7.1, we can introduce sound scheme of rules:

$$(\rho\text{-join-refinement}) \quad \frac{\hat{\rho}(SP_1) \rightsquigarrow'_{\rho^{\mathbf{Sign}}(\Sigma)} \hat{\rho}(SP_2)}{SP_1 \rightsquigarrow_\Sigma SP_2}$$

where $\rho$, $SP_1$ and $SP_2$ satisfy the assumptions of Theorem 7.5. For the above ($\rho$-join-refinement) rule scheme we also can have similar proof strategy as for ($\rho$-join-entailment) rule scheme.

**Example 7.7.** In Example 4.7 we showed that

$$\langle \Sigma, \{\varphi\} \rangle \rightsquigarrow_\Sigma SP$$

cannot be proved in an institution $I$, whenever $SP \vdash_\Sigma \varphi$ cannot be proved, where $\Sigma$ is a signature, $\varphi$ is a $\Sigma$-sentence and $SP$ is a $\Sigma$-specification.

Let us assume that the institution representation $\rho : I \to I'$ satisfies assumptions of Theorem 7.5. We also assume that $I'$ is rich enough to ensure completeness of $\vdash'$. Now we can prove that $\langle \Sigma, \{\varphi\} \rangle \rightsquigarrow_\Sigma SP$ as follows:

$$(\rho\text{-join-refinement})\dfrac{(\text{Basic})\dfrac{\hat{\rho}(SP) \vdash'_{\rho^{\mathbf{Sign}}(\Sigma)} \rho_\Sigma^{\mathbf{Sen}}(\varphi)}{\hat{\rho}(\langle \Sigma, \{\varphi\} \rangle) \rightsquigarrow'_{\rho^{\mathbf{Sign}}(\Sigma)} \hat{\rho}(SP)}}{\langle \Sigma, \{\varphi\} \rangle \rightsquigarrow_\Sigma SP}$$

The proof of $\hat{\rho}(SP) \vdash'_{\rho^{\mathbf{Sign}}(\Sigma)} \rho_\Sigma^{\mathbf{Sen}}(\varphi)$ can be obtained by completeness of $\vdash'_{\rho^{\mathbf{Sign}}(\Sigma)}$ (since $SP \vdash_\Sigma \varphi$ and therefore $\hat{\rho}(SP) \models'_{\rho^{\mathbf{Sign}}(\Sigma)} \rho_\Sigma^{\mathbf{Sen}}(\varphi)$).

## 8. Mapping specifications

In this section we want to show how to obtain results similar to presented in Sections 6 and 7 for *maps of institutions* (see [20]).

Given an entailment system (or an institution) its category $\mathbf{Th}_0$ of *theories* has as objects pairs $T = (\Sigma, \Gamma)$, where $\Sigma$ is a signature and $\Gamma$ a set of sentences on $\Sigma$. Morphisms $\sigma : (\Sigma_1, \Gamma_1) \to (\Sigma_2, \Gamma_2)$ are the signature morphisms $\sigma : \Sigma_1 \to \Sigma_2$ such that $\mathbf{Sen}(\sigma)(\Gamma_1) \subseteq Cl(\Gamma_2)$, where $Cl(\Gamma_2)$ is the closure of $\Sigma_2$-sentences $\Gamma_2$ defined as follows (see [14]):

$$Cl(\Gamma_2) = \{\varphi \in \mathbf{Sen}_I(\Sigma_2) \mid \Gamma_2 \models^I_{\Sigma_2} \varphi\}.$$

We will use auxiliary functor $\mathbf{sign} : \mathbf{Th}_0 \to \mathbf{Sign}_I$ given as follows: $\mathbf{sign}(\Sigma, \Gamma) = \Sigma$, for $(\Sigma, \Gamma) \in |\mathbf{Th}_0|$ and $\mathbf{sign}(\sigma)$ is the signature morphism $\sigma$, for $\sigma \in \mathbf{Th}_0$.

Next, for any institution $I$ we extend the model functor $\mathbf{Mod}_I : \mathbf{Sign}_I^{op} \to \mathbf{Cat}$ to $\mathbf{Mod}_I : \mathbf{Th}_0^{op} \to \mathbf{Cat}$ which for any theory $(\Sigma, \Gamma)$ gives the full subcategory of $\Sigma$-models that satisfy all the sentences $\Gamma$. Similarly, by assigning to each theory $(\Sigma, \Gamma)$ the sentences $\mathbf{Sen}_I(\Sigma)$ we can extend the functor $\mathbf{Sen}_I : \mathbf{Sign}_I \to \mathbf{Set}$ to a functor $\mathbf{Sen}_I : \mathbf{Th}_0 \to \mathbf{Set}$. We can also extend the closure defined above to theories in the obvious way.

**Definition 8.1** (*Map of institutions* [20]). Given institutions $I = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$ and $I' = \langle \mathbf{Sign}', \mathbf{Sen}', \mathbf{Mod}', \langle \models'_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}'|} \rangle$ a *map of institutions* $(\Phi, \alpha, \beta) : I \to I'$ consists of:

- a functor $\Phi : \mathbf{Th}_0 \to \mathbf{Th}'_0$ which is $\alpha$-sensible;[4] and
- *a natural transformation*: $\alpha : \mathbf{Sen} \to \Phi; \mathbf{Sen}'$, that is, a family of functions $\alpha_\Sigma : \mathbf{Sen}(\Sigma) \to \mathbf{Sen}'(\Phi(\Sigma, \emptyset))$, natural in $\Sigma \in |\mathbf{Sign}|$:

$$\begin{array}{ccc} \Sigma_2 & \qquad & \mathbf{Sen}(\Sigma_2) \xrightarrow{\alpha_{\Sigma_2}} \mathbf{Sen}'(\Phi(\Sigma_2, \emptyset)) \\ \uparrow \sigma & & \mathbf{Sen}(\sigma) \uparrow \qquad\qquad\qquad \uparrow \mathbf{Sen}'(\Phi(\sigma)) \\ \Sigma_1 & & \mathbf{Sen}(\Sigma_1) \xrightarrow{\alpha_{\Sigma_1}} \mathbf{Sen}'(\Phi(\Sigma_1, \emptyset)) \end{array}$$

- *a natural transformation* $\beta : \Phi^{op}; \mathbf{Mod}' \to \mathbf{Mod}$, that is, a family of functions $\beta_{th} : \mathbf{Mod}'(\Phi(th)) \to \mathbf{Mod}(th)$, natural in $th \in |\mathbf{Th}_0|$:

$$\begin{array}{ccc} th_2 & \qquad & \mathbf{Mod}(th_2) \xleftarrow{\beta_{th_2}} \mathbf{Mod}'(\Phi(th_2)) \\ \uparrow \sigma & & \mathbf{Mod}(\sigma) \downarrow \qquad\qquad\qquad \downarrow \mathbf{Mod}'(\Phi(\sigma)) \\ th_1 & & \mathbf{Mod}(th_1) \xleftarrow[\beta_{th_1}]{} \mathbf{Mod}'(\Phi(th_1)) \end{array}$$

such that for any signature $\Sigma \in |\mathbf{Sign}|$ the translations $\alpha_\Sigma : \mathbf{Sen}(\Sigma) \to \mathbf{Sen}'(\Phi(\Sigma, \emptyset))$ of sentences and $\beta_{(\Sigma,\emptyset)} : \mathbf{Mod}'(\Phi(\Sigma, \emptyset)) \to \mathbf{Mod}(\Sigma, \emptyset)$ of models preserve the satisfaction relation, that is, for any $\varphi \in \mathbf{Sen}(\Sigma)$ and $M' \in |\mathbf{Mod}'(\Phi(\Sigma, \emptyset))|$:

$$M' \models'_{\mathbf{Sign}'(\Phi(\Sigma,\emptyset))} \alpha_\Sigma(\varphi) \quad \text{iff} \quad \beta_{(\Sigma,\emptyset)}(M') \models_\Sigma \varphi \quad \text{(Map condition)}$$

The above definition of a map of institutions can be easily extended to a map of $(\mathscr{D}, \mathscr{T})$-institutions.

**Definition 8.2** (*Map of* $(\mathscr{D}, \mathscr{T})$*-institutions*). A map of $(\mathscr{D}, \mathscr{T})$-institutions $(\Phi, \alpha, \beta) : I \to I'$ is a usual map of institutions $(\Phi, \alpha, \beta) : I \to I'$ which additionally satisfies

$$\Psi(\mathscr{D}_I) \subseteq \mathscr{D}_{I'} \quad \text{and} \quad \Psi(\mathscr{T}_I) \subseteq \mathscr{T}_{I'},$$

where $\Psi = \Phi; \mathbf{sign}$.

[4] We refer to [20] for detailed definition of $\alpha$-sensible functors. Basically, it is required that the provable consequences of the theory $\Phi(\Sigma, \Gamma)$ are entirely determined by $\Phi(\Sigma, \emptyset)$ and $\alpha$, i.e.

$$Cl(\Phi(\Sigma, \Gamma)) = Cl(\Sigma', \Gamma' \cup \alpha(\Gamma)),$$

where $(\Sigma', \Gamma') = \Phi(\Sigma, \emptyset)$.

We also redefine the $\rho$-expansion and weak-$\mathscr{D}$-amalgamation properties (see Section 5) and obtain

**Definition 8.3** (*$\beta$-Expansion*). A map of institutions $(\Phi,\alpha,\beta): I \to I'$ has the *$\beta$-expansion* property, if for any signature $\Sigma \in |\mathbf{Sign}|$, any $\Sigma$-model $M$ has a *$\beta$-expansion* to a $\Phi(\Sigma,\emptyset)$-model, that is, there exists a $\Phi(\Sigma,\emptyset)$-model $M'$ such that $\beta_{(\Sigma,\emptyset)}(M') = M$.

**Definition 8.4** (*Weak-$\mathscr{D}$-amalgamation*). Let $(\Phi,\alpha,\beta): I \to I'$ be a map of institutions and $\mathscr{D}$ be a class of signature morphisms in $I$. We say that the map of institutions $(\Phi,\alpha,\beta)$ has the *weak-$\mathscr{D}$-amalgamation* property iff for every signatures $\Sigma_1,\Sigma_2 \in |\mathbf{Sign}|$, $(d: \Sigma_2 \to \Sigma_1) \in \mathscr{D}$, $M_1 \in |\mathbf{Mod}(\Sigma_1,\emptyset)|$ and $M_2 \in |\mathbf{Mod}'(\Phi(\Sigma_2,\emptyset))|$, given as in the following diagram

$$\begin{tikzcd}
\mathbf{Mod}(\Sigma_1,\emptyset) \arrow[d, "-|_d"'] & \mathbf{Mod}'(\Phi(\Sigma_1,\emptyset)) \arrow[l, "\beta_{(\Sigma_1,\emptyset)}"'] \arrow[d, "-|_{\Phi(d)}"] \\
\mathbf{Mod}(\Sigma_2,\emptyset) & \mathbf{Mod}'(\Phi(\Sigma_2,\emptyset)) \arrow[l, "\beta_{(\Sigma_2,\emptyset)}"]
\end{tikzcd}$$

if $\beta_{(\Sigma_2,\emptyset)}(M_2) = M_1|_d$ then there exists $M \in |\mathbf{Mod}'(\Phi(\Sigma_1,\emptyset))|$ such that $\beta_{(\Sigma_1,\emptyset)}(M) = M_1$ and $M|_{\Phi(d)} = M_2$.

Now, we extend the notion of map of institutions to specifications.

**Definition 8.5** (*Map of specifications*). For any map of $(\mathscr{D},\mathscr{T})$-institutions $(\Phi,\alpha,\beta)$: $I \to I'$, the *map of specifications* $\gamma$ is a family of functions $\{\gamma_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ between classes of specifications over $(\mathscr{D},\mathscr{T})$-institutions $I$ and $I'$ defined as follows:

1. if $SP$ is a $\Sigma$-specification of the form $\langle\Sigma,\Gamma\rangle$, then $\gamma_\Sigma(SP) = \langle\Sigma',\Gamma' \cup \alpha_\Sigma(\Gamma)\rangle$, where $(\Sigma',\Gamma') = \Phi(\Sigma,\emptyset)$;
2. if $SP$ is a $\Sigma$-specification of the form $SP_1 \cup SP_2$, then $\gamma_\Sigma(SP) = \gamma_\Sigma(SP_1) \cup \gamma_\Sigma(SP_2)$;
3. if $SP$ is a $\Sigma$-specification of the form **translate** $SP_1$ **by** $(t: \Sigma_1 \to \Sigma)$, then $\gamma_\Sigma(SP) =$ **translate** $\gamma_{\Sigma_1}(SP_1)$ **by** $\Psi(t: \Sigma_1 \to \Sigma) \cup \langle\Sigma',\Gamma'\rangle$, where $(\Sigma',\Gamma') = \Phi(\Sigma,\emptyset)$;
4. if $SP$ is a $\Sigma$-specification of the form **derive from** $SP_1$ **by** $(d: \Sigma \to \Sigma_1)$, then $\gamma_\Sigma(SP) =$ **derive from** $\gamma_{\Sigma_1}(SP_1)$ **by** $\Psi(d: \Sigma \to \Sigma_1)$,

where $t \in \mathscr{T}_I$, $d \in \mathscr{D}_I$ and $\Psi = \Phi;\mathbf{sign}$. For a $\Sigma$-specification $SP$ we will write $\gamma(SP)$ as an abbreviation for $\gamma_\Sigma(SP)$.

The *map of specifications* defined above differs form the *specification representation* defined by Definition 5.12 in two cases: presentations and translate SBO. To obtain, for the above defined map of specifications, results similar to presented in Section 6 for the specification representation (especially Lemmas 6.1 and 6.3) we have to repeat the proofs presented there at least for presentations and translate (cf. a similar translation of specifications for *logical institution encodings* in [31]).

**Lemma 8.6.** *For any* $(\mathcal{D},\mathcal{T})$*-institutions* $I$ *and* $I'$, *map of* $(\mathcal{D},\mathcal{T})$*-institutions* $(\Phi,\alpha,\beta): I\to I'$, *signature* $\Sigma\in|\mathbf{Sign}|$, $\Sigma$*-specification SP over the* $(\mathcal{D},\mathcal{T})$*-institution* $I$ *and* $\Phi(\Sigma,\emptyset)$*-model* $M'\in\mathbf{Mod}[\gamma(SP)]$, *we have* $\beta_{(\Sigma,\emptyset)}(M')\in\mathbf{Mod}[SP]$.

**Proof.** By induction on the structure of the specification $SP$. Let us assume that $M'\in\mathbf{Mod}[\gamma(SP)]$.

1. If $SP=\langle\Sigma,\Gamma\rangle$: By assumption and Definition 8.5 we have $M'\in\mathbf{Mod}[\langle\Sigma',\Gamma'\cup\alpha_\Sigma(\Gamma)\rangle]$, where $(\Sigma',\Gamma')=\Phi(\Sigma,\emptyset)$. This implies $M'\models'_{\mathbf{sign}'(\Phi(\Sigma,\emptyset))}\alpha_\Sigma(\Gamma)$. By the map condition we obtain $\beta_{(\Sigma,\emptyset)}(M')\models_\Sigma\Gamma$, which yields $\beta_{(\Sigma,\emptyset)}(M')\in\mathbf{Mod}[\langle\Sigma,\Gamma\rangle]$.
2. If $SP=SP_1\cup SP_2$: Proof similar to case 2 of the proof of Lemma 6.1.
3. If $SP=$ **translate** $SP_1$ **by** $(t:\Sigma_1\to\Sigma)$: By assumption, $M'\in\mathbf{Mod}[$**translate** $\gamma(SP_1)$ **by** $\Psi(t)]\cap\mathbf{Mod}[\langle\Sigma',\Gamma'\rangle]$, where $(\Sigma',\Gamma')=\Phi(\Sigma,\emptyset)$ and $\Psi(t):\Sigma_1'\to\Sigma'$ for $(\Sigma_1',\Gamma_1')=\Phi(\Sigma_1,\emptyset)$ and $\Psi=\Phi;\mathbf{sign}$. By definition, $M'|_{\Psi(t)}\in\mathbf{Mod}[\gamma(SP_1)]$. Now, by the induction hypothesis $\beta_{(\Sigma_1,\emptyset)}(M'|_{\Psi(t)})\in\mathbf{Mod}[SP_1]$ and because the following diagram commutes

$$
\begin{tikzcd}
\mathbf{Mod}(\Sigma,\emptyset) \arrow[d, "-|_d"'] & \mathbf{Mod}'(\Phi(\Sigma,\emptyset)) \arrow[l, "\beta_{(\Sigma,\emptyset)}"'] \arrow[d, "-|_{\Phi(t)}"] \\
\mathbf{Mod}(\Sigma_1,\emptyset) & \mathbf{Mod}'(\Phi(\Sigma_1,\emptyset)) \arrow[l, "\beta_{(\Sigma_1,\emptyset)}"]
\end{tikzcd}
\tag{7}
$$

   we have $\beta_{(\Sigma_1,\emptyset)}(M'|_{\Psi(t)})=(\beta_{(\Sigma,\emptyset)}(M'))|_t$, hence $\beta_{(\Sigma,\emptyset)}(M')|_t\in\mathbf{Mod}[SP_1]$ and finally $\beta_{(\Sigma,\emptyset)}(M')\in\mathbf{Mod}[SP]$.
4. If $SP=$ **derive from** $SP_1$ **by** $d:\Sigma\to\Sigma_1$: Proof similar to case 4 of the proof of Lemma 6.1. □

And similarly as in Section 6 we obtain as a consequence:

**Corollary 8.7.** *For any* $(\mathcal{D},\mathcal{T})$*-institutions* $I$ *and* $I'$, *map of* $(\mathcal{D},\mathcal{T})$*-institutions* $(\Phi,\alpha,\beta): I\to I'$, *signature* $\Sigma\in|\mathbf{Sign}|$ *and* $\Sigma$*-specification SP over* $(\mathcal{D},\mathcal{T})$*-institution* $I$, *we have:*

$$\beta_{(\Sigma,\emptyset)}(\mathbf{Mod}[\gamma(SP)])\subseteq\mathbf{Mod}[SP].$$

The inclusion in the opposite direction we can prove as follows:

**Lemma 8.8.** *For any* $(\mathcal{D},\mathcal{T})$*-institutions* $I$ *and* $I'$ *and map of* $(\mathcal{D},\mathcal{T})$*-institutions* $(\Phi,\alpha,\beta): I\to I'$, *if the map* $(\Phi,\alpha,\beta)$ *satisfies the weak-*$\mathcal{D}$*-amalgamation then for every* *signature* $\Sigma \in |\mathbf{Sign}|$, $\Sigma$-*specification SP over the* $(\mathscr{D},\mathscr{T})$-*institution I and* $\Phi(\Sigma,\emptyset)$-*model* $M'$

$$\beta_{(\Sigma,\emptyset)}(M') \in \mathbf{Mod}[SP] \quad \textit{implies} \quad M' \in \mathbf{Mod}[\gamma(SP)].$$

**Proof.** By induction on the structure of $SP$.

1. If $SP = \langle \Sigma, \Gamma\rangle$: By assumption $\beta_{(\Sigma,\emptyset)}(M') \in \mathbf{Mod}[\langle \Sigma, \Gamma\rangle]$. It is equivalent to $\beta_{(\Sigma,\emptyset)}(M') \models_\Sigma \Gamma$. By the map condition, we obtain $M' \models'_{\Sigma'} \alpha_\Sigma(\Gamma)$ and because $M'$ is $\Phi(\Sigma,\emptyset)$-model we have $M' \models'_{\Sigma'} \Gamma'$, where $(\Sigma',\Gamma') = \Phi(\Sigma,\emptyset)$. Next, by Definition 8.5 we obtain $M' \in \mathbf{Mod}[\gamma\langle \Sigma,\Gamma\rangle]$.
2. If $SP = SP_1 \cup SP_2$: Proof similar to case 2 of the proof of Lemma 6.3.
3. If $SP =$ **translate** $SP_1$ **by** $(t : \Sigma_1 \to \Sigma)$: By assumption $\beta_{(\Sigma,\emptyset)}(M') \in \mathbf{Mod}[$**translate** $SP_1$ **by** $t]$. Next, by Definition 3.1, $\beta_{(\Sigma,\emptyset)}(M')|_t \in \mathbf{Mod}[SP_1]$, which by the commutativity of the diagram (7) (see the proof of Lemma 8.6) is equivalent to $\beta_{(\Sigma_1,\emptyset)}(M'|_{\Psi(t)}) \in \mathbf{Mod}[SP_1]$. Now, since $M'|_{\Psi(t)}$ is a $\Phi(\Sigma_1,\emptyset)$-model and by the induction hypothesis we obtain $M' \in \mathbf{Mod}[$**translate** $\gamma(SP_1)$ **by** $\Psi(t)]$. By assumption $M'$ is a $\Phi(\Sigma,\emptyset)$-model and we have $M' \in \mathbf{Mod}[\langle \Sigma',\Gamma'\rangle]$, where $(\Sigma',\Gamma') = \Phi(\Sigma,\emptyset)$. Finally, by Definition 8.5 we obtain $M' \in \mathbf{Mod}[\gamma(SP)]$.
4. If $SP =$ **derive from** $SP_1$ **by** $(d : \Sigma \to \Sigma_1)$: Proof similar to case 4 of the proof of Lemma 6.3. □

In the next step we just add assumption about $\beta$-expansion and obtain expected inclusion.

**Lemma 8.9.** *For any* $(\mathscr{D},\mathscr{T})$-*institutions I and I′ and map of* $(\mathscr{D},\mathscr{T})$-*institutions* $(\Phi,\alpha,\beta) : I \to I'$, *if* $(\Phi,\alpha,\beta)$ *has the weak-*$\mathscr{D}$*-amalgamation property*, *then for every signature* $\Sigma \in |\mathbf{Sign}|$, $\Sigma$-*specification SP over the* $(\mathscr{D},\mathscr{T})$-*institution I*, *if each model* $M \in \mathbf{Mod}[SP]$ *has a* $\beta$-*expansion to a* $\Phi(\Sigma,\emptyset)$-*model*, *then*

$$\mathbf{Mod}[SP] \subseteq \beta_{(\Sigma,\emptyset)}(\mathbf{Mod}[\gamma(SP)]).$$

**Proof.** By analogy to proof of Lemma 6.4. □

As a consequence we obtain:

**Corollary 8.10.** *For any* $(\mathscr{D},\mathscr{T})$-*institutions I and I′ and map of* $(\mathscr{D},\mathscr{T})$-*institutions* $(\Phi,\alpha,\beta) : I \to I'$, *if the map has the weak-*$\mathscr{D}$*-amalgamation property*, *then for every signature* $\Sigma \in |\mathbf{Sign}|$, $\Sigma$-*specification SP over the* $(\mathscr{D},\mathscr{T})$-*institution I*, *if each model* $M \in \mathbf{Mod}[SP]$ *has a* $\beta$-*expansion to a* $\Phi(\Sigma,\emptyset)$-*model*, *then*

$$\mathbf{Mod}[SP] = \beta_{(\Sigma,\emptyset)}(\mathbf{Mod}[\gamma(SP)]).$$

Having the above equality we can obtain results similar to Theorems 7.1 and 7.5.

**Theorem 8.11.** *For any $(\mathscr{D},\mathscr{T})$-institutions $I$ and $I'$ and map of $(\mathscr{D},\mathscr{T})$-institutions $(\Phi,\alpha,\beta): I \to I'$, if the map has the weak-$\mathscr{D}$-amalgamation property, then for every signature $\Sigma \in |\mathbf{Sign}|$, $\Sigma$-specification SP over $(\mathscr{D},\mathscr{T})$-institution $I$ and $\Sigma$-sentence $\varphi$, if each $\Sigma$-model $M \in \mathbf{Mod}[SP]$ has a $\beta$-expansion to a $\Phi(\Sigma,\emptyset)$-model, then*

$$SP \models_\Sigma \varphi \quad \textit{iff} \quad \gamma(SP) \models'_{\mathbf{sign}'(\Phi(\Sigma,\emptyset))} \alpha_\Sigma(\varphi).$$

**Theorem 8.12.** *For any $(\mathscr{D},\mathscr{T})$-institutions $I$ and $I'$ and map of $(\mathscr{D},\mathscr{T})$-institutions $(\Phi,\alpha,\beta): I \to I'$, if the map has the weak-$\mathscr{D}$-amalgamation property, then for every signature $\Sigma \in |\mathbf{Sign}|$ and $\Sigma$-specifications $SP_1$ and $SP_2$ over $(\mathscr{D},\mathscr{T})$-institution $I$, if each $\Sigma$-model has a $\beta$-expansion to a $\Phi(\Sigma,\emptyset)$-model, then*

$$SP_1 \rightsquigarrow_\Sigma SP_2 \quad \textit{iff} \quad \gamma(SP_1) \rightsquigarrow_{\mathbf{sign}'(\Phi(\Sigma,\emptyset))} \gamma(SP_2).$$

## 9. Conclusions

In this paper we have studied compositional logical systems for reasoning about logical consequences and refinement of structured specifications in an arbitrary institution, based on the logical system presented in [27] and also in [8, 32]. In the first part of the paper we identified the formal properties of the underlying institution that ensure (soundness and) completeness of the logical system considered. Results similar to those presented in this part of the paper were also presented in [8, 32] for the case of first-order logic. Our results generalized this to an arbitrary institution satisfying certain conditions. We showed that the underlying logical system has to satisfy at least weak-$(\mathscr{D},\mathscr{T})$-interpolation, but the question about minimal conditions ensuring completeness of the logical system considered is still open.

We then considered the problem of completing proofs of logical consequences and refinement of structured specifications when the underlying logical system is too weak to satisfy the conditions formulated in the first part, and so need not ensure the completeness of formal systems for reasoning about logical consequences and refinement of structured specifications. We formulated conditions under which we can reuse proof systems built over institutions rich enough to satisfy conditions required for systems completeness for specifications built over poorer institutions (that are too poor to ensure completeness). Similar results to those presented in this part (especially in Theorems 7.1 and 8.11) for the case of flat specifications were presented in [1, 9, 30]. In [1] we can also find a study on a similar topic for the case of structured specifications. As presented in papers mentioned above, the $\rho$-expansion property is a sufficient condition for Theorems 7.1 and 7.5 (and $\beta$-expansion for Theorems 8.11 and 8.12) for flat specifications. In this paper we showed that to extend these results to structured specifications we need an additional condition: weak-$\mathscr{D}$-amalgamation.

In [17] authors presented similar results to our reusing results (Theorems 7.1 and 7.5), but on the theory level. The proof rules given in [17] are more restricted then our proof strategy presented in Section 7. For instance, Example 7.3 is an example of a successful use of our strategy, whereas when using the strategy proposed in [17], we are not able to complete the proof of a judgment similar to presented in Example 7.3 (in fact, this has to be so, since this judgment is not sound under the theory level semantics considered there).

For the future work we consider extensions of results presented in this paper to specifications with more SBOs than presented in Definition 3.1 (see [22, 26] for reference). Other possible directions are extensions of presented results to parameterized specifications (see [8, 26, 32]) and to observational specifications presented in [16].

Before the results presented become practically important some technical definitions and assumptions have to be considerably refined. For example proving the two assumptions about representations considered in Theorems 7.1 and 7.5 ($\rho$-expansion/ $\beta$-expansion and weak-$\mathscr{D}$-amalgamation) may cause problems in practice. Some standard ways of building institution representations from simpler components should be provided so that the two properties of the resulting representation follow from more elementary and quite natural properties of these components.

Also some more efficient proof strategies have to be worked out. For instance, proof system for proving refinement might contain following rules:

$$\frac{SP \rightsquigarrow SP' \quad SP' \rightsquigarrow SP''}{SP \rightsquigarrow SP''} \qquad \frac{SP_1 \rightsquigarrow SP_1' \quad \ldots \quad SP_n \rightsquigarrow SP_n'}{op(SP_1, \ldots, SP_n) \rightsquigarrow op(SP_1', \ldots, SP_n')},$$

where $op$ is an arbitrary (monotonic) SBO. The above rules are known as “vertical composability” and “horizontal composability”, respectively (see [13, 28]).

Another interesting task is to present within our framework some standard examples of universal logics (cf. [30]), in which we will represent simpler logics in order to reuse for them strategies known/worked out for stronger universal logics. Theorem 7.1 together with Theorem 3.9 indicate the interpolation property as one property of a reasonable universal logic. We expect that some of known logical frameworks turn out to satisfy this property. Proper candidates seem to be for instance LF and HOL. It seems also to be possible to prove that assumptions of Corollary 3.10 hold for the structural part of the CASL language (see [22]) or at least for a reasonable part of the CASL language.

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