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An extension of the spectral Tau method for numerical solution of multi-order fractional differential equations with convergence analysis

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ABSTRACT

The main purpose of this paper is to provide an efficient numerical approach for the fractional differential equations (FDEs) based on a spectral Tau method. An extension of the operational approach of the Tau method with the orthogonal polynomial bases is proposed to convert FDEs to its matrix–vector multiplication representation. The fractional derivatives are described in the Caputo sense. The spectral rate of convergence for the proposed method is established in the $\mathfrak{L}^2$ norm. We tested our procedure on several examples and observed that the obtained numerical results confirm the theoretical prediction of the exponential rate of convergence.

1. Introduction

Recently, fractional differential equations have gained much interest in different research areas and engineering applications. Contrary to differential equations of integer order in which derivatives depend only on the local behavior of the function, fractional differential equations accumulate the whole information of the function in a weighted form. This is the so-called memory effect and has many applications in physics, chemistry and other branches of science, for example, to predict the viscoelastic behavior of amorphous polymers in the temperature ranges [1], the nonlinear oscillation of earthquake [2], and the fluid-dynamic traffic model [3]. It has also applications to the diffusion equation which is one of fundamental partial differential equations of mathematical physics [4].

Most fractional differential equations do not have exact analytic solutions, so approximation and numerical techniques must be used. The variational iteration method [5,6], the Adomian decomposition method [7,8], extrapolation method [9], homotopy perturbation method [10] and differential transform method [4,11], are relatively new approaches to provide an analytical approximation to linear and nonlinear problems.

In this paper we consider multi-order fractional differential equations of the form

$$\sum_{i=0}^{N} p_i(t) D^{\alpha_i} u(t) = g(t), \quad p_i(t), g(t) \in C[a, b],\ \alpha_i \in \mathbb{Q}^+,$$

where fractional derivatives are described in the Caputo sense. Our approach is to investigate how numerically one can solve a multi-order fractional differential equation with the spectral Tau method.

Spectral methods are very much efficient for the numerical solution of ordinary or partial differential equations [12]. The Tau method can be described as a spectral method for the numerical solution of differential equations. Ortiz and Samara

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proposed an operational Tau technique for the numerical solution of nonlinear ordinary differential equations [13]. The same technique has been used for the case of linear ordinary differential eigenvalue problems [14], for partial differential equations [15] and also for the iterated solutions of linear operator equations [16].

In this paper, an extension of the operational Tau method is proposed to numerically solve the FDEs. In this procedure, we have used an interpolating polynomial for approximating the integral term of the equation. We will consider an efficient error analysis for the proposed method, which indicates that the numerical errors will decay exponentially.

The organization of this paper is as follows: In Section 2, we introduce some basic definitions. In part (a) of Section 3 we recall the operational Tau method to obtain a matrix form of the differential part. In part (b), converting fractional part of an equation to a matrix form is shown. At the end of this section the corresponding system of linear algebraic equations is given. Section 4 is devoted to convergence analysis. We will show the spectral rate of convergence of the applied method. In Section 5, some numerical results are given to clarify the details and efficiency of the method.

2. Basic definitions

There are several definitions for a fractional derivative of order $\alpha > 0$ e.g. Riemann–Liouville and Caputo [17]. The Riemann–Liouville approach is not suitable for real world physical problems since it requires the definition of fractional order initial conditions. Caputo introduced an alternative definition of initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case in most physical processes.

Now we give some basic definitions and properties of the fractional calculus theory which are used further in this paper [18,19].

Definition 2.1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p\ (>\mu)$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty)$, and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu^m$, $m \in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad \alpha > 0,\ t > 0,$$
$$J^0 f(t) = f(t).$$

For $\gamma > -1$ the following property holds:

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}. \tag{1}$$

Definition 2.3. The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D_*^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s)\mathrm{d}s, \quad t > 0,$$

for $m-1 < \alpha \leq m,\ m \in \mathbb{N},\ t > 0,\ f \in C^{(m)}[0,1]$.

In subsequent numerical analysis, the following definition is needed

$$\langle u(t), v(t)\rangle_w = \int_0^1 u(t)v(t)w(t)\mathrm{d}t,$$

where $u(t)$ and $v(t)$ are any integrable functions on $[0,1]$ and $w(t)$ is a weight function in usual sense.

3. Fractional differential equations

Consider the following linear fractional differential equation together with the given supplementary conditions:

$$\begin{cases} L_D(u(t)) + L_f(u(t)) = g(t), & t \in [0,1],\ u(t) \in C^{(\nu)}[0,1] \\ f_j u = d_j, & j = 1, \ldots, \nu, \end{cases} \tag{2}$$

where $(f_j)_{j=1}^{\nu}$ is a set of linear functionals acting on $u(t)$ and $f_j u = d_j, j = 1, \ldots, \nu$ stands for the initial, boundary or mixed conditions to be satisfied by the required solution $u(t)$ on a part (or the whole) of the set of the endpoints $[0,1]$ (see [13]).

L_D and L_f are linear ordinary and fractional differential operators of order N_d and N_f, respectively. We suppose $p_i(t)$, $q_i(t)$ and $g(t)$ are given polynomials and we consider

$$L_D = \sum_{i=0}^{N_d} p_i(t)\frac{\mathrm{d}^i}{\mathrm{d}t^i}, \qquad L_f = \sum_{i=0}^{N_f} q_i(t) D_*^{\alpha_i}, \quad m_i - 1 < \alpha_i < m_i,\ m_i \in \mathbb{N}, \tag{3}$$

$$p_i(t) = \sum_{j=0}^{N_{p_i}} p_{ij}t^j = \underline{p_i}\underline{X}, \qquad q_i(t) = \sum_{j=0}^{N_{q_i}} q_{ij}t^j = \underline{q_i}\underline{X},$$

$$g(t) = \sum_{j=0}^{N_g} g_j t^j = \underline{g}\underline{X},$$

where

$$\underline{p_i} = (p_{i,0}, \ldots, p_{i,N_{p_i}}, 0, \ldots), \qquad \underline{q_i} = (q_{i,0}, \ldots, q_{i,N_{q_i}}, 0, \ldots),$$
$$\underline{g} = (g_0, \ldots, g_{N_g}, 0, \ldots), \qquad \underline{X} = (1, t, t^2, \ldots)^t$$

with $m_0 < m_1 < \cdots < m_{N_f}$ and $\nu = \max\{m_{N_f}, N_d\}$.

If continuous functions $g(t)$, $p_i(t)$ and $q_i(t)$ in (2) are not polynomials, they can be approximated by polynomials to any degree of accuracy (by interpolation or other suitable methods).

Let $\underline{V} := (v_0(t), v_1(t), \ldots)$, $t \in [0, 1]$ be a set of Jacobi orthogonal polynomial basis given by $\underline{V} := V\underline{X}$, where V is a non-singular lower triangular matrix and degree$(v_i(t)) \le i$, for $i = 0, 1, 2, \ldots$.

Let $u(t) = \sum_{i=0}^{\infty} a_i v_i(t) = \underline{a}V\underline{X}$ be the orthogonal series expansion of the exact solution of (2) where $\underline{a} = (a_0, a_1, a_2, \ldots)$ and $u_N(t)$ is a Tau approximation of degree N for $u(t)$ as

$$u_N(t) = \sum_{j=0}^{N} a_j v_j(t) = \underline{a_N}V\underline{X}, \quad \underline{a_N} = (a_0, \ldots, a_N, 0, \ldots). \tag{4}$$

We have the following results on the Tau approximation for the fractional differential equations based on the Jacobi orthogonal polynomial basis $\underline{V}$.

(a) *Matrix representation of ordinary differential part*

Let us consider the effect of differentiation or multiplication by the variable t on a given polynomial $\phi_N(t) = \underline{\varphi_N}\underline{X}$ by the following relations (see [13])

$$\frac{\mathrm{d}^r}{\mathrm{d}t^r}\phi_N(t) = \underline{\varphi_N}\eta^r\underline{X}, \qquad t^r\phi_N(t) = \underline{\varphi_N}\mu^r\underline{X}, \tag{5}$$

where

$$\eta = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ 0 & 0 & 3 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \mu = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}.$$

We recall the following theorem from [13]:

Theorem 3.1. *The following relation holds*

$$L_D(u_N(t)) = \underline{a_N}V\Pi\underline{X} = \underline{a_N}\Pi_v\underline{V},$$

where

$$\Pi = \sum_{i=0}^{N_d} \eta^i p_i(\mu), \qquad \Pi_v = V\Pi V^{-1}. \quad \square$$

We introduce the vectors $\underline{f} = (f_1, \ldots, f_\nu, 0, \ldots)$ and $\underline{d} = (d_1, \ldots, d_\nu, 0, \ldots)$. Also we define $b_j = f_j(v_i)_{i=0}^{N}$ for $j = 1, \ldots, \nu$, then the supplementary conditions take the form

$$\begin{aligned}\underline{f}u_N &= (f_1u_N, \ldots, f_\nu u_N, 0, \ldots) = \left(f_1\sum_{i=0}^{N} a_iv_i, \ldots, f_\nu\sum_{i=0}^{N} a_iv_i, 0, \ldots\right) \\ &= \underline{a_N}(f_1(v_i)_{i=0}^{\infty}, \ldots, f_\nu(v_i)_{i=0}^{\infty}, 0, \ldots) = \underline{a_N}(b_1, \ldots, b_\nu, 0, \ldots) = \underline{a_N}B = \underline{d},\end{aligned}$$

where

$$B=\begin{pmatrix} f_1v_0 & f_2v_0 & \cdots & f_\nu v_0 & 0 & \cdots \\ f_1v_1 & f_2v_1 & \cdots & f_\nu v_1 & 0 & \cdots \\ \vdots & & \vdots & & & \cdots \\ f_1v_N & f_2v_N & \cdots & f_\nu v_N & 0 & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \end{pmatrix}. \tag{6}$$

(b) *Matrix representation of fractional part*

Firstly we extend property (1) to the Caputo derivative.

Lemma 3.2. *Caputo derivative has the following property for $m-1<\alpha\le m$:*

$$D_*^\alpha t^k=\begin{cases}\dfrac{k!}{\Gamma(k-\alpha+1)}t^{k-\alpha}, & k\ge\alpha\\ 0, \quad k<\alpha.\end{cases} \tag{7}$$

Proof. If $k\ge\alpha$,

$$\begin{aligned}D_*^\alpha t^k &= J^{m-\alpha}D^m t^k = J^{m-\alpha}\frac{k!}{(k-m)!}t^{k-m}=\frac{k!}{(k-m)!}J^{m-\alpha}t^{k-m}\\ &=\frac{k!}{(k-m)!}\cdot\frac{\Gamma(k-m+1)}{\Gamma(m-\alpha+k-m+1)}t^{m-\alpha+k-m}=\frac{k!}{\Gamma(k-\alpha+1)}t^{k-\alpha},\end{aligned}$$

if $k<\alpha$, it is obvious that $\frac{\mathrm{d}^m}{\mathrm{d}t^m}t^k=0$, for $k<\alpha\le m,\ m\in N$. $\square$

Now, consider the Lagrange interpolation polynomial $I_N(u(t))$ with

$$\forall t_i: I_N(u(t_i))=u(t_i),\quad (t_i)_{i=0}^N=\left(t\in[0,1]|t(1-t)\frac{\mathrm{d}}{\mathrm{d}t}T_N^*(t)=0\right),$$

where $(t_i)_{i=0}^N$ represents the shifted Gauss–Lobatto quadrature points.

In this position, we state the main theorem of this section:

Theorem 3.3. *Let $(t_j)_{j=0}^N$ be the set of the $(N+1)$ Gauss or Gauss–Radau, or Gauss–Lobatto points of the shifted Jacobi polynomials in $[0,1]$ and $(w_k)_{k=0}^N$ be the corresponding quadrature weights. Assume that the approximated solution u_N is given by (3) and $\widehat{L_f}(u_N(t))=\sum_{i=0}^{N_f}q_i(t)I_N(D_*^{\alpha_i}u_N(t))$, then*

$$\widehat{L_f}(u_N(t))=\underline{a_N}\sum_{i=0}^{N_f}V\ddot{\Pi}_iVq_i(\mu)\underline{X}=\underline{a_N}V\ddot{\Pi}\underline{X},$$

where $\ddot{\Pi}=\sum_{i=0}^{N_f}\ddot{\Pi}_iVq_i(\mu)$ and

$$\ddot{\Pi}_i=\begin{pmatrix} 0 & \cdots & 0\\ \vdots & \vdots & \vdots\\ 0 & \cdots & 0\\ \dfrac{m_i!}{\Gamma(m_i-\alpha_i+1)}c_{m_i,0} & \cdots & \dfrac{m!}{\Gamma(m_i-\alpha_i+1)}c_{m_i,N}\\ \dfrac{(m_i+1)!}{\Gamma(m_i-\alpha_i+2)}c_{m_i+1,0} & \cdots & \dfrac{(m_i+1)!}{\Gamma(m_i-\alpha_i+2)}c_{m_i+1,N}\\ \vdots & \vdots & \vdots\\ \dfrac{N!}{\Gamma(N-\alpha_i+1)}c_{N,0} & \cdots & \dfrac{N!}{\Gamma(N-\alpha_i+1)}c_{N,N}\\ \vdots & \vdots & \vdots \end{pmatrix},$$

with

$$c_{k,j}=\frac{1}{\sum\limits_{s=0}^N v_j^2(t_s)\omega_s}\sum_{s=0}^N t_s^{k-\alpha_i}v_j(t_s)\omega_s \qquad \begin{array}{l}k\ge\alpha_i\\ j=0,\ldots,N+1-\nu.\end{array}$$

Proof. Assume that $l_j(t), j = 0, 1, \ldots, N$ are Lagrange interpolation polynomials associated with the Gauss points $(t_j)_{j=0}^N$. By substituting $t^{k-\alpha_i}$ with its interpolating polynomial of degree N we may write (7) as

$$I_N(D_*^{\alpha_i} t^k) = \frac{k!}{\Gamma(k-\alpha_i+1)} \sum_{j=0}^{N} t_j^{k-\alpha_i} l_j(t), \quad k \geq \alpha_i,$$

and $I_N(D_*^{\alpha_i} t^k) = 0$ for $k < \alpha_i$.

Using the close relation between orthogonal polynomials and Gauss type integration formulas we way write:

$$I_N(D_*^{\alpha_i} t^k) = \frac{k!}{\Gamma(k-\alpha_i+1)} \sum_{j=0}^{N} c_{k,j} \upsilon_j(t), \quad t \in [0, 1], \tag{8}$$

where

$$c_{k,j} = \frac{1}{\|\upsilon_j(t)\|^2_{\mathcal{L}^2_\omega[0,1]}} \int_0^1 \Big(\sum_{s=0}^{N} t_s^{k-\alpha_i} l_s(t)\Big) \upsilon_j(t)\omega(t)\mathrm{d}t \qquad \begin{array}{l} k \geq \alpha_i \\ j = 0, \ldots, N. \end{array}$$

Again, using the Gauss integration formula we have

$$c_{k,j} = \frac{1}{\sum\limits_{s=0}^{N} \upsilon_j^2(t_s)\omega_s} \sum_{s=0}^{N} t_s^{k-\alpha_i} \upsilon_j(t_s)\omega_s \qquad \begin{array}{l} k \geq \alpha_i \\ j = 0, \ldots, N. \end{array} \tag{9}$$

From (8) we can write

$$\begin{aligned}
I_N(D_*^{\alpha_i} \underline{X}) &= I_N(D_*^{\alpha_i} t^0, \ldots, D_*^{\alpha_i} t^{m_i}, \ldots, D_*^{\alpha_i} t^N, \ldots)^t \\
&= \Bigg(0, \ldots, 0, \frac{m_i!}{\Gamma(m_i-\alpha_i+1)} \sum_{j=0}^{N} c_{m_i,j}\upsilon_j(t), \frac{(m_i+1)!}{\Gamma(m_i-\alpha_i+2)} \sum_{j=0}^{N} c_{m_i+1,j}\upsilon_j(t), \ldots, \\
&\qquad \frac{N!}{\Gamma(N-\alpha_i+1)} \sum_{j=0}^{N} c_{N,j}\upsilon_j(t), \ldots\Bigg)^t \\
&= \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \vdots & \vdots \\
0 & \cdots & 0 \\
\frac{m_i!}{\Gamma(m_i-\alpha_i+1)} c_{m_i,0} & \cdots & \frac{m!}{\Gamma(m_i-\alpha_i+1)} c_{m_i,N} \\
\frac{(m_i+1)!}{\Gamma(m_i-\alpha_i+2)} c_{m_i+1,0} & \cdots & \frac{(m_i+1)!}{\Gamma(m_i-\alpha_i+2)} c_{m_i+1,N} \\
\vdots & \vdots & \vdots \\
\frac{N!}{\Gamma(N-\alpha_i+1)} c_{N,0} & \cdots & \frac{N!}{\Gamma(N-\alpha_i+1)} c_{N,N} \\
\vdots & \vdots & \vdots
\end{pmatrix}
\begin{pmatrix} \upsilon_0(t) \\ \upsilon_1(t) \\ \vdots \\ \upsilon_N(t) \end{pmatrix} \\
&= \ddot{\Pi}_i(\upsilon_j(t))_{j=0}^N = \ddot{\Pi}_i V \underline{X}.
\end{aligned} \tag{10}$$

So we may write

$$I_N(D_*^{\alpha_i} u_N(t)) = D_*^{\alpha_i} \underline{a_N} V \underline{X} = \underline{a_N} V D_*^{\alpha_i} \underline{X} = \underline{a_N} V \ddot{\Pi}_i V \underline{X}.$$

Then from (5) we can write $\widehat{L_f}$ as

$$\widehat{L_f}(u_N(t)) = \sum_{i=0}^{N_f} q_i(t) I_N(D_*^{\alpha_i} u_N(t)) = \sum_{i=0}^{N_f} q_i(t) \underline{a_N} V \ddot{\Pi}_i V \underline{X} = \underline{a_N} \left(\sum_{i=0}^{N_f} V \ddot{\Pi}_i V q_i(\mu)\right) \underline{X}.$$

We denote $\ddot{\Pi} = \sum_{i=0}^{N_f} \ddot{\Pi}_i V q_i(\mu)$, then we conclude that

$$\widehat{L_f}(u_N(t)) = \underline{a_N} V \ddot{\Pi} V^{-1} \underline{V} = \underline{a_N} \ddot{\Pi}_\upsilon \underline{V}. \quad \square$$

For the numerical analysis of the operational Tau approximation to (2) we have to consider a polynomial approximation of $L_f(u_N(t))$, thus we consider $\widehat{L_f}(u_N(t))$ as a polynomial approximation of $L_f(u_N(t))$.

(c) *Tau approximate solution of fractional differential equation*

Following Theorems 3.1 and 3.3 and Eq. (2) we obtain

$$\begin{cases} \underline{a_N}(\Pi_\upsilon + \ddot{\Pi}_\upsilon)\underline{V} = \underline{g}\underline{V}, \\ \underline{a_N}B = \underline{d}. \end{cases} \tag{11}$$

We denote

$$\widehat{\Pi} = \Pi_\upsilon + \ddot{\Pi}_\upsilon. \tag{12}$$

Because of the orthogonality of $(\upsilon_k(t))_{k=0}^{\infty}$, projecting (8) on $(\upsilon_i(t))_{i=0}^{N}$ yields (see [16])

$$\left\langle \sum_{i=0}^{N} \underline{a_N}\widehat{\Pi}_i \upsilon_i(t), \upsilon_j(t) \right\rangle_w = \left\langle \sum_{i=0}^{N} g_i \upsilon_i(t), \upsilon_j(t) \right\rangle_w, \quad j = 0, 1, 2, \ldots, N, \tag{13}$$

where $\widehat{\Pi}_i$ is the ith column of $\widehat{\Pi}$. The orthogonality assumption of $(\upsilon_k(t))_{k=0}^{\infty}$ yields.

$$\underline{a_N}\widehat{\Pi}_j = g_j, \quad j = 0, 1, 2, \ldots, N. \tag{14}$$

By setting

$$G = (b_1, \ldots, b_\nu, \widehat{\Pi}_0, \ldots, \widehat{\Pi}_N),$$
$$R = (d_1, \ldots, d_\nu, g_0, \ldots, g_N),$$

we obtain

$$\underline{a_N}G = R. \tag{15}$$

If we restrict system (12) to its first $N + 1$ columns we obtain a square system $\underline{a_N}G_N = R_N$ which, when solved, gives an unknown vector $(a_0, a_1, \ldots, a_N)$.

The following algorithm summarizes the proposed Tau method:

Algorithm 1 (*The Construction of Tau Approximation System*).
Input: Fractional differential equation of form (2):

$$(p_i(t), q_i(t), N_f, N_d, g(t), f_j, d_j, j = 1, \ldots, \nu)$$

Step 1 Choose N, form the Jacobi orthogonal bases: $\upsilon_i(t), i = 0, 1, \ldots, N, \ldots$.
Step 2 Compute $\Pi_\upsilon = \sum_{i=0}^{N_d} V\eta^i p_i(\mu)V^{-1}$ using Theorem 3.1.
Step 3 Compute $\ddot{\Pi}_i$ and $\ddot{\Pi} = \sum_{i=0}^{N_f} V\ddot{\Pi}_i V q_i(\mu)V^{-1}$ from Theorem 3.3.
Step 4 Compute $\widehat{\Pi}$ and B from (12) and (6), respectively.
Step 5 Form $\mathbf{G_N}$, $\mathbf{R_N}$ and solve system (15) for $(\mathbf{a_0}, \mathbf{a_1}, \ldots, \mathbf{a_N})$.
Step 6 Finally, the solution $u_N(t)$ in the polynomial form can be computed by replacing $(\mathbf{a_0}, \mathbf{a_1}, \ldots, \mathbf{a_N})$ in (4).

4. Convergence analysis

The aim of this section is to analyze the numerical scheme (11). Using multiple integration and integrating by parts and important lemmas we prove our main theorem on exponential rate of convergence.

Consider the FDE of the form

$$\begin{cases} \displaystyle\sum_{i=0}^{n} A_i u^{(i)}(t) + D_*^\alpha u(t) = g(t), \quad m \in \mathbb{N}, \ m-1 < \alpha < m, \ A_n \neq 0, \ t \in [0, 1] \\ u^{(i)}(0) = d_i, \quad i = 1, 2, \ldots, \nu, \end{cases} \tag{16}$$

where $u \in C^{(\nu)}[0, 1]$, $\nu = \max\{m, n\}$ and $g(t) \in C[0, 1]$. Let $e_N(t) = u_N(t) - u(t)$ be the error function of the Tau approximation, where $u(t)$ is the exact solution of (16) and $u_N(t) = \sum_{j=0}^{N} a_j\upsilon_j(t)$ is the Tau approximation for $u(t)$.

Throughout this paper C_i and γ_i will denote the positive constants independent of N but will depend on n, m and α.

As defined in [12,20], we can conclude that the set of shifted Chebyshev (Legendre) polynomials $(T_N^*(t))_{N=0}^{\infty}$ $((L_N^*(t))_{N=0}^{\infty})$ forms a complete $\mathcal{L}_w^2(\rho)$ orthogonal system, where ρ stands for the closed interval $[0, 1]$ and $\mathcal{L}_w^2(\rho)$ is the space of all functions $u : [0, 1] \to \mathbb{R}$ with $\|u\|_{\mathcal{L}_w^2(\rho)} < \infty$. We define

$$\|u\|_{\mathcal{L}_w^2(\rho)}^2 = \langle u, u \rangle_{\mathcal{L}_w^2(\rho)} = \int_0^1 u^2(t)w(t)\mathrm{d}t.$$

$H_w^m(\rho)$ denotes the Sobolev space of all functions $u(t)$ on ρ such that $u(t)$ and all its weak derivatives up to order m are in $\mathcal{L}_w^2(\rho)$. The norm of $H_w^m(\rho)$ is defined by

$$\|u(t)\|^2_{H_w^m(\rho)} = \sum_{k=0}^{m} \left\| \frac{\partial^k}{\partial t^k} u(t) \right\|^2_{\mathcal{L}_w^2(\rho)}.$$

Also we define the semi-norm

$$|u|^2_{H_w^{m;N}(\rho)} = \sum_{j=\min(m;N)}^{N} \|u^{(j)}\|^2_{\mathcal{L}_w^2(\rho)}.$$

Now we consider the following lemmas:

Lemma 4.1. *For multiple integrals the relation*

$$\int_0^t \int_0^{t_n} \dots \int_0^{t_3} \int_0^{t_2} F(t_1)\mathrm{d}t_1\mathrm{d}t_2 \dots \mathrm{d}t_{n-1}\mathrm{d}t_n = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} F(s)\mathrm{d}s,$$

holds.

Proof. See [21] Appendix A.1. □

Let $P_N(\rho)$ be the space of polynomials with degree $\le N$ on ρ and p_N be the orthogonal projective operator from $\mathcal{L}_w^2(\rho)$ onto $P_N(\rho)$. It means that for any function f in $\mathcal{L}_w^2(\rho)$, $p_N f$ belongs to $P_N(\rho)$ and satisfies

$$\forall \psi_N \in P_N(\rho), \quad \int_0^1 (f - p_N f)(\xi)\psi_N(\xi)\mathrm{d}\xi = 0.$$

The following relations with shifted Chebyshev (Legendre) polynomials and shifted Chebyshev–(Legendre)–Gauss–Lobatto nodal points for $k \ge 1$ may readily be obtained in a similar fashion with 5.4.18 and 5.5.22 in [12] as

$$\|u - p_N(u)\|_{H_w^{l;N}(\rho)} \le CN^{2l-1/2-k} |u|_{H_w^{k;N}(\rho)}, \tag{17}$$

$$\|I_N(u) - u\|_{\mathcal{L}_w^2(\rho)} \le CN^{-k} |u|_{H_w^{k;N}(\rho)}, \tag{18}$$

where $u \in H_w^k(\rho)$, and C is a constant independent of N.

Lemma 4.2 (*Gronwall's Lemma*)**.** *Assume that $u, \psi, \beta \in C(\rho)$ with $\beta(t) \ge 0$. If u satisfies the inequality*

$$u(t) \le \psi(t) + \int_0^t \beta(s)u(s)\mathrm{d}s, \quad t \in \rho$$

then

$$u(t) \le \psi(t) + \int_0^t \beta(s)\psi(s) \exp\left(\int_s^t \beta(v)\mathrm{d}v\right)\mathrm{d}s, \quad t \in \rho$$

if ψ is non-decreasing on ρ, the above inequality reduces to

$$u(t) \le \psi(t) \exp\left(\int_0^t \beta(s)\mathrm{d}s\right), \quad t \in \rho.$$

Lemma 4.3 (*Generalized Hardy's Inequality, see [22,23]*)**.** *For all measurable functions $f \ge 0$, the following generalized Hardy's inequality*

$$\left(\int_a^b |(\lambda f)(t)|^q w_1(t)\mathrm{d}t\right)^{1/q} \le C\left(\int_a^b |f(t)|^p w_2(t)\mathrm{d}t\right)^{1/p},$$

holds if and only if

$$\sup_{a<t<b} \left(\int_t^b w_1(t)\mathrm{d}t\right)^{1/q} \left(\int_a^t w_2^{1-p'}(t)\right)^{1/p'} < \infty, \quad p' = \frac{p}{p-1},$$

for $1 < p \le q < \infty$. Here, λ is an operator of the form

$$(\lambda f)(t) = \int_a^t k(t,s)f(s)\mathrm{d}s,$$

with $k(t,s)$ a given kernel, w_1, w_2 weight functions, and $-\infty \le a < b \le \infty$.

Now we state the main result of this section:

Theorem 4.4. *Assume that the exact solution $u(t)$ of (16) is smooth enough. Suppose that the approximate solution $u_N(t)$ is given by the spectral Tau scheme (11) with orthogonal shifted Chebyshev (Legendre) basis, then for sufficiently large N we have*

$$\|e_N(t)\|_{\mathcal{L}^2(\rho)} \leq C_4 N^{1/2-k}|u|_{H^{k;N}(\rho)} + C_5 N^{-1}|u|_{H^{1;N}(\rho)} + C_6 N^{2(\eta-1)-1/2-k}|u|_{H_w^{k;N}(\rho)} + C_g N^{-1}|g|_{H_w^{1;N}(\rho)}, \tag{19}$$

where $\eta = \begin{cases} m, & m \leq 3 \\ 3, & m > 3 \end{cases}$ *and* $u(t) \in H_w^k(\rho)$.

Proof. An extension of the operational Tau scheme was considered for this problem (FDE) in the previous section. The spectral Tau solution u_N was considered as a projection of $u(t)$ on the interval [0, 1]. It is defined through the following equation (see (16))

$$\sum_{i=0}^{n} A_i u_N^{(i)}(t) + I_N\left(\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1} u_N^{(m)}(t)\mathrm{d}s\right) = I_N(g(t)), \quad t \in [0,1],$$

where I_N denotes the interpolating polynomial of the integral term in Theorem 3.3 of part (b). Now with n times integration we obtain

$$\begin{aligned}
&\int_0^t\int_0^{t_n}\cdots\int_0^{t_2} A_n u_N^{(n)}(t_1)\mathrm{d}t_1\mathrm{d}t_2\ldots\mathrm{d}t_n + \sum_{i=0}^{n-1}\int_0^t\int_0^{t_n}\cdots\int_0^{t_2} A_i u_N^{(i)}(t_1)\mathrm{d}t_1\mathrm{d}t_2\ldots\mathrm{d}t_n \\
&\quad + \frac{1}{\Gamma(m-\alpha)}\int_0^t\int_0^{t_n}\cdots\int_0^{t_2} I_N\left(\int_0^{t_1}(t_1-s)^{m-\alpha-1}u_N^{(m)}(s)\mathrm{d}s\right)\mathrm{d}t_1\mathrm{d}t_2\ldots\mathrm{d}t_n \\
&\quad = \int_0^t\int_0^{t_n}\cdots\int_0^{t_2} I_N(g(t_1))\mathrm{d}t_1\mathrm{d}t_2\ldots\mathrm{d}t_n.
\end{aligned} \tag{20}$$

From Lemma 4.1 we may write multiple integrals as single integrals

$$\begin{aligned}
&A_n u_N(t) + C_0(t) + \sum_{i=0}^{n-1}\int_0^t \frac{A_i}{(n-1-i)!}(t-s)^{n-1-i}u_N(s)\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} I_N\left(\int_0^s (s-s_1)^{m-\alpha-1}u_N^{(m)}(s_1)\mathrm{d}s_1\right)\mathrm{d}s \\
&\quad = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} I_N(g(s))\mathrm{d}s,
\end{aligned} \tag{21}$$

where $C_0(t)$ is a polynomial with given initial conditions coefficients. Similarly from (13) we have

$$\begin{aligned}
&A_n u(t) + C_0(t) + \sum_{i=0}^{n-1}\int_0^t \frac{A_i}{(n-1-i)!}(t-s)^{n-1-i}u(s)\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\int_0^s (s-s_1)^{m-\alpha-1}u^{(m)}(s_1)\mathrm{d}s_1\mathrm{d}s = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}g(s)\mathrm{d}s.
\end{aligned} \tag{22}$$

Subtracting (22) from (21), we get

$$A_n e_N(t) + \sum_{i=0}^{n-1}\int_0^t \frac{A_i}{(n-1-i)!}(t-s)^{n-1-i}e_N(s)\mathrm{d}s + \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}E(s)\mathrm{d}s = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}e_g(s)\mathrm{d}s, \tag{23}$$

where

$$\begin{aligned}
E(s) &= I_N\left(\int_0^s (s-s_1)^{m-\alpha-1}u_N^{(m)}(s_1)\mathrm{d}s_1\right) - \int_0^s (s-s_1)^{m-\alpha-1}u^{(m)}(s_1)\mathrm{d}s_1 \\
&= I_N\left(\int_0^s (s-s_1)^{m-\alpha-1}u_N^{(m)}(s_1)\mathrm{d}s_1\right) - \int_0^s (s-s_1)^{m-\alpha-1}u_N^{(m)}(s_1)\mathrm{d}s_1 \\
&\quad + \int_0^s (s-s_1)^{m-\alpha-1}u_N^{(m)}(s_1)\mathrm{d}s_1 - \int_0^s (s-s_1)^{m-\alpha-1}u^{(m)}(s_1)\mathrm{d}s_1 \\
&= e_{I_N} + \int_0^s (s-s_1)^{m-\alpha-1}e_N^{(m)}(s_1)\mathrm{d}s_1,
\end{aligned}$$

$$e_N(s) = u_N(s) - u(s), \qquad e_g(s) = I_N(g(s)) - g(s),$$

$$e_{I_N}(s) = I_N\left(\int_0^s (s-s_1)^{m-\alpha-1}u_N^{(m)}(s_1)\mathrm{d}s_1\right) - \int_0^s (s-s_1)^{m-\alpha-1}u_N^{(m)}(s_1)\mathrm{d}s_1,$$

therefore we may write (23) as follows

$$A_n e_N(t) + \sum_{i=0}^{n-1} \frac{A_i}{(n-1-i)!} \int_0^t (t-s)^{n-1-i} e_N(s)\mathrm{d}s + \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} e_{I_N}(s)\mathrm{d}s$$
$$+ \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \int_0^s (s-s_1)^{m-\alpha-1} e_N^{(m)}(s_1)\mathrm{d}s_1\mathrm{d}s = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} e_g(s)\mathrm{d}s. \tag{24}$$

Now using $(n-1)$ times integrating by parts we obtain

$$\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \int_0^s (s-s_1)^{m-\alpha-1} e_N^{(m)}(s_1)\mathrm{d}s_1\mathrm{d}s = \frac{1}{\Gamma(m-\alpha)} \int_0^t \int_0^s (s-s_1)^{m+n-\alpha-2} e_N^{(m)}(s_1)\mathrm{d}s_1\mathrm{d}s. \tag{25}$$

Let the domain of m be divided into two disjoint subsets, $m < 4$ and $m \geq 4$. First we consider the case where $m \geq 4$. Note that in this case the $(N+1)$ Tau coefficients $(a_j)_{j=0}^N$ are chosen to adjust the approximate solution to the given ν initial conditions. In other words, for $i = 1, 2, \ldots, \nu$ we have $u_N^{(i)}(0) = u^{(i)}(0)$. Thus we can write $e_N(0) = e'_N(0) = \cdots = e_N^{(\nu-1)}(0) = 0$. Now with $(m-3)$ times integration by parts we can write

$$\frac{(n-1)!}{\Gamma(m-\alpha)} \int_0^t \int_0^s (s-s_1)^{m+n-\alpha-2} e_N^{(m)}(s_1)\mathrm{d}s_1\mathrm{d}s = \frac{\prod_{i=4}^{m}(n-\alpha-2+i)}{\Gamma(m-\alpha)} \int_0^t \int_0^s (s-s_1)^{n-\alpha+1} e_N^{(3)}(s_1)\mathrm{d}s_1\mathrm{d}s,$$

where we supposed that $n+1 > \alpha$. From the above two relations we may write

$$\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \int_0^s (s-s_1)^{m-\alpha-1} e_N^{(m)}(s_1)\mathrm{d}s_1\mathrm{d}s$$
$$= \frac{\prod_{i=4}^{m}(n-\alpha-2+i)}{\Gamma(m-\alpha)} \int_0^t \int_0^s (s-s_1)^{n-\alpha+1} e_N^{(3)}(s_1)\mathrm{d}s_1\mathrm{d}s. \tag{26}$$

Substituting (26) in (24) we may write

$$|e_N(t)| \leq \sum_{i=0}^{n-1} \left|\frac{A_i}{-A_n(n-1-i)!}\right| \int_0^t |(t-s)^{n-1-i} e_N(s)|\mathrm{d}s + \left|\frac{1}{-A_n(n-1)!\Gamma(m-\alpha)}\right| \left|\int_0^t (t-s)^{n-1} e_{I_N}(s)\mathrm{d}s\right|$$
$$+ \left|\frac{\prod_{i=4}^{m}(n-\alpha-2+i)}{\Gamma(m-\alpha)}\right| \left|\int_0^t \int_0^s (s-s_1)^{n-\alpha+1} e_N^{(3)}(s_1)\mathrm{d}s_1\mathrm{d}s\right| + \left|\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} e_g(s)\mathrm{d}s\right|,$$
$$\leq \gamma_1 \int_0^t |e_N(s)|\mathrm{d}s + \gamma_2 \int_0^t |e_{I_N}(s)|\mathrm{d}s + \gamma_3 \left|\int_0^t s^{n-\alpha+1} e''_N(s)\mathrm{d}s\right| + \gamma_r \int_0^t |e_g(s)|\mathrm{d}s. \tag{27}$$

Now using Gronwall's lemma we may rewrite the above relation as follows

$$|e_N(t)| \leq \exp\left(\int_0^t (\gamma_1)\mathrm{d}s\right) \left(\gamma_2 \int_0^t |e_{I_N}(s)|\mathrm{d}s + \gamma_3 \left|\int_0^t s^{n-\alpha+1} e''_N(s)\mathrm{d}s\right| + \gamma_r \int_0^t |e_g(s)|\mathrm{d}s\right)$$
$$\leq \gamma_4 \int_0^t |e_{I_N}(s)|\mathrm{d}s + \gamma_5 \left|\int_0^t s^{n-\alpha+1} e''_N(s)\mathrm{d}s\right| + \gamma_r \int_0^t |e_g(s)|\mathrm{d}s,$$

equivalently

$$\|e_N(t)\|_{\mathcal{L}^2_w(\rho)} \leq \gamma_4 \left\|\int_0^t |e_{I_N}(s)|\mathrm{d}s\right\|_{\mathcal{L}^2_w(\rho)} + \gamma_5 \left\|\int_0^t s^{n-\alpha+1} e''_N(s)\mathrm{d}s\right\|_{\mathcal{L}^2_w(\rho)} + \gamma_r \left\|\int_0^t |e_g(s)|\mathrm{d}s\right\|_{\mathcal{L}^2_w(\rho)}.$$

Using generalized Hardy's inequality we may write

$$\|e_N(t)\|_{\mathcal{L}^2_w(\rho)} \leq \gamma_6 \|e_{I_N}(s)\|_{\mathcal{L}^2_w(\rho)} + \gamma_7 \|s^{n-\alpha+1} e''_N(s)\|_{\mathcal{L}^2_w(\rho)} + \gamma_8 \|e_{I_N}(s)\|_{\mathcal{L}^2_w(\rho)}. \tag{28}$$

From (17) we get

$$\|s^{n-\alpha} e''_N(s)\|_{\mathcal{L}^2_w(\rho)} \leq \|s^{n-\alpha}\|_{\mathcal{L}^2_w(\rho)} \|e''_N(s)\|_{\mathcal{L}^2_w(\rho)} \leq \gamma_9 \|e_N(s)\|_{H^2_w(\rho)}$$
$$\leq \gamma_{10} N^{7/2-k} |u|_{H^{k;N}_w(\rho)}. \tag{29}$$

From (18) we obtain

$$\|e_{I_N}(t)\|_{\mathcal{L}^2_w(\rho)} \le C_3 N^{-1} \left\| \int_0^t (t-s)^{m-\alpha-1} u_N^{(m)}(s)\mathrm{d}s \right\|_{H_w^{1;N}(\rho)} = C_4 N^{-1} \|D_*^\alpha u_N\|_{H_w^{1;N}(\rho)}$$

$$\|e_g(t)\|_{\mathcal{L}^2_w(\rho)} \le C_g N^{-1} \|g\|_{H_w^{1;N}(\rho)}.$$

Linear operator $D_*^\alpha : \mathbb{P}_N \to \mathbb{P}_N$ is continuous and bounded (see [24]), thus there exists a constant C^* such that $\|D_*^\alpha u_N\|_{H_w^{K;N}(\rho)} \le C^* \|u_N\|_{H_w^{k;N}(\rho)}$. Using this and (18) we proceed with the above relation as

$$\begin{aligned}
\|e_{I_N}(t)\|_{\mathcal{L}^2_w(\rho)} &\le C^* N^{-1} \|u_N(s)\|_{H_w^{1;N}(\rho)} \\
&\le C^* N^{-1} (\|e_N(s)\|_{H_w^{1;N}(\rho)} + \|u\|_{H_w^{1;N}(\rho)}) \\
&\le C^* N^{-1} (C_2 N^{3/2-k} \|u\|_{H_w^{k;N}(\rho)} + \|u\|_{H_w^{1;N}(\rho)}) \\
&\le C_3 N^{1/2-k} \|u\|_{H_w^{k;N}(\rho)} + C^* N^{-1} \|u\|_{H^{1;N}(\rho)}.
\end{aligned} \tag{30}$$

Finally from (28)–(30) for $m \ge 4$ we obtain

$$\|e_N(t)\|_{\mathcal{L}^2(\rho)} \le C_4 N^{1/2-k} \|u\|_{H^{k;N}(\rho)} + C_5 N^{-1} \|u\|_{H^{1;N}(\rho)} + C_6 N^{2(m-1)-1/2-k} |u|_{H_w^{k;N}(\rho)} + C_g N^{-1} \|g\|_{H_w^{1;N}(\rho)}. \tag{31}$$

This inequality proves that the approximation is convergent and the error decays faster than algebraically when the solution is sufficiently smooth.

We now consider the case where $m \le 3$. Substituting (25) in (24) in a similar manner with (23) we may write

$$|e_N(t)| \le \gamma_1 \int_0^t |e_N(s)|\mathrm{d}s + \gamma_2 \int_0^t |e_{I_N}(s)|\mathrm{d}s + \gamma_3 \left| \int_0^t s^{n-\alpha-2+m} e_N^{(m-1)}(s)\mathrm{d}s \right| + \gamma_r \int_0^t |e_g(s)|\mathrm{d}s, \tag{32}$$

where we supposed that $m - \alpha - 2 + n > 0$. By an argument similar to that used for proving (27), the following error bound between the exact solution and the Tau approximate solution of (13) can be established:

$$\|e_N(t)\|_{\mathcal{L}^2(\rho)} \le C_4 N^{1/2-k} \|u\|_{H^{k;N}(\rho)} + C_5 N^{-1} \|u\|_{H^{1;N}(\rho)} + C_6 N^{2(m-1)-1/2-k} |u|_{H_w^{k;N}(\rho)} + C_g N^{-1} \|g\|_{H_w^{1;N}(\rho)}. \quad \square \tag{33}$$

In the previous section we have considered Jacobi polynomials as basis functions for the spectral Tau method. But in this section convergence analysis of the applied method has been discussed using Chebyshev (Legendre) basis functions.

Note that, the convergence analysis of the proposed scheme can be established in a similar manner with Theorem 4.4, using the Jacobi truncation error bound [25] and Jacobi polynomial approximation error bound [26] instead of (17) and (18), respectively.

5. Numerical experiments

In this section, three test problems were solved using Chebyshev and Legendre Tau methods. All calculations were performed on a PC running Mathematica® software. In tables "Max. Err. in Chebyshev–Gauss–Lobatto (CGL) points" (Legendre–Gauss–Lobatto (LGL) points) always refers to the maximal difference between approximation and exact solution at the Gauss–Lobatto points and "Max Err. in [0, 1]" shows the maximum value of the error function in [0, 1]. In all cases any non-polynomial coefficient was replaced by a suitable truncated polynomial expansion. Results obtained using the presented scheme, agree well with the analytical solutions and the numerical results of the compared papers. Also, they confirm the convergence of the presented scheme.

Example 5.1. The Bagley–Torvik equation, one of the most popular FDEs, studied in [11,27] is in the form of

$$A\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + B\frac{\mathrm{d}^{3/2}}{\mathrm{d}t^{3/2}}x(t) + Cx(t) = f(t).$$

Consider the case where $f(t) = 1 + t,\ A = 1,\ B = 1$ and $C = 1$ with initial conditions

$$x(0) = x'(0) = 1.$$

The solution process has five steps, according to Algorithm 1;

$$p_0(t) = 1, \qquad p_1(t) = 0, \qquad p_2(t) = 1, \qquad q_0(t) = 1.$$

Table 1
The Tau approximation errors of example 2 with Chebyshev basis functions.

N	Max. Err. in CGL points	Max. Err. in [0, 1]
4	7.51×10^{-16}	8.03×10^{-16}
6	5.40×10^{-16}	7.33×10^{-16}
8	1.01×10^{-16}	3.50×10^{-16}

1.

$$N = 3, \qquad v_i(t) = T_i^*(t), \quad i = 1, 2, 3, \qquad x_N(t) = \sum_{i=0}^{3} a_i T_i^*(t), \quad t \in [0, 1],$$

where $T_i^*(t) = T_i(2x - 1)$, $i = 0, \ldots, 3$, stands for the shifted Chebyshev polynomial of degree i defined in [0, 1].

2.

$$L_D(x(t)) = \frac{d^2}{dt^2}x(t) + x(t), \qquad \Pi_v = \sum_{i=0}^{2} \eta^i p_i(\mu) V^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 16 & 0 & 1 & 0 \\ 0 & 96 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

3.

$$L_f(x(t)) = \frac{d^{3/2}}{dt^{3/2}}x(t), \qquad \ddot{\Pi}_v = \left(\sum_{i=0}^{N_f} V\ddot{\Pi}_i V q_i(\mu)\right) V^{-1} = V\ddot{\Pi}_0 V q_0(\mu) V^{-1},$$

$$\ddot{\Pi}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.564 & 0.564 & 0 & 0 \\ 0.634 & 0.846 & 0.211 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \qquad \ddot{\Pi}_v = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4.513 & 4.513 & 0 & 0 \\ -6.770 & 0 & 6.770 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

4.

$$\widehat{\Pi} = \Pi_v + \ddot{\Pi}_v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 20.513 & 4.513 & 1 & 0 \\ -6.770 & 96 & 6.770 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -1 & 2 & 0 & \cdots \\ 1 & -8 & 0 & \cdots \\ -1 & 18 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

5.

$$G_N = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 1 & -8 & 20.513 & 4.513 \\ -1 & 18 & -6.770 & 96 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \qquad R_N = \begin{pmatrix} 1 \\ 1 \\ 1.5 \\ 0.5 \end{pmatrix}^t.$$

We have obtained a square system $(a_0, \ldots, a_3, 0, \ldots)G_N = R_N$ which, when solved gives us $a_0 = 1.5$, $a_1 = 0.5$, $a_2 = 0$, $a_3 = 0$ and unknown function $x_N(t) = 1 + t$. Numerical results will not be presented since the exact solution is obtained.

Example 5.2 (*From [28]*). Consider the equation

$$\begin{aligned} &aD^2x(t) + b(t)D_*^{\alpha_2}(x(t)) + c(t)Dx(t) + e(t)D_*^{\alpha_1}(x(t)) + k(t)x(t) \\ &\quad = -a - \frac{b(t)}{\Gamma(3-\alpha_2)}t^{2-\alpha_2} - c(t)t - \frac{e(t)}{\Gamma(3-\alpha_1)}t^{2-\alpha_1} + k(t)\left(2 - \frac{1}{2}t^2\right), \end{aligned}$$

where $x(0) = 2$ and $x'(0) = 0$ with exact solution $x(t) = 2 - \frac{1}{2}t^2$.

For $a = 1$, $b(t) = t$, $c(t) = t + 1$, $e(t) = t^2$, $k(t) = (t + 1)^2$, $\alpha_1 = \frac{\sqrt{3}}{30}$, and $\alpha_2 = \sqrt{3}$ we obtained the following results.

Numerical results are shown in Table 1. The exact solution has been obtained with $N = 6$. Our scheme produces better results in comparison to El-Mesiry et al. in [28].

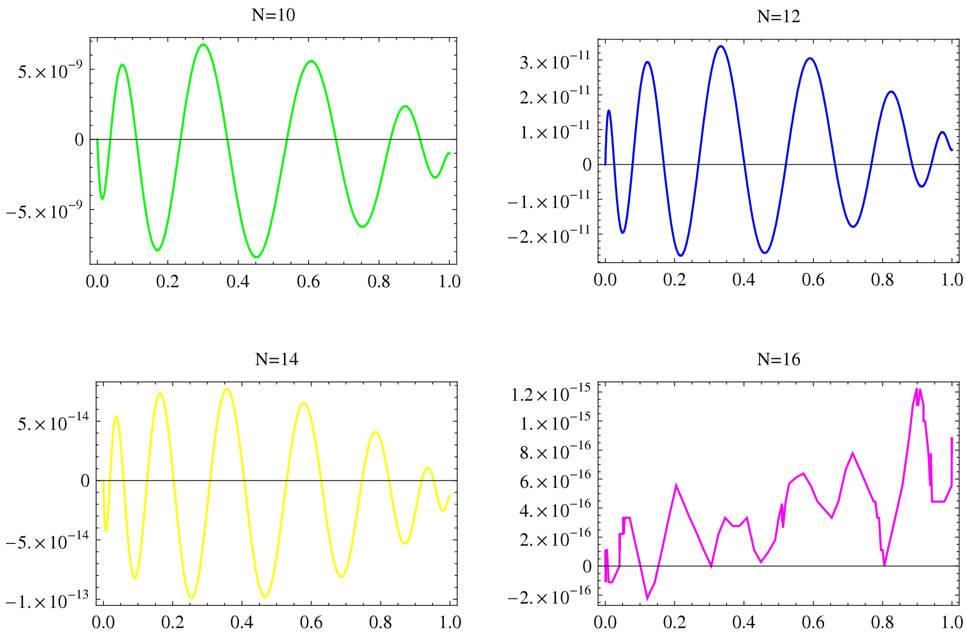

Fig. 1. Graph of the Tau approximation error of example 3 for Chebyshev bases.

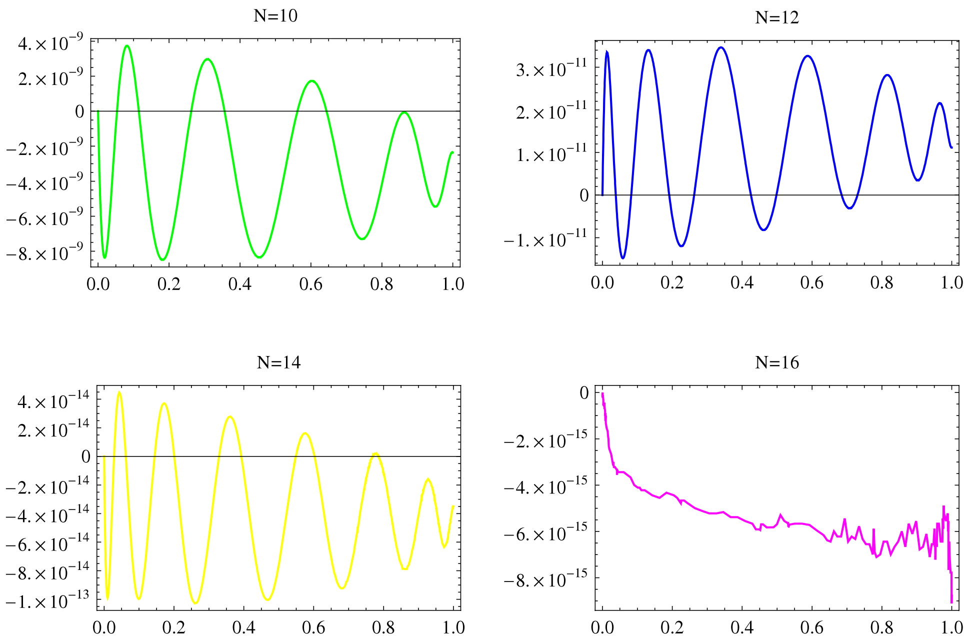

Fig. 2. Graph of the Tau approximation error of example 3 for Legendre bases.

Again, for $a = 1$, $b(t) = \frac{1}{(t^2+1)}$, $c(t) = t^2 + 3$, $e(t) = \frac{1}{(t-2)}$, $k(t) = t - 1$, $\alpha_1 = \frac{1}{3}$, and $\alpha_2 = \frac{4}{5}$, computational results have been reported in Table 2.

Example 5.3 (*From [9]*)**.** Consider the equation

$$D_*^{\alpha}x(t) + 2x(t) = 2\cos \pi t + \frac{t^{-\alpha}}{2\Gamma(1-\alpha)}({}_1F_1(1; 1-\alpha; \mathrm{i}\pi t).{}_1F_1(1; 1-\alpha; -\mathrm{i}\pi t)) - 2,$$

where $x(0) = 1$ and $pFq(a; b; z)$ is the generalized hypergeometric function with exact solution $x(t) = \cos \pi t$ and $\alpha = 0.5$.

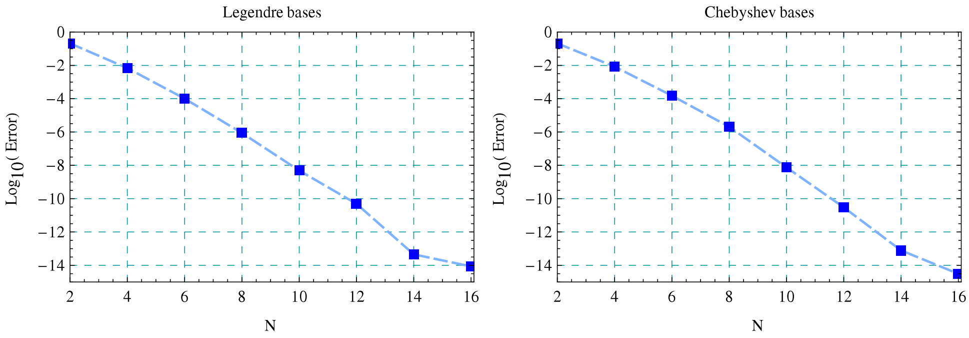

Fig. 3. L^∞ Tau approximation errors of example 3.

Table 2
The Tau approximation errors of example 2 with Legendre basis functions.

N	Max. Err. in LGL points	Max. Err. in [0, 1]
4	7.47×10^{-16}	9.41×10^{-16}
6	3.40×10^{-16}	6.26×10^{-16}
8	2.14×10^{-16}	4.01×10^{-16}

Table 3
The Tau approximation errors of example 3 with Legendre basis functions.

N	Max. Err. in LGL points	Max. Err. in [0, 1]
8	3.20×10^{-6}	1.21×10^{-6}
10	1.41×10^{-9}	3.23×10^{-9}
12	1.92×10^{-11}	3.70×10^{-11}

Table 4
The Tau approximation errors of example 3 with Chebyshev basis functions.

N	Max. Err. in CGL points	Max. Err. in [0, 1]
8	9.14×10^{-7}	1.61×10^{-6}
10	8.68×10^{-9}	6.61×10^{-9}
12	7.14×10^{-11}	4.53×10^{-11}

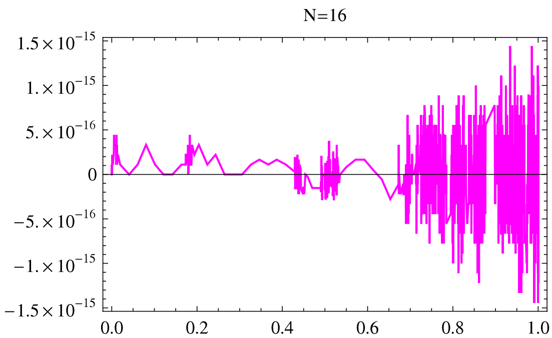

Fig. 4. The Tau approximation error of example 3 with LGL points and double precision.

The computational results for various N, with Chebyshev and Legendre bases have been reported in Tables 3 and 4 and Figs. 1–3. Numerical results show the robustness of the method. We observe that the extended Tau approximation is an extremely good approximation to solution of FDEs and is much better in fact than the other existing methods.

Fig. 3 shows that the Tau approximation error is less for the case of Chebyshev polynomial basis than in the case of Legendre polynomial basis when N is greater than or equal to 14 (Figs. 1 and 2). In the Tau Legendre method, we apply Legendre–Gauss–Lobatto points. The main problem with those points is that they are not given explicitly and their evaluation for large N is not robust due to roundoff errors [29,30].

It is possible to avoid this difficulty by increasing the machine precision. This enables us to modify our computational results even for large approximation degree N. (See Fig. 4.)

6. Conclusion

Numerical solution of multi-order FDEs with an extension of the Tau method is presented in this paper. If we make a comparison with other existing methods, we find that this scheme is completely superior. Finally, we established the convergence of the Tau approximation. In order to derive the convergence analysis of the scheme, for $m \geq 3$ we assumed that $n+1 > \alpha$ and otherwise $n-2+m > \alpha$. This assumption on n is restrictive. The application of this discussion to more general problems, seems to be a worthy area of further research.

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