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Certain results for a class of convex functions related to a shell-like curve connected with Fibonacci numbers

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ABSTRACT

This paper investigates some basic geometric properties for the class $\mathcal{KSL}$ of functions f analytic in the open unit disc $\Delta = \{z : |z| < 1\}$ (which is related to a shell-like curve and associated with Fibonacci numbers) satisfying the condition that

$$f(0) = 0, \quad f'(0) = 1 \quad \text{and} \quad \frac{zf''(z)}{f'(z)} \prec \frac{\tau z + 2\tau^2 z^2}{1 - \tau z - \tau^2 z^2} \quad (z \in \Delta),$$

where, the number $\tau = (1-\sqrt{5})/2$ is such that $|\tau|$ fulfils the golden section of the segment $[0, 1]$. Some relevant remarks and useful connections of the main results are also pointed out.

1. Introduction

Assume that $\mathcal{A}$ is the class of all holomorphic functions f in the open unit disc Δ with normalization $f(0) = 0, f'(0) = 1$, and let $\mathcal{S}$ (as is customary) be the subclass of $\mathcal{A}$ which consists of univalent functions. We say that f is subordinate to F in Δ, written as $f \prec F$, if and only if $f(z) = F(\omega(z))$ for some holomorphic function ω such that $|\omega(z)| \le |z|,\ z \in \Delta$. The class $\mathcal{SL}$ of shell-like functions is the set of functions $f \in \mathcal{A}$ satisfying the condition that

$$\frac{zf'(z)}{f(z)} \prec \widetilde{p}(z) \quad (z \in \Delta), \tag{1.1}$$

where

$$\widetilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}, \quad \tau = \frac{1 - \sqrt{5}}{2} \approx -0.618,\ (z \in \Delta). \tag{1.2}$$

It should be observed that $\mathcal{SL}$ is a subclass of the class of the starlike functions $\mathcal{S}^*$. The class of shell-like functions $\mathcal{SL}$ was defined in [1] and further examined in [2]. The function (1.2) has some nice properties. The name attributed to the class $\mathcal{SL}$ is motivated by the shape of the curve

$$\mathfrak{C} = \left\{\widetilde{p}(e^{it}),\ t \in [0, 2\pi) \setminus \{\pi\}\right\}, \tag{1.3}$$

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which is a shell-like curve and a simple transformation converts it into a curve called the conchoid of de Sluze (René François Baron de Sluze 1622–1685). For details, see Section 2. Moreover, the coefficients of (1.2) are connected with the Fibonacci numbers as explained in Lemma 1.1 and in the next section.

A geometric description of the conchoid of de Sluze is given here as follows:

A ray OB is drawn from the point $O(0, 0)$ and it cuts the directrix $x = a$, where $a > 0$ at the point $B(a, b)$. From the point $B(a, b)$, segments BM and BN are laid off in either direction along the ray such that

$$|OB| \cdot |BM| = k^2 \quad \text{and} \quad |OB| \cdot |BN| = k^2,$$

where $k > 0$ is given. As b changes, the ray revolves, and the point M describes a curve (called the conchoid of de Sluze) given by

$$a(x-a)(x^2+y^2)+k^2x^2=0, \tag{1.4}$$

while the point N describes a curve (called the conjugate of the conchoid of de Sluze) given by

$$a(x-a)(x^2+y^2)-k^2x^2=0. \tag{1.5}$$

Motivated by these ideas, we consider for the purpose of this paper a certain class $\mathcal{KSL}$ of functions analytic in the open unit disc which connects (by means of the subordination) the class $\mathcal{K}$ of convex functions with the class $\mathcal{SL}$ of shell-like functions. The various results studied depict some of the basic geometric properties for this function class $\mathcal{KSL}$. Some relevant cases and useful remarks are also mentioned.

Definition 1. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{KSL}$ of convex shell-like functions if it satisfies the condition that

$$1+\frac{zf''(z)}{f'(z)} \prec \widetilde{p}(z) \quad (z \in \Delta), \tag{1.6}$$

where the function $\widetilde{p}$ is defined in (1.2).

It may be pointed out here that Ma and Minda in [3] defined $\mathcal{C}(\varphi)$, (or $\mathcal{S}^*(\varphi)$) to be the class of all normalized functions $f(z) = z + a_2z^2 + \cdots$ such that $1+zf''(z)/f'(z) \prec \varphi(z)$, (or $1+zf'(z)/f(z) \prec \varphi(z)$), where φ is fixed function analytic and univalent in the unit disc Δ with $\mathfrak{Re}\varphi(z) > 0$, $\varphi(\Delta)$ is symmetric with a real axis and φ is starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. Several subclasses of convex and starlike functions were unified in this way. Because the function $\widetilde{p}$ is not univalent in Δ, we cannot use the results from [3] to obtain some theorems on the class $\mathcal{KSL}$. The radius of univalence and several other properties of the function $\widetilde{p}$ were found in [4]. Let us recall some of them.

Lemma 1.1 ([4]). *Let the function $\widetilde{p}$ be given by (1.2), then it satisfies the following:*

(1) *$\widetilde{p}$ is univalent in the disc $|z| < (3-\sqrt{5})/2 \approx 0.38$, any increase in the greater side makes the assertion false,*
(2) *$\widetilde{p}(z) = 1+\sum_{n=1}^{\infty}(u_{n-1}+u_{n+1})\tau^n z^n = 1+\tau z+3\tau^2z^2+4\tau^3z^3+7\tau^4z^4+11\tau^5z^5+\cdots$, where $\{u_n\}$ is the sequence of Fibonacci numbers $u_0 = 0,\ u_1 = 1,\ u_{n+2} = u_n + u_{n+1}$ $(n = 0, 1, 2, 3, \ldots)$,*
(3) *$\lim_{\varphi\to\pi^-} \mathfrak{Im}[\widetilde{p}(e^{i\varphi})] = -\infty$, and $\lim_{\varphi\to\pi^+} \mathfrak{Im}[\widetilde{p}(e^{i\varphi})] = \infty$,*
(4) *$\mathfrak{Re}\widetilde{p}(e^{i\varphi}) = \frac{\sqrt{5}}{2(3-2\cos\varphi)} \geq \frac{\sqrt{5}}{10} = \gamma$ for all $\varphi \in [0, 2\pi)$.*

In [4], the authors presented a class $\mathcal{SLM}_\alpha$, $\alpha \in [0, 1]$, of functions that are analytic in the open unit disc such that

$$f(0)=0, \qquad f'(0)=1 \quad \text{and} \quad \alpha\left[1+\frac{zf''(z)}{f'(z)}\right]+(1-\alpha)\frac{zf'(z)}{f(z)} \in \widetilde{p}(\Delta) \quad \text{for all } z \in \Delta.$$

This class $\mathcal{SLM}_\alpha$ is related to the presently investigated class $\mathcal{KSL}$ only through the function $\widetilde{p}$ and $\mathcal{SLM}_\alpha \neq \mathcal{KSL}$ for all α. It easy to see that $\mathcal{KSL} \subset \mathcal{SLM}_1$ but $\mathcal{KSL} \neq \mathcal{SLM}_1$ because $\widetilde{p}$ is not univalent function. The present paper deals with ideas and techniques used in geometric function theory. The central problem considered here is the coefficient estimates for this class depicted by the Fibonacci numbers. Besides the coefficient problems, we also provide some interesting corollaries concerning the connections of our defined class with other well known classes.

2. Preliminary lemmas

For some in-depth understanding of the class $\mathcal{KSL}$ it would be worthwhile here to find the shape of the curve $\mathfrak{C} = \{\widetilde{p}(e^{it}),\ t \in [0, 2\pi) \setminus \{\pi\}\}$. We begin our study by noting that

$$\widetilde{p}(0) = \widetilde{p}\left(-\frac{1}{2\tau}\right) = 1 \quad \text{and} \quad \widetilde{p}(1) = \widetilde{p}(\tau^4) = \frac{\sqrt{5}}{2}.$$

Moreover, because

$$\widetilde{p}\left(e^{\pm i\arccos(1/4)}\right) = \frac{\sqrt{5}}{5} = 2\gamma,$$

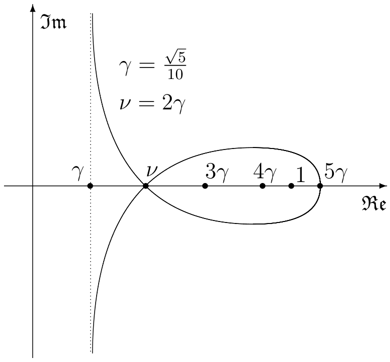

Fig. 1. Curve $\mathfrak{C}: (10x-\sqrt{5})y^2 = (\sqrt{5}-2x)(\sqrt{5}x-1)^2$.

so the curve $\mathfrak{C}$ intersects itself on the real axis at the point 2γ. If we denote

$$\mathfrak{Re}\widetilde{p}(e^{i\varphi}) = x \quad \text{and} \quad \mathfrak{Im}\widetilde{p}(e^{i\varphi}) = y, \quad \varphi \in [0, 2\pi) \setminus \{\pi\},$$

then after simple calculations, we get

$$x = \frac{\sqrt{5}}{2(3-2\cos\varphi)}, \qquad y = \frac{\sin\varphi(4\cos\varphi - 1)}{2(3-2\cos\varphi)(1+\cos\varphi)}, \quad \varphi \in [0, 2\pi) \setminus \{\pi\}. \tag{2.1}$$

It is useful here to use (2.1) to find the corresponding Cartesian equation of the curve $\mathfrak{C}$. This curve is described by the equation

$$(10x-\sqrt{5})y^2 = (\sqrt{5}-2x)(\sqrt{5}x-1)^2. \tag{2.2}$$

It is worthy to point out that for $k = 2a$, the conchoid of de Sluze (1.4) becomes the trisectrix of Maclaurin (Colin Maclaurin 1698–1746):

$$x^3 + 3ax^2 + (x-a)y^2 = 0, \tag{2.3}$$

while the conjugate of the conchoid (1.5) becomes the conjugate of the trisectrix of Maclaurin given by

$$x^3 - 5ax^2 + (x-a)y^2 = 0. \tag{2.4}$$

If we rewrite (2.2) in the following form

$$\left(\frac{\sqrt{5}}{5} - x\right)^3 + \frac{3\sqrt{5}}{10}\left(\frac{\sqrt{5}}{5} - x\right)^2 + \left[\left(\frac{\sqrt{5}}{5} - x\right) - \frac{\sqrt{5}}{10}\right]y^2 = 0,$$

then the image of the unit circle under the function $\widetilde{p}$ is translated into a trisectrix of Maclaurin (2.3) (with $a = (1-2\tau)/10 = \sqrt{5}/10$). Therefore the curve $\mathfrak{C}$ has a shell-like shape, see Fig. 1.

Let us refer the class $\mathcal{K}(\alpha) \subset \mathcal{S}$, $(0 \le \alpha < 1)$, of convex functions of order α, introduced in [5] and defined by

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \mathfrak{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha \text{ for all } z \in \Delta \right\}.$$

If $f \in \mathcal{KSL}$, then by (1.6) we have

$$\mathfrak{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \min\{\mathfrak{Re}\widetilde{p}(z),\ z \in \partial\Delta\} > \frac{\sqrt{5}}{10}$$

by condition (4) of Lemma 1.1. Therefore, we obtain

$$\mathcal{KSL} \subset \mathcal{K}(\gamma), \tag{2.5}$$

where $\gamma = \sqrt{5}/10 \approx 0.2236$, which means that if $f \in \mathcal{KSL}$, then it is convex of order γ and hence univalent in the unit disc Δ.

Corollary 2.1. *If f is in $\mathcal{KSL}$, then f is univalent in Δ.*

Let us recall the relevant connection of the function defined by (1.2) with the Fibonacci numbers

$$u_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}}, \quad \tau = \frac{1-\sqrt{5}}{2}, \; (n = 0, 1, 2, 3, \ldots) \tag{2.6}$$

contained in (2) of Lemma 1.1. Moreover if

$$\widetilde{p}(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

then the coefficients p_n satisfy

$$p_n = \begin{cases} \tau & \text{for } n = 1, \\ 3\tau^2 & \text{for } n = 2, \\ \tau p_{n-1} + \tau^2 p_{n-2}, & \text{for } n = 3, 4, 5, \ldots, \end{cases} \tag{2.7}$$

where $\tau = \frac{1-\sqrt{5}}{2}$.

It is also worth noting here that since

$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|},$$

so the number $|\tau|$ divides [0, 1] such that it fulfils the golden section of this segment.

Lemma 2.2 ([2]). *If a function*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta) \tag{2.8}$$

is in the class $\mathcal{SL}$, then

$$|a_n| \le |\tau|^{n-1} u_n \quad n = 2, 3, 4, \ldots, \tag{2.9}$$

where u_n are given in (2.6). This result is sharp and the equality in (2.9) is attained by the function

$$\widetilde{g}(z) = \frac{z}{1 - \tau z - \tau^2 z^2} = \frac{1}{\tau\sqrt{5}} \left[\frac{1}{1+z} - \frac{1}{1-\tau^2 z} \right] = z + \sum_{n=2}^{\infty} u_n \tau^{n-1} z^n \quad (z \in \Delta). \tag{2.10}$$

3. Main results

Corollary 3.1. *If a function f of the form (2.8) belongs to the class $\mathcal{KSL}$, then*

$$|a_n| \le \frac{|\tau|^{n-1} u_n}{n} \quad n = 2, 3, 4, \ldots, \tag{3.1}$$

where u_n are given in (2.6). This result is sharp and the equality in (3.1) is attained by the function

$$\widetilde{f}(z) = \frac{1}{1+\tau^2} \log \frac{1+z}{1-\tau^2 z} \quad (z \in \Delta). \tag{3.2}$$

Proof. A function f is in the class $\mathcal{KSL}$ if and only if there exists a function $g \in \mathcal{SL}$ such that

$$g(z) = zf'(z) \quad \text{for all } z \in \Delta, \tag{3.3}$$

or equivalently

$$f(z) = \int_0^z \frac{g(t)}{t} \, dt \quad \text{for all } z \in \Delta. \tag{3.4}$$

The relations (3.3) and (3.4) follow directly from (1.1) and (1.6). Therefore, if

$$zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n \quad (z \in \Delta) \tag{3.5}$$

belongs to the class $\mathcal{SL}$, then by Lemma 2.2, we conclude that $|na_n| \leq |\tau|^{n-1}u_n$, which establishes (3.1). The function (3.2) is such that $z\widetilde{f}'(z) = \widetilde{g}(z)$, where the function $\widetilde{g}$ is given in (2.10), and hence by (3.3), it follows that $\widetilde{f} \in \mathcal{KSL}$. Moreover, by (2.10) we get

$$\widetilde{f}(z) = z + \sum_{n=2}^{\infty} \frac{u_n \tau^{n-1}}{n} z^n \quad (z \in \Delta). \tag{3.6}$$

Thus the result (3.1) is sharp. □

Theorem 3.2. *A function f belongs to the class $\mathcal{KSL}$ if and only if there exists an analytic function q, $q \prec \widetilde{p}$, such that*

$$f(z) = \int_0^z \left[\exp \int_0^w \frac{q(t) - 1}{t} \, \mathrm{d}t \right] \mathrm{d}w. \tag{3.7}$$

Proof. It is known that a function $g \in \mathcal{SL}$ can be expressed by

$$g(z) = z \exp \int_0^z \frac{q(t) - 1}{t} \, \mathrm{d}t, \tag{3.8}$$

where $q \prec \widetilde{p}$. In view of (3.3), we infer that $f \in \mathcal{KSL}$ if and only if $zf'(z) = g(z)$ (with $g \in \mathcal{SL}$), and integrating (3.8), we obtain (3.7). □

Theorem 3.2 provides us a method of finding the members of the class $\mathcal{KSL}$. Let us refer the class $\mathcal{S}^*(\alpha)(0 \leq \alpha < 1)$ of starlike functions of order α, introduced in [5] and defined by

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z} \right\}$$

and the class (see [6,7])

$$\mathcal{S}^*(A, B) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \right\},$$

where $-1 < B < A \leq 1$.

Theorem 3.3. *If a function f belongs to the class $\mathcal{KSL}$, then there exists a function $g \in \mathcal{S}^*(0, -\tau^2)$ and a function $h \in \mathcal{S}^*(1/2)$ such that*

$$z^2 f'(z) = g(z)h(z) \quad (z \in \Delta). \tag{3.9}$$

Proof. Let $f \in \mathcal{KSL}$, then by Theorem 3.2, there exists a holomorphic function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \Delta$ such that

$$z^2 f'(z) = z^2 \exp \int_0^z \frac{\widetilde{p}\,(\omega(t)) - 1}{t} \, \mathrm{d}t. \tag{3.10}$$

Observe that

$$\widetilde{p}\,(\omega(t)) = \frac{1}{1 - \tau^2 \omega(t)} + \frac{1}{1 + \omega(t)} - 1, \tag{3.11}$$

so we can rewrite (3.10) in the form

$$\begin{aligned} z^2 f'(z) &= z^2 \exp \left[\int_0^z \frac{\frac{1}{1-\tau^2\omega(t)} - 1}{t} \, \mathrm{d}t + \int_0^z \frac{\frac{1}{1+\omega(t)} - 1}{t} \, \mathrm{d}t \right] \\ &= z \exp \int_0^z \frac{\frac{1}{1-\tau^2\omega(t)} - 1}{t} \, \mathrm{d}t \cdot z \exp \int_0^z \frac{\frac{1}{1+\omega(t)} - 1}{t} \, \mathrm{d}t \\ &=: g(z)h(z). \end{aligned} \tag{3.12}$$

Using the structural formulas for the class $\mathcal{S}^*(A, B)$ (see [7, p. 315]) and for the class $\mathcal{S}^*(\alpha)$ (see [8, p. 172]), we can find that the functions g and h defined by (3.12) satisfy $g \in \mathcal{S}^*(0, -\tau^2)$ and $h \in \mathcal{S}^*(1/2)$, respectively. □

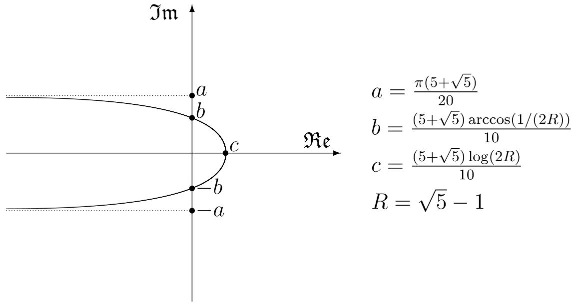

Fig. 2. The curve $\mathfrak{D} = \{\widetilde{f}(e^{i\varphi}),\ \varphi \in [0, 2\pi) \setminus \{\pi\}\}$.

If the functions $g \in \mathcal{S}^*(0, -\tau^2)$ and $h \in \mathcal{S}^*(1/2)$ are generated by their structural formulas with the same function ω, then we can reverse the above steps of the proof, and by (3.11) and (3.12), we can get the function f such that

$$z^2 f'(z) = g(z)h(z) \quad (z \in \Delta)$$

is in the class $\mathcal{KSL}$. For example, if $\omega(z) = xz$, $|x| \le 1$, then (3.12) gives

$$z^2 f'(z) = \frac{z^2}{(1+xz)(1-\tau^2 xz)} \quad (z \in \Delta).$$

Upon integrating we obtain that the functions

$$f(z) = \frac{1}{1+\tau^2} \log \frac{1+xz}{1-\tau^2 xz} \quad (z \in \Delta),$$

belong to the class $\mathcal{KSL}$ for all $|x| \le 1$. For $x = 1$ it becomes the function $\widetilde{f}$ given by (3.2), which shows the sharpness of the estimation (3.1) of coefficients in the class $\mathcal{KSL}$, for its Taylor expansion see (3.2). This function is also extremal for the other problems in this class. Let us see what it looks like $\widetilde{f}(\Delta)$. To find the image of Δ under the function

$$\widetilde{f}(z) = \frac{1}{1+\tau^2} \log \frac{1+z}{1-\tau^2 z} \quad (z \in \Delta),$$

we observe that the function

$$w(z) = \frac{1+z}{1-\tau^2 z} \quad (z \in \Delta)$$

maps the circle $|z| = 1$ onto the circle

$$|w - R| = R, \quad \text{where } R = \sqrt{5} - 1 \approx 1.236.$$

If we denote $\operatorname{Arg} w(e^{it}) = \psi(t),\ t \in [0, 2\pi)$, then it is easy to observe that $\psi(t) \in (-\pi/2, \pi/2)$ and $|w(e^{it})| = 2R\cos\psi(t)$. By setting $\Re\mathfrak{e}\widetilde{f}(e^{it}) = x$ and $\Im\mathfrak{m}\widetilde{f}(e^{it}) = y,\ t \in [0, 2\pi) \setminus \{\pi\}$, then after some calculations, we get from (3.2) that

$$x = \frac{5+\sqrt{5}}{10} \log[2R\cos\psi], \qquad y = \frac{5+\sqrt{5}}{10}\psi, \quad \psi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{3.13}$$

Consequently, the function (3.2) maps the unit circle onto a curve $\mathfrak{D}$ described by (3.13), see Fig. 2. The functions $\widetilde{p}$ and $\widetilde{f}$ are connected by the relation:

$$1 + \frac{z\widetilde{f}''(z)}{\widetilde{f}'(z)} = \widetilde{p}(z) \quad (z \in \Delta).$$

Just as the Koebe function plays a central role in the class $\mathcal{S}$ the function $\widetilde{f}$ plays a central role in the class $\mathcal{KSL}$.

Theorem 3.2 provides us a method of finding the members of the class $\mathcal{KSL}$ for given function $q,\ q \prec \widetilde{p}$. Much more hard to verify is the question if given f belongs to the class $\mathcal{KSL}$. Even when we consider a simple polynomial, e.g. $g(z) = z + cz^n$, then (1.6) becomes

$$\frac{1 + n^2 c z^{n-1}}{1 + ncz^{n-1}} \prec \widetilde{p}(z) \quad (z \in \Delta)$$

and it is difficult to find all c satisfying this subordination because $\widetilde{p}$ is not univalent. The next theorem excludes some polynomials of $\mathcal{KSL}$ and somewhat solves this problem.

Theorem 3.4. *If* $n \in \{2, 3, 4, \ldots\}$ *and*

$$|c| > \frac{\sqrt{5}-1}{n(\sqrt{5}n-1)}, \tag{3.14}$$

then the function $g(z) = z + cz^n$ *does not belong to the class* $\mathcal{KSL}$.

Proof. Let us denote

$$G(z) := 1 + \frac{zg''(z)}{g'(z)} = \frac{1+n^2cz^{n-1}}{1+ncz^{n-1}} \quad (z \in \Delta).$$

We prove that if (3.14) is satisfied, then $G(z) \not\prec \widetilde{p}(z)$. It suffices to show that $G(\Delta) \not\subset \widetilde{p}(\Delta)$. The set $\widetilde{p}(\Delta)$ is on the right of the curve in Fig. 1. The set $G(\Delta)$ is a disc with the diameter from $x_1 = \frac{1-n^2|c|}{1-n|c|}$ to $x_2 = \frac{1+n^2|c|}{1+n|c|}$. If (3.14) is satisfied, then the one of x_i, where $i = 1, 2$, satisfies $x_i < 2\gamma = \sqrt{5}/5$, and then $G(\Delta) \not\subset \widetilde{p}(\Delta)$. This proves Theorem 3.4. □

Thus for $n = 2$: if $g(z) = z + cz^2 \in \mathcal{KSL}$, then $|c| \le (\sqrt{5}-1)/(4\sqrt{5}-2) \approx 0.18$.

Theorem 3.5. *If* $f \in \mathcal{KSL}(|z| = r, 0 \le r < 1)$, *then*

$$\frac{(1+r)^{2\gamma-1}-1}{2\gamma-1} \le |f(z)| \le |\widetilde{f}(-r)| = \frac{1}{1+\tau^2}\log\frac{1+\tau^2 r}{1-r} \tag{3.15}$$

and

$$\frac{1}{1+(1+\tau^2)r+\tau^2r^2} \le |f'(z)| \le \widetilde{f}'(-r) = \frac{1}{1+\tau r-\tau^2r^2}, \tag{3.16}$$

where $\gamma = \sqrt{5}/10$ *and* $\tau = (1-\sqrt{5})/2$. *The upper bounds are sharp.*

Proof. Let $f \in \mathcal{KSL}$ be of the form (2.8), and let us denote by b_n the coefficients of the function $\widetilde{f}$ given by (3.6). Then by (3.1), we have $|a_n| \le |b_n|$ for each integer $n \ge 0$. Notice that the even coefficients $b_n = u_n\tau^{n-1}/n$ are negative, while the odd coefficients are positive. If $z = re^{i\theta}$, then from the coefficient inequality $|a_n| \le |b_n|$, we obtain

$$|f(z)| \le r + \sum_{n=2}^{\infty}|a_n|r^n \le r + \sum_{n=2}^{\infty}|b_n|r^n = r + \sum_{n=2}^{\infty}b_n(-1)^{n-1}r^n = -\widetilde{f}(-r)$$

which yields the upper bound of (3.15). Analogously, we can obtain

$$\begin{aligned} |f'(z)| &= \left|1 + \sum_{n=2}^{\infty}na_nz^{n-1}\right| \le 1 + \sum_{n=2}^{\infty}n|a_n|\,|z|^{n-1} \\ &\le 1 + \sum_{n=2}^{\infty}n|b_n|\,|z|^{n-1} = 1 + \sum_{n=2}^{\infty}n|b_n|r^{n-1} \\ &= 1 - 2b_2r + 3b_3r^2 - 4b_4r^3 + \cdots = \widetilde{f}'(-r), \end{aligned}$$

and we get the upper bound of the inequality (3.16). It is easy to see that the upper bounds are sharp, being attained by the function $\widetilde{f} \in \mathcal{KSL}$ at the point $z = -r$. To find the left-hand side of the inequality (3.15), let us recall (see [7, pp. 315–317]), that if $g \in \mathcal{S}^*(0, -\tau^2)$, then for $|z| = r$ $(0 \le r < 1)$, we have

$$\frac{r}{1+\tau^2r} \le |g(z)|. \tag{3.17}$$

Moreover, if $h \in \mathcal{S}^*(1/2)$, then for $|z| = r$ $(0 \le r < 1)$, we get

$$\frac{r}{1+r} \le |h(z)|. \tag{3.18}$$

By Theorem 3.3, we have $|z^2f'(z)| = |g(z)|\,|h(z)|$ with $g \in \mathcal{S}^*(0, -\tau^2)$ and $h \in \mathcal{S}^*(1/2)$, and multiplying the respective sides of (3.17) with those of (3.18), we thus obtain the left-hand side of (3.16). To prove the left-hand side of the inequality (3.15), we note that by (2.5) if $f \in \mathcal{KSL}$, then $f \in \mathcal{K}(\gamma)$ is convex of order γ. The desired inequality will now follow from the well-known inequality for $|f(z)|$ with $f \in \mathcal{K}(\gamma)$ (see, for example [8, p. 139]). □

Theorem 3.6. *If a function f of the form (2.8) belongs to $\mathcal{KSL}$, then*

$$n|a_n| \le 3b_3 = 3-\sqrt{5} \approx 0.764, \quad n = 2,3,4,\ldots, \tag{3.19}$$

where $b_n = u_n\tau^{n-1}/n$ are the coefficients of $\widetilde{f} \in \mathcal{KSL}$ given by (3.6). Moreover, if the coefficient sequence $\{a_n\}$ of the function f converges, then

$$\lim_{n\to\infty} n|a_n| \le \frac{5+\sqrt{5}}{10} \approx 0.72, \tag{3.20}$$

with equality holding for the coefficients b_n of the function $\widetilde{f}$.

Proof. If $\widetilde{f}(z) = z + b_2z^2 + b_3z^3 + \cdots$, then by (3.6) we have

$$nb_n = \tau^{n-1}u_n = \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right],$$

which implies that

$$nb_n = (n-1)b_{n-1}\tau + (n-2)b_{n-2}\tau^2$$

and

$$2|b_2| \le n|b_n| \le 3b_3 \quad \text{for } n \ge 2,$$

where

$$b_2 = ((1-\sqrt{5}-1)/4) \approx -0.31, \qquad b_3 = (3-\sqrt{5})/3 \approx 0.26.$$

Using (3.1), we have $|a_n| \le |b_n|$, which readily yields the inequality (3.19). The coefficient inequality $|a_n| \le |b_n|$ and the limiting case that

$$\lim_{n\to\infty} n|b_n| = \frac{\sqrt{5}+5}{10}$$

leads at once to the other inequality (3.20). □

By putting $z = re^{i\varphi}$, $\varphi \in [0,2\pi) \setminus \{\pi\}$, and performing simple calculations, we get

$$\begin{aligned}\widetilde{p}\left(re^{i\varphi}\right) &= \frac{1+\tau^2r^2e^{2i\varphi}}{1-\tau re^{i\varphi}-\tau^2r^2e^{2i\varphi}} \\ &= \frac{(1+\tau^2r^2)(1-\tau^2r^2-\tau r\cos\varphi)}{|1-\tau re^{i\varphi}-\tau^2r^2e^{2i\varphi}|^2} + \frac{i\tau r(1-\tau^2r^2+4\tau r\cos\varphi)\sin\varphi}{|1-\tau re^{i\varphi}-\tau^2r^2e^{2i\varphi}|^2}.\end{aligned}$$

Hence

$$\begin{aligned}\left|\frac{\mathfrak{Im}[\widetilde{p}\left(re^{i\varphi}\right)]}{\mathfrak{Re}[\widetilde{p}\left(re^{i\varphi}\right)]}\right| &= \left|\frac{\tau r(1-\tau^2r^2+4\tau r\cos\varphi)\sin\varphi}{(1+\tau^2r^2)(1-\tau^2r^2-\tau r\cos\varphi)}\right| \\ &\le \frac{-\tau r(1-\tau^2r^2-4\tau r)}{(1-\tau^2r^2+\tau r)(1+\tau^2r^2)} := \phi(r),\end{aligned} \tag{3.21}$$

whenever $r < r_0 = \frac{3-\sqrt{5}}{2}$. By Lemma 1.1 for such r the curve $\widetilde{p}(re^{it})$, $t \in [0,2\pi) \setminus \{\pi\}$, has no loops and $\widetilde{p}$ is univalent in $\Delta_{r_0} = \{z : |z| < r_0\}$. Therefore

$$\left[\frac{zf'(z)}{f(z)} \prec \widetilde{p}(z),\ \left(z \in \Delta_{r_0}\right)\right] \Leftrightarrow \left[\frac{zf'(z)}{f(z)} \in \widetilde{p}\left(\Delta_{r_0}\right) \text{ for all } z \in \Delta_{r_0}\right],$$

where the subordination $F \prec G$ in a disc $|z| < r$ denote that $F(z) = G(\omega(z))$ for some holomorphic function ω, $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $|z| < r$. A simple geometric observation of (3.21) gives the following theorem.

Theorem 7. *If $f \in \mathcal{KSL}$, then*

$$\left|\operatorname{Arg}\left[1+\frac{zf''(z)}{f'(z)}\right]\right| < \arctan\phi(r) \quad \left(|z| < r < r_0 = \frac{3-\sqrt{5}}{2}\right), \tag{3.22}$$

where $\phi(r)$ is given by (3.21).

This theorem says that if $f \in \mathcal{KSL}$, then f is strongly convex of order $\frac{2}{\pi}\arctan\phi(r)$ (see [9]) in the disc $|z| < r$, whenever $r < r_0$.

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