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Vectorial form of Ekeland-type variational principle with applications to vector equilibrium problems and fixed point theory

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Abstract

This paper introduces a vectorial form of equilibrium version of Ekeland-type variational principle. Some equivalent results to our variational principle are given. As applications, we derive the existence of solutions of a vector equilibrium problem in the setting of complete quasi-metric spaces with a W-distance. Caristi–Kirk fixed point theorem for multivalued maps is also established in a more general setting.

Keywords: Vectorial form of Ekeland-type variational principle; Vector equilibrium problems; Complete quasi-metric spaces; Caristi–Kirk fixed point theorem

1. Introduction

An existence result for an approximate minimizer of a lower semicontinuous and bounded below function was given by Ekeland in 1972 [14] (see also [15,16]), now it is known as Ekeland's variational principle (in short, EVP). It appeared as one of the most useful tools to solve the problems in optimization, optimal control theory, game theory, nonlinear equations, dynamical systems, etc. See, for example, [4–6,17,23,30] and references therein.

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The equilibrium problem is a unified model of several problems, for example, optimization problem, saddle point problem, Nash equilibrium problem, variational inequality problem, nonlinear complementarity problem, fixed point problem, etc. In the last decade, it has emerged as a new direction of research in nonlinear analysis, optimization, optimal control, game theory, mathematical economics, etc. Most of the results on the existence of solutions of equilibrium problems are studied in the setting of topological vector spaces by using some kind of fixed point (Fan–Browder type fixed point) theorem or KKM type theorem. In [9,28], Blum, Oettli and Théra first gave the existence of a solution of an equilibrium problem in the setting of complete metric spaces. They have also showed that their existence result for a solution of equilibrium problem is equivalent to Ekeland-type variational principle for bifunctions, Caristi–Kirk fixed point theorem for multivalued maps [10] and a maximal element theorem.

After the introduction of vector variational inequalities by F. Giannessi [21] in 1980, equilibrium problems have been extensively studied for vector valued functions in the setting of topological vector spaces. To the best of our knowledge, so far there is no existence result for solutions of vector equilibrium problems in the setting of complete metric spaces. This paper is an effort in this direction.

In this paper, we first establish Ekeland-type variational principle for vector valued functions in the setting of complete quasi-metric spaces with a W-distance. Then by using this result, we derive the existence results for a solution of a vector equilibrium problem. We also establish some equivalent results to our Ekeland-type variational principle. Caristi–Kirk fixed point theorem for multivalued maps is established in a more general setting.

2. Preliminaries

Throughout the paper, unless otherwise specified, we denote by $\mathbb{N}$ the set of positive integers, $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+ = [0, \infty)$.

Let X be a nonempty set. A real valued function $d : X \times X \to \mathbb{R}_+$ is said to be a *quasi-metric* on X if the following conditions are satisfied:

(M1) $d(x, y) \geqslant 0$, for all $x, y \in X$;
(M2) $d(x, y) = 0$ if and only if $x = y$;
(M3) $d(x, y) \leqslant d(x, z) + d(z, y)$, for all $x, y, z \in X$.

A set X together with a quasi-metric d is called a *quasi-metric space* and it is denoted by (X, d). Therefore, the concept of a quasi-metric space generalizes the concept of a metric space by lifting the symmetry condition.

Let Y be a locally convex Hausdorff topological vector space with its zero vector $\mathbf{0}$, C a proper, closed and convex cone in Y and $\operatorname{int} C \neq \emptyset$, and e a fixed vector in Y such that $e \in \operatorname{int} C$, where $\operatorname{int} C$ denotes the interior of C. Recall that $C \subseteq Y$ is said to be *closed and convex cone* if C is closed, $\alpha C \subseteq C$ for all $\alpha > 0$ and $C + C \subseteq C$. In addition, if $C \neq Y$, then C is called a *proper, closed and convex cone*. A closed convex cone is *pointed* if $C \cap (-C) = \{\mathbf{0}\}$.

The nonlinear *scalarization function* [13,20] (see also [23,24]) $\xi_e : Y \to \mathbb{R}$ is defined as

$$\xi_e(y) := \inf\{r \in \mathbb{R}:\ y \in re - C\}, \quad \text{for all } y \in Y.$$

We present some properties of scalarization function which will be used in the sequel.

Lemma 2.1. *(See [13,20]. See also [23,24].) For each $r \in \mathbb{R}$ and $y \in Y$, the following statements are satisfied*:

(i) $\xi_e(y) \leqslant r \Leftrightarrow y \in re - C$.
(ii) $\xi_e(y) > r \Leftrightarrow y \notin re - C$.
(iii) $\xi_e(y) \geqslant r \Leftrightarrow y \notin re - \operatorname{int} C$.
(iv) $\xi_e(y) < r \Leftrightarrow y \in re - \operatorname{int} C$.
(v) $\xi_e(\cdot)$ *is positively homogeneous and continuous on* Y.
(vi) $\xi_e(y_1 + y_2) \leqslant \xi_e(y_1) + \xi_e(y_2)$, *for all* $y_1, y_2 \in Y$.
(vii) $\xi_e(\cdot)$ *is monotone, that is, if* $y_1 \in y_2 + C$, *then* $\xi_e(y_2) \leqslant \xi_e(y_1)$.

Definition 2.1. (See [12,27].) A function $\phi : X \to Y$ is said to be

(i) *C-bounded below* if there exists $y \in Y$ such that $\phi(x) \subseteq y + C$, for all $x \in X$;
(ii) *(e, C)-lower semicontinuous* if for all $r \in \mathbb{R}$, the set $\{x \in X\colon \phi(x) \in re - C\}$ is closed;
(iii) *(e, C)-upper semicontinuous* if for all $r \in \mathbb{R}$, the set $\{x \in X\colon \phi(x) \in re + C\}$ is closed;
(iv) *(e, C)-continuous* if it is both (e, C)-lower semicontinuous as well as (e, C)-upper semicontinuous;
(v) *C-lower semicontinuous* if for all $y \in Y$, the set $\{x \in X\colon \phi(x) \in y - C\}$ is closed;
(vi) *C-upper semicontinuous* if for all $y \in Y$, the set $\{x \in X\colon \phi(x) \in y + C\}$ is closed;
(vii) *C-continuous* if it is both C-lower semicontinuous as well as C-upper semicontinuous.

Remark 2.1. It is easy to see that the C-lower (respectively upper) semicontinuity of ϕ implies the (e, C)-lower (respectively upper) semicontinuity.

Lemma 2.2. *(See [31].)*

(i) *If ϕ is (e, C)-lower semicontinuous and C-bounded below, then $\xi_e \circ \phi$ is lower semicontinuous and bounded below.*
(ii) *If ϕ is (e, C)-upper semicontinuous, then $\xi_e \circ \phi$ is upper semicontinuous.*

Now, we define the concept of a W-distance for a quasi-metric space.

Definition 2.2. Let (X, d) be a quasi-metric space. A function $\omega : X \times X \to \mathbb{R}_+$ is called a *W-distance* on X if the following conditions are satisfied:

(i) $\omega(x_1, x_3) \leqslant \omega(x_1, x_2) + \omega(x_2, x_3)$, for all $x_1, x_2, x_3 \in X$.
(ii) For any fixed $x \in X$, $\omega(x, \cdot) \mapsto \mathbb{R}_+$ is lower semicontinuous.
(iii) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\omega(x_3, x_1) \leqslant \delta$ and $\omega(x_3, x_2) \leqslant \delta$ imply $d(x_1, x_2) \leqslant \varepsilon$.

If we replace quasi-metric space by the metric space, above definition reduces to the concept of a W-distance for a metric space which is introduced by Kada et al. [25]. Several examples and properties of a W-distance for metric spaces are given in [25,30]. For the quasi-metric space (X, d), the concept of Cauchy sequences and completeness can be defined in the same manner as in the setting of metric spaces.

The following lemma describes some properties of a W-distance and it will be used in the sequel.

Lemma 2.3. *(See [25].) Let ω be a W-distance on X and, $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in X. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}_+$ converging to 0, and let $x, y, z \in X$. Then the following conditions hold*:

(i) *if $\omega(x_n, y) \leqslant \alpha_n$ and $\omega(x_n, z) \leqslant \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $\omega(x, y) = 0$ and $\omega(x, z) = 0$, then $y = z$;*
(ii) *if $\omega(x_n, x_m) \leqslant \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence*;
(iii) *if $t > 0$, then $t\omega$ is also a W-distance on X.*

We introduce the following concept of a W-diameter of a nonempty subset of a quasi-metric space.

Definition 2.3. Let $\omega: X \times X \to \mathbb{R}_+$ be a W-distance on X. The *W-diameter* of a nonempty subset D of X is, denoted by $\omega(D)$,

$$\omega(D) = \sup_{x,y\in D} \omega(x, y).$$

Remark 2.2. (a) Since every metric d on a metric space X is a W-distance (see [25, Example 1]), the diameter of a nonempty subset of a metric space is equal to the W-diameter with respect to the W-distance d. But converse need not be true in general.

We establish the following intersection theorem in the setting of a W-distance on a complete quasi-metric space.

Proposition 2.1. *Let (X, d) be a complete quasi-metric space and let $\omega: X \times X \to \mathbb{R}_+$ be a W-distance on X. Let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty closed subsets of X such that*

(i) *for every $n \in \mathbb{N}$, $D_{n+1} \subseteq D_n$,*
(ii) *$\omega(D_n) \leqslant \alpha_n$, where $\alpha_n \geqslant 0$ and $\alpha_n \to 0$ as $n \to \infty$.*

Then there exists exactly one point $\bar{x} \in X$ such that $D = \bigcap_{n=1}^{\infty} D_n = \{\bar{x}\}$.

Proof. For every $n \in \mathbb{N}$, let $x_n \in D_n$, since D_n is nonempty. By (i), for $m > n$, $D_m \subseteq D_n$ and for each $x_m \in D_m$, we have $x_m \in D_n$. Then

$$\omega(x_n, x_m) \leqslant \sup_{x,x'\in D_n} \omega(x, x') = \omega(D_n) \quad \text{whenever } m > n.$$

It follows from (ii) that

$$\omega(x_n, x_m) \leqslant \omega(D_n) \leqslant \alpha_n, \quad \text{for } n, m \in \mathbb{N} \text{ and } m > n.$$

Since $\alpha_n \in \mathbb{R}_+$ and $\alpha_n \to 0$ as $n \to \infty$, we have by Lemma 2.3(ii) that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete, there exists $\bar{x} \in X$ such that $x_n \to \bar{x}$ as $n \to \infty$. For every $n \in \mathbb{N}$, it is easy to see that $\bar{x}$ is an accumulation point of D_n, so that $\bar{x} \in D_n$ since D_n is closed. It follows that $\bar{x} \in D = \bigcap_{n=1}^{\infty} D_n$.

It remains to show that $\bar{x}$ is the only point in D. Suppose that $\bar{y} \in D$, then $\bar{y} \in D_n$ for all $n \in \mathbb{N}$, and so

$$\sup_{x \in D_n} \omega(x, \bar{y}) \leqslant \omega(D_n) \leqslant \alpha_n.$$

Also $\bar{x} \in D$, we have $\bar{x} \in D_n$ for all $n \in \mathbb{N}$ and so

$$\sup_{x \in D_n} \omega(x, \bar{x}) \leqslant \omega(D_n) \leqslant \alpha_n.$$

Then by Lemma 2.3(i), $\bar{y} = \bar{x}$. □

3. Vectorial form of Ekeland-type variational principle

We present the following vectorial form of equilibrium version of Ekeland-type variational principle in the setting of complete quasi-metric spaces and W-distances which is one of the main motivations of this paper.

Theorem 3.1. *Let (X, d) be a complete quasi-metric space, $\omega : X \times X \to \mathbb{R}_+$ a W-distance on X, Y a locally convex Hausdorff topological vector space, C a proper, closed and convex cone in Y with apex at origin and* $\operatorname{int} C \neq \emptyset$*, and $e \in Y$ a fixed vector such that $e \in \operatorname{int} C$. Let $F : X \times X \to Y$ be a function satisfying the following conditions*:

(i) $F(x, x) = \mathbf{0}$, *for all $x \in X$;*
(ii) $F(x, y) + F(y, z) \in F(x, z) + C$, *for all $x, y, z \in X$;*
(iii) *for each fixed $x \in X$, the function $F(x, \cdot) : X \mapsto Y$ is (e, C)-lower semicontinuous and C-bounded below.*

Then for every $\varepsilon > 0$ and for every $\hat{x} \in X$, there exists $\bar{x} \in X$ such that

(a) $F(\hat{x}, \bar{x}) + \varepsilon\omega(\hat{x}, \bar{x})e \in -C$,
(b) $F(\bar{x}, x) + \varepsilon\omega(\bar{x}, x)e \notin -C$, *for all $x \in X$, $x \neq \bar{x}$.*

Proof. For the sake of convenience, we set $\omega_\varepsilon(x, y) = (1/\varepsilon)\omega(x, y)$. Then by Lemma 2.3(iii), ω_ε is a W-distance on X. For all $x \in X$, define

$$S(x) = \bigl\{y \in X\colon\ x = y \text{ or } \xi_e\bigl(F(x, y)\bigr) + \omega_\varepsilon(x, y) \leqslant 0\bigr\}$$

and set

$$\mathcal{V}(x) := \inf_{y \in S(x)} \xi_e\bigl(F(x, y)\bigr).$$

Then clearly $x \in S(x)$, so $S(x)$ is nonempty for all $x \in X$. Also, $\mathcal{V}(x) \leqslant 0$ for all $x \in X$. Since $F(x, \cdot)$ is (e, C)-lower semicontinuous and C-bounded below, Lemma 2.2(i) implies that $\xi_e(F(x, \cdot))$ is lower semicontinuous and bounded below. Since $\omega_\varepsilon(x, \cdot)$ is lower semicontinuous, $S(x)$ is closed for all $x \in X$.

Let $x_0 = \hat{x} \in X$. Since $\xi_e(F(x, \cdot))$ is bounded below, we have

$$\mathcal{V}(x_0) = \inf_{y \in X} \xi_e\bigl(F(x_0, y)\bigr) > -\infty.$$

Let $n \in \mathbb{N}$ and assume that x_{n-1} has been defined with $\mathcal{V}(x_{n-1}) > -\infty$. Choose $x_n \in S(x_{n-1})$ such that

$$\xi_e\big(F(x_{n-1}, x_n)\big) \leqslant \mathcal{V}(x_{n-1}) + \frac{1}{n}.$$

Let $y \in S(x_n) \setminus \{x_n\}$ then

$$\xi_e\big(F(x_n, y)\big) + \omega_\varepsilon(x_n, y) \leqslant 0. \tag{3.1}$$

Since $x_n \in S(x_{n-1})$, we have

$$\xi_e\big(F(x_{n-1}, x_n)\big) + \omega_\varepsilon(x_{n-1}, x_n) \leqslant 0. \tag{3.2}$$

Adding (3.1) and (3.2), we obtain

$$\xi_e\big(F(x_{n-1}, x_n)\big) + \xi_e\big(F(x_n, y)\big) + \omega_\varepsilon(x_{n-1}, x_n) + \omega_\varepsilon(x_n, y) \leqslant 0.$$

Using the triangle inequality for W-distances, we obtain from above inequality

$$\xi_e\big(F(x_{n-1}, x_n)\big) + \xi_e\big(F(x_n, y)\big) + \omega_\varepsilon(x_{n-1}, y) \leqslant 0. \tag{3.3}$$

By condition (ii) and using Lemma 2.1(vi) and (vii), we have

$$\xi_e\big(F(x_{n-1}, y)\big) \leqslant \xi_e\big(F(x_{n-1}, x_n)\big) + \xi_e\big(F(x_n, y)\big). \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\xi_e\big(F(x_{n-1}, y)\big) + \omega_\varepsilon(x_{n-1}, y) \leqslant 0$$

and so $y \in S(x_{n-1})$ which implies that $S(x_n) \subseteq S(x_{n-1})$. Therefore, we obtain

$$\begin{aligned}
\mathcal{V}(x_n) &= \inf_{y \in S(x_n)} \xi_e\big(F(x_n, y)\big) \geqslant \inf_{y \in S(x_{n-1})} \xi_e\big(F(x_n, y)\big) \\
&\geqslant \inf_{y \in S(x_{n-1})} \big[\xi_e\big(F(x_{n-1}, y)\big) - \xi_e\big(F(x_{n-1}, x_n)\big)\big] \\
&= \mathcal{V}(x_{n-1}) - \xi_e\big(F(x_{n-1}, x_n)\big) \geqslant -\frac{1}{n}.
\end{aligned}$$

Thus, for $y \in S(x_n) \setminus \{x_n\}$, from (3.1) and by definition of $\mathcal{V}$, we have

$$\omega_\varepsilon(x_n, y) \leqslant -\xi_e\big(F(x_n, y)\big) \leqslant -\mathcal{V}(x_n) \leqslant \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

This shows that $\omega_\varepsilon(x_n, y) \to 0$ as $n \to \infty$. Since $x_n \in S(x_n)$, the W-diameter of $S(x_n)$ $\omega(S(x_n)) \to 0$ as $n \to \infty$. By Proposition 2.1, there exists exactly one point $\bar{x} \in X$ such that $\bigcap_{n=0}^{\infty} S(x_n) = \{\bar{x}\}$.

This implies that $\bar{x} \in S(x_0) = S(\hat{x})$, that is,

$$\xi_e\big(F(\hat{x}, \bar{x})\big) + \omega_\varepsilon(\hat{x}, \bar{x}) \leqslant 0, \quad \text{that is,} \quad \xi_e\big(F(\hat{x}, \bar{x})\big) \leqslant -\omega_\varepsilon(\hat{x}, \bar{x}).$$

From Lemma 2.1(i), we have

$$F(\hat{x}, \bar{x}) \in -\omega_\varepsilon(\hat{x}, \bar{x})e - C, \quad \text{that is,} \quad F(\hat{x}, \bar{x}) + \omega_\varepsilon(\hat{x}, \bar{x})e \in -C$$

and so (a) holds.

Moreover, $\bar{x}$ also belongs to all $S(x_n)$ and, since $S(\bar{x}) \subseteq S(x_n)$ for all n, we have

$$S(\bar{x}) = \{\bar{x}\}.$$

It follows that $x \notin S(\bar{x})$ whenever $x \neq \bar{x}$ implying that

$$\xi_e\big(F(\bar{x}, x)\big) + \omega_\varepsilon(\bar{x}, x) > 0 \quad \text{or} \quad \xi_e\big(F(\bar{x}, x)\big) > -\omega_\varepsilon(\bar{x}, x).$$

From Lemma 2.1(ii), we have

$$F(\bar{x}, x) \notin -\omega_\varepsilon(\bar{x}, x)e - C,$$

that is,

$$F(\bar{x}, x) + \omega_\varepsilon(\bar{x}, x)e \notin -C, \quad \text{for all } x \in X \text{ and } x \neq \bar{x},$$

that is, (b) holds. □

Remark 3.1. (i) If (X, d) is a quasi-metric space, $\preccurlyeq$ is a quasi-order on X defined as

$$x \preccurlyeq y \quad \text{if and only if} \quad x = y \quad \text{or} \quad \xi_e\big(F(x, y)\big) + \omega_\varepsilon(x, y) \leqslant 0,$$

and the set $S(x) = \{y \in X\colon\ x \preccurlyeq y\}$ is $\preccurlyeq$-complete, even then the conclusion of Theorem 3.1 holds. In this case, Theorem 3.1 extends Theorem 1(i) in [29] for vector valued functions. Also, Theorem 3.1 extends Theorem 2(i) in [29] for vector valued functions.

(ii) If X is replaced by a nonempty closed subset K of X, even then the conclusion of Theorem 3.1 holds.

(iii) The conclusion (b) of Theorem 3.1 implies that

$$F(\bar{x}, x) + \varepsilon\omega(\bar{x}, x)e \notin -\operatorname{int} C, \quad \text{for all } x \in X.$$

Indeed, from (b) we have

$$F(\bar{x}, x) + \varepsilon\omega(\bar{x}, x)e \notin -\operatorname{int} C, \quad \text{for all } x \in X,\ x \neq \bar{x}.$$

Suppose that $F(\bar{x}, \bar{x}) + \varepsilon\omega(\bar{x}, \bar{x})e \in -\operatorname{int} C$. Since $F(\bar{x}, \bar{x}) = \mathbf{0}$ by condition (i), $\omega(\bar{x}, \bar{x}) \geqslant 0$ and $e \in \operatorname{int} C$, we obtain $F(\bar{x}, \bar{x}) + \varepsilon\omega(\bar{x}, \bar{x})e \in C$, a contradiction of our supposition.

Corollary 3.1. *Let (X, d), ω, Y, C, and e be the same as in Theorem 3.1 and let $f : X \to Y$ be a (e, C)-lower semicontinuous and C bounded below function. For every given $\varepsilon > 0$, there is $\hat{x} \in X$ such that $f(x) - f(\hat{x}) \notin \varepsilon e - C$ for all $x \in X$, then there exists $\bar{x} \in X$ such that*

(i) *$f(x) - f(\bar{x}) \notin -\varepsilon e - C$, for all $x \in X$,*
(ii) *$f(x) - f(\bar{x}) + \varepsilon\omega(\bar{x}, x)e \notin -C$, for all $x \in X$, $x \neq \bar{x}$.*

Proof. Set $F(x, y) = f(y) - f(x)$ for all $x, y \in X$. Then all the conditions of Theorem 3.1 are satisfied and hence there exists $\bar{x} \in X$ such that

$$f(\bar{x}) - f(\hat{x}) + \varepsilon\omega(\hat{x}, \bar{x})e \in -C \tag{3.5}$$

and

$$f(x) - f(\bar{x}) + \varepsilon\omega(\bar{x}, x)e \notin -C, \quad \text{for all } x \in X,\ x \neq \bar{x},$$

that is, (ii) holds.

Since $\omega(\hat{x}, \bar{x}) \geqslant 0$ and $e \in \operatorname{int} C$, we have $\varepsilon\omega(\hat{x}, \bar{x})e \in C$. Then (3.5) implies that

$$f(\bar{x}) - f(\hat{x}) \in -C - C \subseteq -C. \tag{3.6}$$

By hypothesis, $\hat{x}$ satisfies

$$f(x) - f(\hat{x}) \notin -\varepsilon e - C, \quad \text{for all } x \in X. \tag{3.7}$$

We claim that $f(x) - f(\bar{x}) \notin -\varepsilon e - C$, for all $x \in X$.

Suppose, to the contrary, that

$$f(x) - f(\bar{x}) \in -\varepsilon e - C, \quad \text{for some } x \in X. \tag{3.8}$$

Then from (3.6) and (3.8), we have

$$f(x) - f(\bar{x}) \in -\varepsilon e - C - C \subseteq -\varepsilon e - C,$$

contradicting (3.7). Hence (i) holds. □

Remark 3.2. (i) Corollary 3.1 extends and generalizes Corollary 2.1 in [11] in the following manner:

(1) (X, d) is a complete quasi-metric space instead of a Banach space.
(2) ω is a W-distance instead of a norm $\|\cdot\|$ on X.

(ii) If ω is a W-distance and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, lower semicontinuous and subadditive function such that $\eta^{-1}(\{0\}) = \{0\}$, then $\eta \circ \varepsilon$ is a W-distance, too, and Theorem 3.1 applies also in this case.

If $C = \mathbb{R}_+$ and $Y = \mathbb{R}$, then from Theorem 3.1, we have the following result.

Corollary 3.2. *Let (X, d) and ω be the same as in Theorem* 3.1. *Let $F : X \times X \to \mathbb{R}$ be a function satisfying the following conditions*:

(i) $F(x, x) = 0$, *for all $x \in X$*;
(ii) $F(x, z) \leqslant F(x, y) + F(y, z)$, *for all $x, y, z \in X$*;
(iii) *for each fixed $x \in X$, the function $F(x, \cdot) : X \mapsto \mathbb{R}$ is lower semicontinuous and bounded below.*

Then for every $\varepsilon > 0$ and for every $\hat{x} \in X$, there exists $\bar{x} \in X$ such that

(a) $F(\hat{x}, \bar{x}) + \varepsilon\omega(\hat{x}, \bar{x}) \leqslant 0$;
(b) $F(\bar{x}, x) + \varepsilon\omega(\bar{x}, x) > 0$, *for all $x \in X$, $x \neq \bar{x}$.*

Remark 3.3. (i) If we replace X by a nonempty closed subset K of X, even then the conclusion of Corollary 3.2 holds. In this case, Corollary 3.2 extends Theorem 2.1 in [8] to the complete quasi-metric spaces and a W-distance setting. Corollary 3.2 also generalizes Theorem 3 in [9]. In general, Corollary 3.2 is a well-known Ekeland's variational principle in a more general setting.

(ii) If we consider $F(x, y) = f(y) - f(x)$, where $f : X \to \mathbb{R}$ is lower semicontinuous and bounded below, then Corollary 3.1 is the extension of Theorem 3.3.1 in [6] for complete quasi-metric spaces and a W-distance setting.

4. Existence of solutions of VEP

Throughout this section, unless otherwise specified, Y, C, and e are the same as in the previous section and (X, d) is a complete metric space.

Let K be a nonempty subset of X and let $F : K \times K \to Y$ be a vector valued function. The *vector equilibrium problem* (in short, VEP) is to find $\bar{x} \in K$ such that

$$F(\bar{x}, x) \notin -\operatorname{int} C, \quad \text{for all } x \in K. \tag{4.1}$$

If $C = \mathbb{R}_+$, then VEP is called an *equilibrium problem*. For further details on equilibrium problems, we refer to a survey article by Flores-Bazán [19] and references therein.

It is well known that VEP contains as special cases several problems, namely, vector optimization problem, vector saddle point problem, vector variational inequality problem, vector complementarity problem, etc. For further details on VEP, we refer to [1–3,7,22] and references therein. A direct application of VEP to generalized semi-infinite programming can be found in [26]. A variational principle, different from Ekeland's variational principle, for VEP is studied in [2,3].

Definition 4.1. Let $F : K \times K \to Y$ and $\lambda \in Y$ be given. A point $\bar{x} \in K$ is called a λ-*equilibrium point* of F if

$$F(\bar{x}, y) + \lambda d(\bar{x}, y) \notin -\operatorname{int} C, \quad \text{for all } y \in K.$$

Theorem 4.1. *Let K be a nonempty compact* (*not necessarily convex*) *subset of X and $F : K \times K \to Y$ satisfy conditions* (i)–(iii) *of Theorem* 3.1 *and for each fixed $y \in K$, the map $x \mapsto F(x, y)$ is (e, C)-upper semicontinuous. Then there exists a solution $\bar{x} \in K$ of VEP.*

Proof. By Theorem 3.1 along with Remark 3.1(iii), for each $n \in \mathbb{N}$, there exists $x_n \in K$ such that

$$F(x_n, y) + \frac{1}{n} d(x_n, y)e \notin -\operatorname{int} C, \quad \text{for all } y \in K,$$

that is, for each $n \in \mathbb{N}$, $x_n \in K$ is a λ-equilibrium point of F for $\lambda = \frac{1}{n}e$. By Lemma 2.1(iii), we have

$$\xi_e\big(F(x_n, y)\big) + \frac{1}{n} d(x_n, y) \geqslant 0, \quad \text{for all } y \in K \text{ and } n \in \mathbb{N}.$$

Since K is compact, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$ as $n \to \infty$. Then by (e, C)-upper semicontinuity of $F(\cdot, y)$ on K, we have $\xi_e \circ F(\cdot, y)$ is upper semicontinuous and thus

$$\xi_e\big(F(\bar{x}, y)\big) \geqslant \limsup_{k \to \infty} \left(\xi_e\big(F(x_{n_k}, y)\big) + \frac{1}{n_k} d(x_{n_k}, y) \right) \geqslant 0, \quad \text{for all } y \in K.$$

Hence again by Lemma 2.1(iii),

$$F(\bar{x}, y) \notin -\operatorname{int} C, \quad \text{for all } y \in K,$$

and thus $\bar{x}$ is a solution of VEP. $\square$

When K is not necessarily compact, we have the following existence result for a solution of VEP.

Theorem 4.2. *Let $(X, \|\cdot\|)$ be a Banach space equipped with weak topology, K a nonempty closed subset of X and $F : K \times K \to Y$ satisfy conditions* (i)–(iii) *of Theorem* 3.1 *and for each fixed $y \in K$, the map $x \mapsto F(x, y)$ is (e, C)-upper semicontinuous. Let the following coercivity condition holds*:

there exists $r > 0$ such that for all $x \in K \setminus K_r$, there exists $y \in K$ with $\|y\| < \|x\|$ satisfying $F(x, y) \in -C$, where $K_r = \{x \in K\colon \|x\| \leqslant r\}$.

Then there exists a solution $\bar{x} \in K$ of VEP.

Proof. For all $x \in K$, define

$$S(x) = \bigl\{y \in K\colon \|y\| \leqslant \|x\| \text{ and } \xi_e\bigl(F(x, y)\bigr) \leqslant 0\bigr\}.$$

Then for all $x \in K$, $S(x) \neq \emptyset$, and for each $x, y \in K$, $y \in S(x)$ implies that $S(y) \subseteq S(x)$. Indeed, for $z \in S(y)$, we have $\|z\| \leqslant \|y\| \leqslant \|x\|$. Condition (iii) in Theorem 3.1 implies that

$$\xi_e\bigl(F(x, z)\bigr) \leqslant \xi_e\bigl(F(x, y)\bigr) + \xi_e\bigl(F(y, z)\bigr) \leqslant 0.$$

Since $\xi_e \circ F(x, \cdot)$ is lower semicontinuous on K, $S(x)$ is closed for all $x \in K$. Also, since $K_{\|x\|}$ is weakly compact, $S(x)$ is weakly compact subset of $K_{\|x\|}$ for all $x \in K$. Then by Theorem 4.1, there exists $\bar{x}_r \in K_r$ such that

$$F(\bar{x}_r, y) \notin -\operatorname{int} C, \quad \text{for all } y \in K_r. \tag{4.2}$$

Assume that there exists $x \in K$ such that $F(\bar{x}_r, x) \in -\operatorname{int} C$. Set $a = \min_{y \in S(x)} \|y\|$ (the minimum is achieved because $S(x)$ is nonempty and weakly compact and the norm is continuous). We consider the following two cases:

Case 1. ($a \leqslant r$). Assume that $y_0 \in S(x)$ such that $\|y_0\| = a$. Then $\|y_0\| = a \leqslant r$ and $\xi_e(F(x, y_0)) \leqslant 0$. Since $F(\bar{x}_r, x) \in -\operatorname{int} C$, we have $\xi_e((\bar{x}_r, x)) < 0$ by Lemma 2.1(iv) and thus

$$\xi_e\bigl(F(\bar{x}_r, x)\bigr) + \xi_e\bigl(F(x, y_0)\bigr) < 0. \tag{4.3}$$

By condition (iii), we obtain

$$\xi_e\bigl(F(\bar{x}_r, y_0)\bigr) \leqslant \xi_e\bigl(F(\bar{x}_r, x)\bigr) + \xi_e\bigl(F(x, y_0)\bigr). \tag{4.4}$$

Combining (4.3) and (4.4), we get

$$\xi_e\bigl(F(\bar{x}_r, y_0)\bigr) < 0 \quad \Rightarrow \quad F(\bar{x}_r, y_0) \in -\operatorname{int} C$$

contradicting (4.2).

Case 2. ($a > r$). Again, assume that $y_0 \in S(x)$ such that $\|y_0\| = a$. Then $\|y_0\| = a > r$ and by coercivity condition we can choose an element $y_1 \in K$ such that $\|y_1\| < \|y_0\| = a$ and satisfying $F(y_0, y_1) \in -C$, that is, $\xi_e(F(y_0, y_1)) \leqslant 0$. Therefore, $y_1 \in S(y_0) \subseteq S(x)$ contradicting $\|y_1\| < a = \min_{y \in S(x)} \|y\|$.

Thus, there is no $x \in K$ such that $F(\bar{x}_r, x) \in -\operatorname{int} C$, that is, $\bar{x}_r$ is a solution of VEP. □

Remark 4.1. Theorems 4.1 and 4.2 can be seen as vectorial forms of Proposition 3.2 and Theorem 4.1, respectively, in [8].

The following results can be easily derived from Theorems 4.1 and 4.2, respectively, by taking $F(x, y) = \phi(y) - \phi(x)$ for all $x, y \in K$, where $\phi : K \to Y$ is a function. These results are the vectorial form of the Weierstrass existence theorem.

Corollary 4.1. *Let K be a nonempty compact subset of X and $\phi : K \to Y$ be a (e, C)-lower semicontinuous and C-bounded below. Then there exists $\bar{x} \in K$ such that $\phi(y) - \phi(\bar{x}) \notin -\operatorname{int} C$ for all $y \in K$.*

Corollary 4.2. *Let $(X, \|\cdot\|)$ be a Banach space equipped with weak topology, K a nonempty closed subset of X and $\phi : K \to Y$ be a (e, C)-lower semicontinuous and C-bounded below. Let the following coercivity condition holds*:

there exists $r > 0$ such that for all $x \in K \setminus K_r$, there exists $y \in K$ with $\|y\| < \|x\|$ satisfying $\phi(y) - \phi(x) \in -C$, where $K_r = \{x \in K \colon \|x\| \leqslant r\}$.

Then there exists $\bar{x} \in K$ such that $\phi(y) - \phi(\bar{x}) \notin -\operatorname{int} C$ for all $y \in K$.

In the rest of the section, (X, d) is a complete quasi-metric space.

Definition 4.2. We say that $x_0 \in X$ satisfies *Condition* (A) if and only if every sequence $\{x_n\} \subseteq X$ satisfying $F(x_0, x_n) \in -C$ for all $n \in \mathbb{N}$ and $F(x_n, x) + \frac{1}{n}\omega(x_n, x)e \notin -\operatorname{int} C$ for all $x \in X$ and $n \in \mathbb{N}$, has a convergent subsequence.

Theorem 4.3. *Let (X, d) be a complete quasi-metric space, ω a W-distance on X and $F : X \times X \to Y$ satisfy conditions* (i)–(iii) *of Theorem* 3.1 *and (e, C)-upper semicontinuous in the first argument. If some $x_0 \in X$ satisfies Condition* (A)*, then there exists $\bar{x} \in X$ such that $F(\bar{x}, x) \notin -\operatorname{int} C$ for all $x \in X$.*

Proof. From Theorem 3.1 along with Remark 3.1(iii), for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that

$$F(x_n, x) + \frac{1}{n}\omega(x_n, x)e \notin -\operatorname{int} C, \quad \text{for all } x \in X, \tag{4.5}$$

and

$$F(\hat{x}, x_n) + \frac{1}{n}\omega(\hat{x}, x_n)e \in -C. \tag{4.6}$$

In view of Lemma 2.1, (4.5) and (4.6), respectively, can be rewritten as

$$\xi_e\big(F(x_n, x)\big) \geqslant -\frac{1}{n}\omega(x_n, x), \quad \text{for all } x \in X, \tag{4.7}$$

and

$$\xi_e\big(F(\hat{x}, x_n)\big) \leqslant -\frac{1}{n}\omega(\hat{x}, x_n). \tag{4.8}$$

Since $\omega(\hat{x}, x_n) \geqslant 0$, we have

$$\xi_e\big(F(\hat{x}, x_n)\big) \leqslant 0 \quad \Leftrightarrow \quad F(\hat{x}, x_n) \in -C, \quad \text{for all } n \in \mathbb{N}.$$

From Condition (A), there exists a subsequence of $\{x_n\}$ which converges to some $\bar{x} \in X$. Then by using the upper semicontinuity of $\xi_e \circ F(\cdot, x)$ and (4.7), we obtain

$$\xi_e\big(F(\bar{x}, x)\big) \geqslant 0, \quad \text{for all } x \in X.$$

Again by applying Lemma 2.1, we have

$$F(\bar{x}, x) \notin -\operatorname{int} C, \quad \text{for all } x \in X. \qquad \square$$

Remark 4.2. (i) If we replace X by a nonempty closed subset K of X in Definition 4.2 and Theorem 4.3, then the conclusion of Theorem 4.3 also holds and gives the existence of a solution of VEP.

(ii) Theorem 4.3 extends Theorem 6(a) in [28] for vector valued functions in the setting of complete quasi-metric spaces.

(iii) Most of the results appearing in the literature on the existence of solutions of VEP, some kind of convexity condition on the underlying function F is required along with convexity structure on the underlying set K; see, for example, [1–3,7,19,22,26] and references therein. But in Theorems 4.1–4.3, neither any kind of convexity condition is required on the function F nor convexity structure on the set K. Therefore, the results of this section are new in the literature.

5. Some equivalences and fixed point results

In this section, we prove some equivalences among our Ekeland-type variational principle, existence of solutions of VEP, Caristi–Kirk type fixed point theorem, and Oettli and Théra type theorem. As an application of our variational principle, we derive a more general version of Caristi and Kirk's fixed point theorem for multivalued maps.

Theorem 5.1. *Let* (X,d), ω, Y, C, *and* e *be the same as in Theorem* 3.1. *Let* $F : X \times X \to Y$ *be a function satisfying the conditions* (i)–(iii) *of Theorem* 3.1. *Then the following statements are equivalent*:

(i) (*Vectorial form of Ekeland-type variational principle*) *For every* $\hat{x} \in X$, *there exists* $\bar{x} \in X$ *such that*

$$\bar{x} \in \hat{S} := \{x \in X\colon\ F(\hat{x}, x) + \omega(\hat{x}, x)e \in -C,\ x \neq \hat{x}\}$$

and

$$F(\bar{x}, x) + \omega(\bar{x}, x)e \notin -C, \quad \text{for all } x \in X \text{ and } x \neq \bar{x}. \tag{5.1}$$

(ii) (*Existence of solutions of VEP*) *Assume that*

$$\begin{cases} \text{for every } \tilde{x} \in \hat{S} \text{ with } F(\tilde{x}, y) \in -C \text{ for all } y \in X, \text{ there exists } x \in X \\ \text{such that } x \neq \tilde{x} \text{ and } F(\tilde{x}, x) + \omega(\tilde{x}, x)e \in -C. \end{cases} \tag{5.2}$$

Then there exists $\bar{x} \in \hat{S}$ *such that* $F(\bar{x}, x) \notin -C$, *for all* $x \in X$.

(iii) (*Caristi–Kirk type fixed point theorem*) *Let* $\Phi : X \to 2^X$ *be a multivalued mapping such that*

$$\begin{cases} \text{for every } \tilde{x} \in \hat{S}, \text{ there exists } x \in \Phi(\tilde{x}) \text{ satisfying} \\ F(\tilde{x}, x) + \omega(\tilde{x}, x)e \in -C. \end{cases} \tag{5.3}$$

Then there exists $\bar{x} \in \hat{S}$ *such that* $\bar{x} \in \Phi(\bar{x})$.

(iv) (*Oettli and Théra type theorem*) *Let* D *be a subset of* X *such that*

$$\begin{cases} \text{for every } \tilde{x} \in \hat{S} \setminus D, \text{ there exists } x \in X \\ \text{such that } x \neq \tilde{x} \text{ and } F(\tilde{x}, x) + \omega(\hat{x}, x)e \in -C. \end{cases} \tag{5.4}$$

Then there exists $\bar{x} \in \hat{S} \cap D$.

Proof. (i) $\Rightarrow$ (iv): Let (i) and the hypothesis of (iv) hold. Then (i) gives $\bar{x} \in \hat{S}$ such that

$$F(\bar{x}, x) + \omega(\bar{x}, x)e \notin -C, \quad \text{for all } x \in X \text{ and } x \neq \bar{x}.$$

From (5.4), we have $\bar{x} \in D$. Hence $\bar{x} \in \hat{S} \cap D$, and (iv) holds.

(iv) $\Rightarrow$ (i): Let (iv) hold. For all $\hat{x} \in X$, define

$$\Gamma(\hat{x}) = \{x \in X\colon\ F(\hat{x}, x) + \omega(\hat{x}, x)e \in -C,\ x \neq \hat{x}\}.$$

Choose $D := \{\hat{x} \in X\colon\ \Gamma(\hat{x}) = \emptyset\}$. If $\hat{x} \notin D$, then from the definition of D, there exists $x \in \Gamma(\hat{x})$. That is, for $\hat{x} \notin D$, there exists $x \in X$ such that

$$x \neq \hat{x} \quad \text{and} \quad F(\hat{x}, x) + \omega(\hat{x}, x)e \in -C.$$

Hence (5.4) is satisfied, and by (iv), there exists $\bar{x} \in \hat{S} \cap D$. Then $\Gamma(\bar{x}) = \emptyset$, that is, $F(\bar{x}, x) + \omega(\bar{x}, x)e \notin -C$ for all $x \neq \bar{x}$. Hence (i) holds.

(ii) $\Rightarrow$ (iv): Suppose that both (ii) and the hypothesis of (iv) hold. Assume, for contradiction, that $\tilde{x} \notin D$ for all $\tilde{x} \in \hat{S}$ satisfying $F(\tilde{x}, y) \in -C$ for all $y \in K$. Then by (5.4), for all $\tilde{x} \in \hat{S}$

$$\text{there exists } x \in X \text{ such that } x \neq \tilde{x} \quad \text{and} \quad F(\tilde{x}, x) + \omega(\tilde{x}, x)e \in -C. \tag{5.5}$$

Hence (5.2) is satisfied and by (ii), there exists $\bar{x} \in \hat{S}$ such that

$$F(\bar{x}, x) \notin -C, \quad \text{for all } x \in X. \tag{5.6}$$

We claim that $F(\bar{x}, x) + \omega(\bar{x}, x)e \notin -C$ for all $x \in X$, $x \neq \bar{x}$ which leads to a contradiction of (5.5). Assume, contrary that, there exists $x \in X$ such that $x \neq \bar{x}$ and

$$F(\bar{x}, x) + \omega(\bar{x}, x)e \in -C, \quad \text{that is,} \quad F(\bar{x}, x) \in -\omega(\bar{x}, x)e - C. \tag{5.7}$$

Since $e \in \operatorname{int} C$ and $\omega(\bar{x}, x) \geqslant 0$, we have

$$\omega(\bar{x}, x)e \in C. \tag{5.8}$$

Combining (5.7) and (5.8), we obtain

$$F(\bar{x}, x) \in -C - C \subseteq -C$$

a contradiction of (5.6).

(iv) $\Rightarrow$ (ii): Suppose that both (iv) and the hypothesis of (ii) hold. Choose $D := \{\tilde{x} \in X\colon F(\tilde{x}, y) \notin -C,\ \text{for all } y \in X\}$. Then by hypothesis (5.2), for every $\tilde{x} \in \hat{S}$ with $F(\tilde{x}, y) \in -C$ for all $y \in X$, there exists $x \in X$ such that $x \neq \tilde{x}$ and $F(\tilde{x}, x) + \omega(\tilde{x}, x)e \in -C$, that is, for every $\tilde{x} \in \hat{S} \setminus D$, there exists $x \in X$ such that $x \neq \tilde{x}$ and $F(\tilde{x}, x) + \omega(\hat{x}, x)e \in -C$. Then by (iv), there exists $\bar{x} \in \hat{S} \cap D$. This implies that $\bar{x} \in \hat{S}$ and $F(\bar{x}, y) \notin -C$ for all $y \in X$. Hence (ii) holds.

(iii) $\Rightarrow$ (iv): Let (iii) and the hypothesis of (iv) hold. Define a multivalued map $\Phi : X \to 2^X$ by

$$\Phi(\tilde{x}) = \{x \in X\colon\ x \neq \tilde{x}\}.$$

Assume, for contradiction, that $\tilde{x} \notin D$ for all $\tilde{x} \in \hat{S}$. By (5.4), for every $\tilde{x} \in \hat{S} \setminus D$, there exists $x \in X$ such that $x \neq \tilde{x}$ and $F(\tilde{x}, x) + \omega(\hat{x}, x)e \in -C$, that is, for every $\tilde{x} \in \hat{S}$, there exists $x \in \Phi(\tilde{x})$ satisfying $F(\tilde{x}, x) + \omega(\hat{x}, x)e \in -C$. Then (iii) implies that there exists $\bar{x} \in \hat{S}$ such that $\bar{x} \in \Phi(\bar{x})$. But this is clearly impossible from the definition of Φ. Hence $\tilde{x} \in D$ for some $\tilde{x} \in \hat{S}$, and (iv) holds.

(iv) $\Rightarrow$ (iii): Suppose that both (iv) and the hypothesis of (iii) hold. Choose $D := \{\tilde{x} \in X\colon \tilde{x} \in \Phi(\tilde{x})\}$. By (5.3), for every $\tilde{x} \in \hat{S}$, there exists $x \in \Phi(\tilde{x})$ satisfying $F(\tilde{x}, x) + \omega(\tilde{x}, x)e \in -C$, that is, for every $\tilde{x} \in \hat{S} \setminus D$, there exists $x \in X$ such that $x \neq \tilde{x}$ satisfying $F(\tilde{x}, x) + \omega(\tilde{x}, x)e \in -C$. Then by (iv) furnishes some $\bar{x} \in \hat{S} \cap D$. From the definition of D, we have $\bar{x} \in \Phi(\bar{x})$. Hence (iii) holds. □

As another application of our Ekeland-type variational principle, we derive a more general version of Caristi and Kirk's fixed point theorem for multivalued maps.

Theorem 5.2 *(Caristi and Kirk's fixed point theorem for multivalued maps). Let (X, d) be a complete quasi-metric space, ω and F be the same as in Corollary* 3.2. *Let $\Phi : X \to 2^X$ be a multivalued map such that*

$$F(x, y) + \omega(x, y) \leqslant 0, \quad \text{for all } x \in X \text{ and } y \in \Phi(x). \tag{5.9}$$

Then there exists $\bar{x} \in X$ such that $\bar{x} \in \Phi(\bar{x})$.

Proof. By using Corollary 3.2 (with $\varepsilon = 1$), we obtain $\bar{x} \in X$ such that

$$F(\bar{x}, x) + \omega(\bar{x}, x) > 0, \quad \text{for all } x \in X \text{ and } x \neq \bar{x}. \tag{5.10}$$

We claim that $\bar{x} \in \Phi(\bar{x})$. Otherwise all $y \in \Phi(\bar{x}) \subseteq X$ are such that $y \neq \bar{x}$. Then from (5.9) and (5.10), we have

$$F(\bar{x}, y) + \omega(\bar{x}, y) \leqslant 0 \quad \text{and} \quad F(\bar{x}, y) + \omega(\bar{x}, y) > 0$$

which can not hold simultaneously. □

Remark 5.1. If ω is a W-distance and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, lower semicontinuous and subadditive function such that $\eta^{-1}(\{0\}) = \{0\}$. Then $\eta \circ \varepsilon$ is a W-distance, too, and Theorem 5.2 applies also in this case.

Remark 5.2. Theorem 5.2 along with Remark 5.1 generalizes Theorem 4.2 in [18] in the following ways:

(1) (X, d) is a complete quasi-metric space instead of a complete metric space.
(2) η is lower semicontinuous instead of continuous.
(3) ω is a W-distance instead of a metric distance d.

In fact, in [18], $\eta : [0, \infty) \to [0, \infty)$ is assumed to be a continuous, nondecreasing and subadditive function such that $\eta^{-1}(\{0\}) = \{0\}$ and (X, d) is a complete metric space. These properties of η imply that the function $d_\eta : X \times X \to \mathbb{R}_+$ defined as $d_\eta(x, y) = \eta(d(x, y))$ is a metric on X and (X, d_η) is complete. Therefore, Theorem 4.2 in [18] is nothing but it is a Caristi and Kirk's fixed point theorem [10].

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