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Asymptotically Efficient Adaptive Allocation Rules*

T. L. LAI AND HERBERT ROBBINS

Department of Statistics, Columbia University, New York, New York 10027

1. INTRODUCTION

Let Π_j $(j = 1, \dots, k)$ denote statistical populations (treatments, manufacturing processes, etc.) specified respectively by univariate density functions $f(x; \theta_j)$ with respect to some measure ν, where $f(\cdot\,; \cdot)$ is known and the θ_j are unknown parameters belonging to some set Θ. Assume that $\int_{-\infty}^{\infty} |x| f(x; \theta)\, d\nu(x) < \infty$ for all $\theta \in \Theta$. How should we sample $x_1, x_2, \dots$ sequentially from the k populations in order to achieve the greatest possible expected value of the sum $S_n = x_1 + \cdots + x_n$ as $n \to \infty$? Starting with [3] there has been a considerable literature on this subject, which is often called the multi-armed bandit problem. The name derives from an imagined slot machine with $k \geq 2$ arms. (Ordinary slot machines with one arm are one-armed bandits, since in the long run they are as effective as human bandits in separating the victim from his money.) When an arm is pulled, the player wins a random reward. For each arm j there is an unknown probability distribution Π_j of the reward. The player wants to choose at each stage one of the k arms, the choice depending in some way on the record of previous trials, so as to maximize the long-run total expected reward. A more worthy setting for this problem is in the context of sequential clinical trials, where there are k treatments of unknown efficacy to be used in treating a long sequence of patients.

An *adaptive allocation rule* φ is a sequence of random variables $\varphi_1, \varphi_2, \dots$ taking values in the set $\{1, \dots, k\}$ and such that the event $\{\varphi_n = j\}$ ("sample from Π_j at stage n") belongs to the σ-field $\mathscr{F}_{n-1}$ generated by the previous values $\varphi_1, x_1, \dots, \varphi_{n-1}, x_{n-1}$. Let $\mu(\theta) = \int_{-\infty}^{\infty} x f(x; \theta)\, d\nu(x)$.

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Then

$$ES_n = \sum_{i=1}^{n} \sum_{j=1}^{k} E\Big(E\big[x_i I_{\{\varphi_i = j\}} | \mathscr{F}_{i-1}\big]\Big)$$
$$= \sum_{j=1}^{k} \mu(\theta_j) ET_n(j), \tag{1.1}$$

where

$$T_n(j) = \sum_{i=1}^{n} I_{\{\varphi_i = j\}} \tag{1.2}$$

is the number of times that φ samples from Π_j up to stage n. The problem of maximizing ES_n is therefore equivalent to that of minimizing the "regret"

$$R_n(\theta_1,\ldots,\theta_k) = n\mu^* - ES_n = \sum_{j:\mu(\theta_j)<\mu^*} \big(\mu^* - \mu(\theta_j)\big) ET_n(j), \tag{1.3}$$

where by definition

$$\mu^* = \max\{\mu(\theta_1),\ldots,\mu(\theta_k)\} = \mu(\theta^*) \qquad \text{for some } \theta^* \in \{\theta_1,\ldots,\theta_k\}. \tag{1.4}$$

Let $I(\theta,\lambda)$ denote the Kullback–Leibler number

$$I(\theta,\lambda) = \int_{-\infty}^{\infty} [\log(f(x;\theta)/f(x;\lambda))] f(x;\theta)\, d\nu(x). \tag{1.5}$$

Then $0 \le I(\theta,\lambda) \le \infty$, and we shall always assume that $f(\cdot\,;\cdot)$ is such that

$$0 < I(\theta,\lambda) < \infty \qquad \text{whenever } \mu(\lambda) > \mu(\theta), \tag{1.6}$$

and

$$\forall\epsilon > 0 \text{ and } \forall\theta,\lambda \text{ such that } \mu(\lambda) > \mu(\theta), \qquad \exists\delta = \delta(\epsilon,\theta,\lambda) > 0$$
$$\text{for which } |I(\theta,\lambda) - I(\theta,\lambda')| < \epsilon \text{ whenever } \mu(\lambda) \le \mu(\lambda') \le \mu(\lambda) + \delta. \tag{1.7}$$

In Sections 3 and 4 we construct adaptive allocation rules φ such that for any fixed values $\theta_1,\ldots,\theta_k$ for which the $\mu(\theta_j)$ are not all equal,

$$R_n(\theta_1,\ldots,\theta_k) \sim \Big\{\sum_{j:\mu(\theta_j)<\mu^*} \big(\mu^* - \mu(\theta_j)\big)/I(\theta_j,\theta^*)\Big\}\log n \qquad \text{as } n \to \infty. \tag{1.8}$$

(Here and in the sequel we use the notations μ^* and θ^* defined in (1.4).) Note that by (1.7), $I(\theta_j, \theta^*) = I(\theta_j, \lambda)$ whenever $\mu(\theta_j) < \mu(\theta^*) = \mu(\lambda)$.

The asymptotic behavior (1.8) of the regret will be shown in Sect. 2 to be *optimal* in the sense of

THEOREM 1. *Assume that $I(\theta, \lambda)$ satisfies* (1.6) *and* (1.7), *and that Θ is such that*

$$\forall \lambda \in \Theta \text{ and } \forall \delta > 0, \exists \lambda' \in \Theta \text{ such that} \qquad \mu(\lambda) < \mu(\lambda') < \mu(\lambda) + \delta. \tag{1.9}$$

Let φ be a rule whose regret satisfies, for each fixed $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$, the condition that as $n \to \infty$

$$R_n(\boldsymbol{\theta}) = o(n^a) \qquad \text{for every } a > 0. \tag{1.10}$$

Then for every $\boldsymbol{\theta}$ such that the $\mu(\theta_j)$ are not all equal,

$$\liminf_{n \to \infty} R_n(\boldsymbol{\theta})/\log n \geq \sum_{j: \mu(\theta_j) < \mu^*} (\mu^* - \mu(\theta_j))/I(\theta_j, \theta^*). \tag{1.11}$$

Condition (1.10) of Theorem 1 implies that for every $\boldsymbol{\theta}$

$$\lim_{n \to \infty} n^{-1} ES_n = \mu^*. \tag{1.12}$$

We shall call rules that satisfy (1.12) *consistent*. Under the assumptions of Theorem 1, we shall call rules that satisfy (1.8) whenever the $\mu(\theta_j)$ are not all equal *asymptotically efficient*. In the case $k = 2$ Robbins [3] proposed a simple procedure for constructing consistent rules. Let $a_1 = 1 < a_2 < \ldots$ and $b_1 = 2 < b_2 < \ldots$ be any two disjoint, increasing sequences of positive integers such that $a_n/n \to \infty$ and $b_n/n \to \infty$ as $n \to \infty$. At stage n, sample from Π_1 if $n \in \{a_1, a_2, \ldots\}$, sample from Π_2 if $n \in \{b_1, b_2, \ldots\}$, and if $n \notin \{a_1, a_2, \ldots, b_1, b_2, \ldots\}$ sample from Π_1 or Π_2 according as the arithmetic mean of all previous observations from Π_1 exceeds or does not exceed the arithmetic mean of all previous observations from Π_2. The consistency of this rule follows easily from the strong law of large numbers.

The sequences a_n and b_n above are assumed to be prescribed in advance, and a natural question is how to choose them so that $n^{-1}ES_n$ approaches μ^* as rapidly as possible. However, the choice clearly involves the unknown parameters $\theta_1, \ldots, \theta_k$. It is therefore desirable to let the sequences a_n, b_n be generated adaptively from the data rather than prescribed in advance. Such an approach was recently followed by Reimnitz [2] who, in the case of two Bernoulli populations, constructed an allocation rule with regret $R_n(\theta_1, \theta_2) = O(\log n)$. His rule, however, does not attain the asymptotically optimal rate (1.8).

In this paper we develop a new approach for constructing rather simple rules that are asymptotically efficient. Our approach is based on a certain class of upper confidence bounds, and the idea is described in general in Section 3. Applications to the special cases of normal, Bernoulli, Poisson, and exponential populations are discussed in Section 4.

2. A Lower Bound for the Expected Sample Size from an Inferior Population

Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$ and let $P_{\boldsymbol{\theta}}$ denote the probability measure under which θ_j is the parameter corresponding to population Π_j, $j = 1, \dots, k$. Define for $j = 1, \dots, k$ the parameter sets

$$\begin{aligned} \Theta_j &= \left\{\boldsymbol{\theta} : \mu(\theta_j) < \max_{i \neq j} \mu(\theta_i)\right\} \qquad (\text{“}\theta_j \text{ is not best”}), \\ \Theta_j^* &= \left\{\boldsymbol{\theta} : \mu(\theta_j) > \max_{i \neq j} \mu(\theta_i)\right\} \qquad (\text{“}\theta_j \text{ is the unique best”}). \end{aligned} \tag{2.1}$$

The main result of this section is given by

Theorem 2. *Assume that $I(\theta, \lambda)$ satisfies* (1.6) *and* (1.7) *and that Θ satisfies* (1.9). *Fix $j \in \{1, \dots, k\}$, and define Θ_j and Θ_j^* by* (2.1). *Let φ be any rule such that for every $\boldsymbol{\theta} \in \Theta_j^*$, as $n \to \infty$*

$$\sum_{i \neq j} E_{\boldsymbol{\theta}} T_n(i) = o(n^a) \qquad \text{for every } a > 0, \tag{2.2}$$

where $T_n(i)$, defined in (1.2), *is the number of times that the rule φ samples from Π_i up to stage n. Then for every $\boldsymbol{\theta} \in \Theta_j$ and every $\epsilon > 0$,*

$$\lim_{n \to \infty} P_{\boldsymbol{\theta}}\left\{T_n(j) \geq (1 - \epsilon)(\log n)/I(\theta_j, \theta^*)\right\} = 1, \tag{2.3}$$

where θ^* is defined in (1.4), *and hence*

$$\liminf_{n \to \infty} E_{\boldsymbol{\theta}} T_n(j)/\log n \geq 1/I(\theta_j, \theta^*).$$

Proof. To fix the ideas let $j = 1$, $\boldsymbol{\theta} \in \Theta_1$, and $\theta^* = \theta_2$. Then $\mu(\theta_2) > \mu(\theta_1)$ and $\mu(\theta_2) \geq \mu(\theta_i)$ for $3 \leq i \leq k$. Fix any $0 < \delta < 1$. In view of (1.6), (1.7), and (1.9), we can choose $\lambda \in \Theta$ such that

$$\mu(\lambda) > \mu(\theta_2) \qquad \text{and} \qquad |I(\theta_1, \lambda) - I(\theta_1, \theta_2)| < \delta I(\theta_1, \theta_2). \tag{2.4}$$

Define the new parameter vector $\gamma = (\lambda, \theta_2, \ldots, \theta_k)$. Then $\gamma \in \Theta_1^*$, so by (2.2)

$$E_\gamma(n - T_n(1)) = \sum_{h \neq 1} E_\gamma(T_n(h)) = o(n^a)$$

with $0 < a < \delta$, and therefore

$$(n - O(\log n)) P_\gamma\{T_n(1) < (1 - \delta)(\log n)/I(\theta_1, \lambda)\}$$
$$\leq E_\gamma(n - T_n(1)) = o(n^a).$$

Letting $Y_1, Y_2, \ldots$ denote successive observations from Π_1, and defining $L_m = \sum_{i=1}^m \log(f(Y_i; \theta_1)/f(Y_i; \lambda))$, it follows that

$$P_\gamma(C_n) = o(n^{a-1}), \qquad \text{where}$$
$$C_n = \left\{T_n(1) < (1 - \delta)(\log n)/I(\theta_1, \lambda) \text{ and } L_{T_n(1)} \leq (1 - a)\log n\right\}. \quad (2.5)$$

Note that

$$P_\gamma\{T_n(1) = n_1, \ldots, T_n(k) = n_k,\ L_{n_1} \leq (1 - a)\log n\}$$
$$= \int_{\{T_n(1) = n_1, \ldots, T_n(k) = n_k,\ L_{n_1} \leq (1-a)\log n\}} \prod_{i=1}^{n_1} \frac{f(Y_i; \lambda)}{f(Y_i; \theta_1)}\, dP_\theta$$
$$\geq \exp(-(1 - a)\log n)$$
$$\cdot P_\theta\{T_n(1) = n_1, \ldots, T_n(k) = n_k,\ L_{n_1} \leq (1 - a)\log n\}. \quad (2.6)$$

Since C_n is a disjoint union of events of the form $\{T_n(1) = n_1, \ldots, T_n(k) = n_k,\ L_{n_1} \leq (1 - a)\log n\}$ with $n_1 + \cdots + n_k = n$ and $n_1 < (1 - \delta)(\log n)/I(\theta_1, \lambda)$, it follows from (2.5) and (2.6) that as $n \to \infty$

$$P_\theta(C_n) \leq n^{1-a} P_\gamma(C_n) \to 0. \quad (2.7)$$

By the strong law of large numbers, $L_m/m \to I(\theta_1, \lambda) > 0$, and therefore $\max_{i \leq m} L_i/m \to I(\theta_1, \lambda)$, a.s. $[P_\theta]$. Since $1 - a > 1 - \delta$, it then follows that

$$P_\theta\{L_i > (1 - a)\log n \text{ for some } i < (1 - \delta)(\log n)/I(\theta_1, \lambda)\} \to 0$$
$$\text{as } n \to \infty. \quad (2.8)$$

From (2.7) and (2.8) we see that

$$\lim_{n \to \infty} P_\theta\{T_n(1) < (1 - \delta)(\log n)/I(\theta_1, \lambda)\} = 0.$$

In view of (2.4), this implies that

$$\lim_{n \to \infty} P_\theta\{T_n(1) < (1 - \delta)(\log n)/[(1 + \delta)I(\theta_1, \theta_2)]\} = 0,$$

from which (2.3) for $j = 1$ follows.

Proof of Theorem 1. In view of condition (1.10) on the rule φ, it follows from Theorem 2 that for any fixed $\boldsymbol{\theta} = (\theta_1, \ldots \theta_k)$, if $\mu(\theta_j) < \mu^*$ then

$$\liminf_{n \to \infty} \{ E_{\boldsymbol{\theta}} T_n(j) / \log n \} \geq 1 / I(\theta_j, \theta^*). \tag{2.9}$$

Since $R_n(\boldsymbol{\theta}) = \sum_{j : \mu(\theta_j) < \mu^*} (\mu^* - \mu(\theta_j)) E_{\boldsymbol{\theta}} T_n(j)$, (1.11) follows from (2.9).

3. Construction of Asymptotically Efficient Allocation Rules

We describe here a general method of constructing adaptive allocation rules that attain the asymptotic lower bound for the regret given by the right-hand side of (1.11). We first outline briefly the motivation of our approach. In order to attain asymptotic efficiency we shall sample from the population with the largest estimated mean, provided that we have sampled enough from each population to be reasonably confident that this population is indeed superior. Our degree of confidence will depend on the number n of observations that we have taken so far, and as n increases we should be increasingly confident that we are not sampling from an inferior population. In view of Theorem 2, if φ is an asymptotically efficient rule, then the number of observations that φ takes from any inferior population Π_j up to stage n is about $(\log n)/I(\theta_j, \theta^*)$. Thus, at stage n, we need about $(\log n)/I(\theta_j, \theta^*)$ observations from Π_j to be reasonably confident that it is not a contender. These considerations suggest the following modified "sample-from-the-leader" rule. First, define the "leader" at stage n as the population with the largest estimated mean among all populations that have been sampled at least δn times, for some predetermined positive number $\delta < 1/k$. While we would like to sample from this apparently superior population, we need to make sure that the other populations have been sampled enough for us to be reasonably confident that they are indeed inferior. One way of doing this is to compare certain upper confidence bounds for the mean of an apparently inferior population with the estimated mean of the leader. These confidence bounds are required to satisfy conditions (3.1), (3.2), and (3.3) below to ensure that the allocation rule obtained thereby is asymptotically efficient.

To fix the ideas, let $Y_1, Y_2, \ldots$ be i.i.d. random variables with a common density function $f(y; \theta)$ with respect to some measure ν, where $\theta \in \Theta$ denotes an unknown parameter. We shall use "upper confidence bounds" for the mean $\mu(\theta)$, defined by Borel functions $g_{ni} : R^i \to R$ $(n = 1, 2, \ldots;$

$i = 1, \ldots, n$) such that for every $\theta \in \Theta$,

$$P_\theta\{r \le g_{ni}(Y_1, \ldots, Y_i) \text{ for all } i \le n\} = 1 - o(n^{-1}) \qquad \text{for every } r < \mu(\theta), \tag{3.1}$$

$$\lim_{\epsilon \downarrow 0}\left(\limsup_{n\to\infty} \sum_{i=1}^{n} P_\theta\{g_{ni}(Y_1, \ldots, Y_i) \ge \mu(\lambda) - \epsilon\}/\log n\right) \le 1/I(\theta, \lambda) \qquad \text{whenever } \mu(\lambda) > \mu(\theta), \tag{3.2}$$

and

$$g_{ni} \text{ is nondecreasing in } n \ge i \text{ for every fixed } i = 1, 2, \ldots. \tag{3.3}$$

Examples of such confidence bounds in the special cases of normal, Bernoulli, exponential, and Poisson distributions are given in Section 4, along with a general method for their construction.

In addition to these sequences of upper confidence bounds for $\mu(\theta)$, we shall also use point estimates $h_i(Y_1, \ldots, Y_i)$ of $\mu(\theta)$, where the $h_i\colon R^i \to R$ ($i = 1, 2, \ldots$) are Borel functions such that

$$h_i \le g_{ni} \qquad \text{for all } n \ge i, \tag{3.4}$$

and for every $\theta \in \Theta$,

$$P_\theta\left\{\max_{\delta n \le i \le n} |h_i(Y_1, \ldots, Y_i) - \mu(\theta)| > \epsilon\right\} = o(n^{-1}) \qquad \text{for all } \epsilon > 0 \text{ and } 0 < \delta < 1. \tag{3.5}$$

Note that (3.5) holds for the sample mean $h_i(Y_1, \ldots, Y_i) = (Y_1 + \cdots + Y_i)/i$ under the assumption that $E_\theta Y_1^2 < \infty$ (cf. the proof of Theorem 1 of [1]).

We now make use of the functions g_{ni} and h_i to construct an asymptotically efficient rule for sampling $x_1, x_2, \ldots$ sequentially from populations $\Pi_1, \ldots, \Pi_k$ with respective density functions $f(x; \theta_1), \ldots, f(x; \theta_k)$. For $j = 1, \ldots, k$, let $T_n(j)$ denote the number of times that the rule samples from Π_j up to stage n, and let $Y_{j1}, \ldots, Y_{j, T_n(j)}$ denote the successive observations from Π_j up to stage n. Define

$$\begin{aligned} \hat{\mu}_n(j) &= h_{T_n(j)}\left(Y_{j1}, \ldots, Y_{j, T_n(j)}\right), \\ U_n(j) &= g_{n, T_n(j)}\left(Y_{j1}, \ldots, Y_{j, T_n(j)}\right), \end{aligned} \tag{3.6}$$

and let $0 < \delta < 1/k$. To begin with, at stage $j = 1, 2, \ldots, k$, the rule takes one observation from Π_j. Now suppose that the rule has taken $n \ge k$

observations. Since $T_n(1) + \cdots + T_n(k) = n$, we can choose $j_n \in \{1, \ldots, k\}$ such that

$$\hat{\mu}_n(j_n) = \max\{\hat{\mu}_n(j) : T_n(j) \geq \delta n\}. \tag{3.7}$$

At stage $n + 1$, writing $n + 1 = km + j$, where m is a positive integer and $j \in \{1, \ldots, k\}$, we take an observation from Π_j only if

$$\hat{\mu}_n(j_n) \leq U_n(j), \tag{3.8}$$

and sample from Π_{j_n} otherwise. This sampling rule will be denoted by φ^*.

The population Π_{j_n} defined by (3.7) can be regarded as the "leader" at the end of stage n; it has the largest estimated mean among all populations that have been sampled at least δn times. The rule φ^*, therefore, compares at stage $n + 1 = km + j$ the population Π_j with the leader Π_{j_n}. It samples from Π_j if the upper confidence bound $U_n(j)$ for the mean of Π_j does not fall below the estimated mean of Π_{j_n}; otherwise it samples from Π_{j_n}. We now establish the asymptotic efficiency of this rule.

THEOREM 3. *Assume that $I(\theta, \lambda)$ satisfies* (1.6) *and* (1.7) *and that the functions g_{ni} and h_i satisfy* (3.1)–(3.5). *For $j = 1, \ldots, k$, let $T_n(j)$ be the number of times that the rule φ^* samples from Π_j up to stage n, as defined in* (1.2). *Define θ^* as in* (1.4).

(i) *For every $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$ and every j such that $\mu(\theta_j) < \mu(\theta^*)$,*

$$E_{\boldsymbol{\theta}} T_n(j) \leq \left(\frac{1}{I(\theta_j, \theta^*)} + o(1) \right) \log n. \tag{3.9}$$

(ii) *Assume also that Θ satisfies* (1.9). *Then $E_{\theta} T_n(j) \sim (\log n)/I(\theta_j, \theta^*)$ for every j such that $\mu(\theta_j) < \mu(\theta^*)$, and the regret of φ^* satisfies* (1.8).

Proof. To prove (3.9), let $L = \{1 \leq l \leq k : \mu(\theta_l) = \mu(\theta^*)\}$. Let $0 < \epsilon < \mu(\theta^*) - \max_{j \notin L} \mu(\theta_j)$. Using the notation $\#A$ to denote the number of elements of a set A, we note that for any fixed $j \notin L$,

$$\begin{aligned} T_N(j) \leq \#\{&1 \leq n \leq N - 1 : j_n \in L,\ |\hat{\mu}_n(j_n) - \mu(\theta^*)| \leq \epsilon, \text{ and} \\ &\varphi^* \text{ samples from } \Pi_j \text{ at stage } n + 1\} + 1 \\ &+ \#\{1 \leq n \leq N - 1 : j_n \in L \text{ and } |\hat{\mu}_n(j_n) - \mu(\theta^*)| > \epsilon\} \\ &+ \#\{1 \leq n \leq N - 1 : j_n \notin L\}. \end{aligned} \tag{3.10}$$

Let $Y_{j1}, Y_{j2}, \ldots$ denote successive i.i.d. observations from Π_j. From the definition of φ^* it follows that

$$\begin{aligned}
&\#\{2 \le m \le N: j_{m-1} \in L, |\hat{\mu}_{m-1}(j_{m-1}) - \mu(\theta^*)| \le \epsilon, \text{ and} \\
&\quad \varphi^* \text{ samples from } \Pi_j \text{ at stage } m\} \\
&\quad\quad \le \#\{1 \le \nu \le N: \varphi^* \text{ samples } Y_{j\nu} \text{ at some stage } m \text{ with} \\
&\quad\quad\quad \nu \le m \le N \text{ and } j_{m-1} \in L, |\hat{\mu}_{m-1}(j_{m-1}) - \mu(\theta^*)| \le \epsilon\} \\
&\quad\quad \le 1 + \#\{1 \le i \le N-1: g_{ni}(Y_{j1}, \dots, Y_{ji}) \ge \mu(\theta^*) - \epsilon \\
&\quad\quad\quad \text{for some } i \le n \le N-1\} \\
&\le 1 + \#\{1 \le i \le N-1: g_{Ni}(Y_{j1}, \dots, Y_{ji}) \ge \mu(\theta^*) - \epsilon\}, \\
&\qquad\qquad\qquad\qquad \text{by (3.3)}. \quad (3.11)
\end{aligned}$$

In view of (3.2), for every $\rho > 0$, we can choose $\epsilon > 0$ so small that

$$\begin{aligned}
&E_\theta(\#\{1 \le i \le N-1: g_{Ni}(Y_{j1}, \dots, Y_{ji}) \ge \mu(\theta^*) - \epsilon\}) \\
&\quad \le \sum_{i=1}^{N} P_\theta\{g_{Ni}(Y_{j1}, \dots, Y_{ji}) \ge \mu(\theta^*) - \epsilon\} \\
&\quad \le \frac{1 + \rho + o(1)}{I(\theta_j, \theta^*)} \log N. \qquad (3.12)
\end{aligned}$$

Since $T_n(j_n) \ge \delta n$ by (3.7), it follows that

$$\begin{aligned}
&P_\theta\{j_n \in L \text{ and } |\hat{\mu}_n(j_n) - \mu(\theta^*)| > \epsilon\} \\
&\quad \le P_\theta\left\{\max_{l \in L} \max_{\delta n \le i \le n} |h_i(Y_{l1}, \dots, Y_{li}) - \mu(\theta^*)| > \epsilon\right\} \\
&\quad = o(n^{-1}) \qquad \text{by (3.5)},
\end{aligned}$$

and therefore

$$E_\theta(\#\{1 \le n \le N-1: j_n \in L \text{ and } |\hat{\mu}_n(j_n) - \mu(\theta^*)| > \epsilon\}) = o(\log N). \qquad (3.13)$$

It will be shown in Lemma 1 below that

$$E_\theta(\#\{1 \le n \le N-1: j_n \notin L\}) = o(\log N). \qquad (3.14)$$

From (3.10)–(3.14), (3.9) follows.

If Θ also satisfies (1.9), then by Theorem 2, $E_\theta T_n(j) \ge (1/I(\theta_j, \theta^*) + o(1))\log n$. This and (3.9) imply that

$$E_\theta T_n(j) \sim (\log n)/I(\theta_j, \theta^*)$$

for every j such that $\mu(\theta_j) < \mu(\theta^*)$, and (1.8) follows.

LEMMA 1. *With the same notation and assumptions as in Theorem* 3(*i*), *let* $L = \{1 \le l \le k : \mu(\theta_l) = \mu(\theta^*)\}$. *Let* $0 < \epsilon < \{\mu(\theta^*) - \max_{j \notin L}\mu(\theta_j)\}/2$, *and let c be a positive integer. For* $r = 0, 1, \ldots,$ *define*

$$A_r = \bigcap_{1 \le j \le k} \left\{ \max_{\delta c^{r-1} \le n \le c^{r+1}} |h_n(Y_{j1}, \ldots, Y_{jn}) - \mu(\theta_j)| \le \epsilon \right\},$$

$$B_r = \bigcap_{l \in L} \Big\{ g_{ni}(Y_{l1}, \ldots, Y_{li}) \ge \mu(\theta^*) - \epsilon \text{ for all } 1 \le i \le \delta n$$
$$\text{and } c^{r-1} \le n \le c^{r+1} \Big\},$$

where $0 < \delta < 1/k$ *is the same as that used in the rule* φ^*. *Then*

(i) $P_\theta(\bar{A}_r) = o(c^{-r}), \qquad P_\theta(\bar{B}_r) = o(c^{-r}),$

where $\bar{A}$ *denotes the complement of an event A. Moreover, if* $c > (1 - k\delta)^{-1}$ *and* $r \ge r_0$ (*sufficiently large*), *then*

(ii) *on* $A_r \cap B_r$, $j_n \in L$ *for all* $c^r \le n \le c^{r+1}$.

Consequently,

(iii) $E_\theta(\#\{1 \le n \le N : j_n \notin L\}) = \sum_{n=1}^{N} P_\theta\{j_n \notin L\} = o(\log N)$.

Proof. (i) From (3.5), it follows that $P(\bar{A}_r) = o(c^{-r})$. Let $[x]$ denote the largest integer $\le x$, and let p be the smallest positive integer such that $[c^{r-1}/\delta^p] \ge c^{r+1}$. For $t = 0, \ldots, p$, let $n_t = [c^{r-1}/\delta^t]$, and define

$$D_t = \bigcap_{l \in L} \left\{ g_{n_t, i}(Y_{l1}, \ldots, Y_{li}) \ge \mu(\theta^*) - \epsilon \text{ for all } i \le n_t \right\}.$$

Then by (3.1),

$$P_\theta(\bar{D}_t) = o(n_t^{-1}) = o(c^{-r}) \qquad \text{for } t = 0, \ldots, p. \tag{3.15}$$

Given $c^{r-1} \le n < c^{r+1}$ and $1 \le i \le \delta n$, there exists $t \in \{0, \ldots, p-1\}$ such that $n_{t+1} > n \ge n_t \ge i$, and therefore by (3.3)

$$g_{ni}(Y_{l1}, \ldots, Y_{li}) \ge g_{n_t, i}(Y_{l1}, \ldots, Y_{li}) \ge \mu(\theta^*) - \epsilon$$

for all $l \in L$ on the event $\bigcap_{0 \le t \le p} D_t$. It then follows that $B_r \supset \bigcap_{0 \le t \le p} D_t$, and therefore by (3.15), $P_\theta(\bar{B}_r) = o(c^{-r})$.

(ii) We now assume that $(1 - c^{-1})/k > \delta$. We shall say that at stage n the rule φ^* samples from L if it samples from Π_l for some $l \in L$. Let

$$\nu_L(n) = \sum_{l \in L} T_n(l)$$

be the number of times that φ^* samples from L up to stage n. We note that

$$\max_{l \in L} T_n(l) \ge \nu_L(n)/\#L. \tag{3.16}$$

Consider the stage $n + 1 = km + l$ with $l \in L$ and $c^{r-1} \leq n < c^{r+1}$. We now show that at this stage φ^* must sample from L on the event $A_r \cap B_r$. First note that if $j_n \in L$, then φ^* samples from either Π_{j_n} or Π_l at stage $n + 1 = km + l$. Now assume that $j_n \notin L$. Then since $T_n(j_n) \geq \delta n$,

$$\hat{\mu}_n(j_n) \leq \max_{j \notin L} \mu(\theta_j) + \epsilon < \mu(\theta^*) - \epsilon \qquad \text{on } A_r. \tag{3.17}$$

In the case $T_n(l) \geq \delta n$, we have on A_r

$$\mu(\theta^*) - \epsilon \leq h_{T_n(l)}\left(Y_{l1}, \ldots, Y_{l,T_n(l)}\right) \leq g_{n,T_n(l)}\left(Y_{l1}, \ldots, Y_{l,T_n(l)}\right) \tag{3.18}$$

by (3.4), and therefore by (3.17) and (3.18), φ^* samples from Π_l at stage $n + 1$. In the case $T_n(l) < \delta n$, we have on the event B_r

$$\mu(\theta^*) - \epsilon \leq g_{n,T_n(l)}\left(Y_{l1}, \ldots, Y_{l,T_n(l)}\right), \tag{3.19}$$

and therefore by (3.17) and (3.19), φ^* also samples from Π_l at stage $n + 1$ on $A_r \cap B_r$.

On the event $A_r \cap B_r$, since φ^* must sample from L at stage $n + 1 = km + l$ with $l \in L$ and $c^{r-1} \leq n \leq c^{r+1}$, and since $(1 - c^{-1})/k > \delta$, it follows that

$$\nu_L(n) \geq (\#L/k)(n - c^{r-1} - 2k) > (\#L)\delta n \tag{3.20}$$

for all $c^r \leq n \leq c^{r+1}$ and $r \geq r_0$ (sufficiently large). From (3.16) and (3.20), we obtain that on $A_r \cap B_r$

$$\max_{l \in L} T_n(l) > \delta n \qquad \text{for all } c^r \leq n \leq c^{r+1}, \tag{3.21}$$

if $r \geq r_0$. We note that for $r \geq r_0$ and $c^r \leq n \leq c^{r+1}$, on the event $A_r \cap B_r$,

$$\begin{aligned} &\max\{\hat{\mu}_n(j) : T_n(j) \geq \delta n \text{ and } j \notin L\} \\ &\quad \leq \max_{j \notin L} \mu(\theta_j) + \epsilon < \mu(\theta^*) - \epsilon \\ &\quad \leq \min\{\hat{\mu}_n(l) : T_n(l) \geq \delta n \text{ and } l \in L\}, \end{aligned}$$

the last set being nonempty by (3.21). Hence $j_n \in L$ for all $c^r \leq n \leq c^{r+1}$ on $A_r \cap B_r$ if $r \geq r_0$.

(iii) Let $c > (1 - k\delta)^{-1}$. Then it follows from (i) and (ii) that for $r \geq r_0$ and $c^r \leq n \leq c^{r+1}$,

$$P_\theta\{j_n \notin L\} \leq P_\theta(\bar{A}_r) + P_\theta(\bar{B}_r) = o(c^{-r}),$$

and therefore $\Sigma_{c^r \le n \le c^{r+1}} P_\theta\{ j_n \notin L\} = o(1)$. Hence $\Sigma_{n=1}^{N} P_\theta\{ j_n \notin L\} = o(\log N)$.

4. Confidence Sequences and Allocation Rules For Special Distributions

In this section we make use of certain generalized likelihood ratios to construct confidence sequences that satisfy the conditions (3.1)–(3.3). These generalized likelihood ratios are described in the following lemma.

Lemma 2. *Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with a common density $f(y;\theta)$ with respect to some measure ν, where θ is a real parameter.*

(i) *Let $\hat\theta_n = \hat\theta_n(Y_1, \ldots, Y_n)$ be an estimate of θ at stage n. Then for every $a \ge 1$,*

$$P_\theta\left\{ \prod_{i=1}^{n} f(Y_i; \hat\theta_{i-1}) \Big/ \prod_{i=1}^{n} f(Y_i;\theta) \ge a \text{ for some } n \ge 1 \right\} \le a^{-1}. \quad (4.1)$$

(ii) *Let C be a compact set of real numbers such that for every $\lambda \in C$ and some θ*

$$\lim_{\delta \downarrow 0} E_\theta\left(\sup\{ f(Y_1;r)/f(Y_1;\theta) : r \in C, |r-\lambda| < \delta\}\right) = 1. \quad (4.2)$$

Then for every $d > 0$,

$$\limsup_{a\to\infty} (\log a)^{-1} \log P_\theta\left\{ \sup_{\lambda\in C} \prod_{i=1}^{n} f(Y_i;\lambda) \Big/ \prod_{i=1}^{n} f(Y_i;\theta) \ge a \text{ for some } n \le d \log a \right\} \le -1. \quad (4.3)$$

Moreover, for every $d > 0$ and $0 < \rho < 1$,

$$\limsup_{a\to\infty} (\log a)^{-1} \log P_\gamma\left\{ \prod_{i=1}^{n} f(Y_i;\gamma) \Big/ \prod_{i=1}^{n} f(Y_i;\theta) \le \rho^n \text{ and } \sup_{\lambda\in C} \prod_{i=1}^{n} f(Y_i;\lambda) \Big/ \prod_{i=1}^{n} f(Y_i;\theta) \ge a \text{ for some } n \ge d \log a \right\}$$
$$\le -1 + d\log\rho < -1. \quad (4.4)$$

Proof. (i) Under P_θ, the sequence $\{\prod_{i=1}^{n} f(Y_i; \hat\theta_{i-1}) / \prod_{i=1}^{n} f(Y_i;\theta),\ n \ge 1\}$ is a nonnegative martingale with mean 1, and therefore (4.1) follows (cf. [4]).

(ii) To prove (4.3), in view of (4.2) we can choose for every $\epsilon > 0$ and $\lambda \in C$ a positive constant $\delta(\epsilon, \lambda)$ such that

$$E_\theta\Big(\sup_{r\in B(\lambda)} f(Y_1; r)/f(Y_1; \theta)\Big) < 1 + \epsilon, \tag{4.5}$$

where $B(\lambda) = \{r \in C: |r - \lambda| < \delta(\epsilon, \lambda)\}$. From (4.5) it follows that

$$P_\theta\left\{\sup_{r\in B(\lambda)} \prod_{i=1}^n f(Y_i; r)\Big/\prod_{i=1}^n f(Y_i; \theta) \ge a\right\} \le a^{-1}(1+\epsilon)^n.$$

Since C is compact, we can choose a finite covering $B(\lambda_1), \dots, B(\lambda_m)$ of C, and therefore

$$P_\theta\left\{\sup_{r\in C} \prod_{i=1}^n f(Y_i; r)\Big/\prod_{i=1}^n f(Y_i; \theta) \ge a\right\} \le ma^{-1}(1+\epsilon)^n. \tag{4.6}$$

Since ϵ can be arbitrarily small, (4.3) follows from (4.6).

To prove (4.4), let $F_n = \{\prod_{i=1}^n f(Y_i; \gamma)/\prod_{i=1}^n f(Y_i; \theta) \le \rho^n\}$, $G_n = \{\sup_{\lambda\in C}\prod_{i=1}^n f(Y_i; \lambda)/\prod_{i=1}^n f(Y_i; \theta) \ge a\}$. We note that

$$\begin{aligned} P_\gamma(F_n \cap G_n) &= \int_{F_n\cap G_n} \prod_{i=1}^n \big(f(Y_i; \gamma)/f(Y_i; \theta)\big)\, dP_\theta \\ &\le \rho^n P_\theta(G_n) \le ma^{-1}\{\rho(1+\epsilon)\}^n, \qquad \text{by (4.6).} \end{aligned} \tag{4.7}$$

Choosing ϵ so small that $\rho(1+\epsilon) < 1$, we have

$$\sum_{n\ge d\log a} \{\rho(1+\epsilon)\}^n = O\big(\{\rho(1+\epsilon)\}^{d\log a}\big),$$

and therefore (4.4) follows from (4.7).

We now apply the preceding results to construct confidence sequences and asymptotically efficient allocation rules for normal, Bernoulli, Poisson, and double exponential populations.

EXAMPLE 1. Let Y_{ji}, $j = 1, 2, \dots, k$, $i = 1, 2, \dots$, be independent normal random variables with known common variance $\sigma^2 > 0$ and unknown means $EY_{ji} = \theta_j$. Thus, $\mu(\theta) = \theta$, $\Theta = (-\infty, \infty)$, ν = Lebesgue measure,

$$f(y, \theta) = (2\pi\sigma^2)^{-1/2}\exp\{-(y-\theta)^2/2\sigma^2\}, \tag{4.8}$$

and

$$I(\theta, \lambda) = (\theta - \lambda)^2/(2\sigma^2). \tag{4.9}$$

Conditions (1.6), (1.7), and (1.9) are clearly satisfied.

Let a_{ni} $(n = 1, 2, \ldots, i = 1, \ldots, n)$ be positive constants such that for every fixed i

$$a_{ni} \text{ is nondecreasing in } n \geq i, \tag{4.10}$$

and there exist $\epsilon_n \to 0$ for which

$$|a_{ni} - (\log n)/i| \leq \epsilon_n (\log n)^{1/2}/i^{1/2} \text{ for all } i \leq n. \tag{4.11}$$

For $j = 1, \ldots, k$, define

$$\overline{Y}_i(j) = (Y_{j1} + \cdots + Y_{ji})/i, \qquad h_i(Y_{j1}, \ldots, Y_{ji}) = \overline{Y}_i(j), \tag{4.12}$$

$$g_{ni}(Y_{j1}, \ldots, Y_{ji}) = \overline{Y}_i(j) + \sigma(2a_{ni})^{1/2} \qquad \text{for } n \geq i. \tag{4.13}$$

Obviously, for every $0 < \delta < 1$ and $\epsilon > 0$,

$$P\Big\{\max_{\delta n \leq i \leq n} |\overline{Y}_i(j) - \theta_j| > \epsilon\Big\} = o(n^{-1}), \tag{4.14}$$

so condition (3.5) is satisfied. From (4.11) and the tail probability of the normal distribution it follows easily that for $r < \theta_j$

$$\sum_{i=1}^{n} P\{r > \overline{Y}_i(j) + \sigma(2a_{ni})^{1/2}\} = o(n^{-1}),$$

and therefore condition (3.1) is satisfied.

Conditions (3.3) and (3.4) are obviously satisfied. We now show that (3.2) also holds. Let $\lambda > \theta_j$ and define

$$L_n = \sup\{1 \leq i \leq n : \overline{Y}_i(j) + \sigma(2a_{ni})^{1/2} \geq \lambda\} \qquad (\sup \varnothing = 0),$$
$$T_\epsilon = \sup\{i \geq 1 : |\overline{Y}_i(j) - \theta_j| \geq \epsilon\}.$$

Then $ET_\epsilon < \infty$ for all $\epsilon > 0$ (cf. [1]). Moreover, it follows from (4.11) that for $0 < \epsilon < \lambda - \theta_j$

$$E\big(L_n I_{\{L_n > T_\epsilon\}}\big) \leq 2\sigma^2(\lambda - \theta_j - \epsilon)^{-2}(1 + o(1))\log n \qquad \text{as } n \to \infty.$$

Obviously, $E(L_n I_{\{L_n \leq T_\epsilon\}}) \leq ET_\epsilon$. By the strong law of large numbers and Fatou's lemma,

$$EL_n \geq 2\sigma^2(\lambda - \theta_j)^{-2}(1 + o(1))\log n.$$

Hence, letting $\epsilon \downarrow 0$, we obtain that

$$EL_n \sim 2\sigma^2(\lambda - \theta_j)^{-2}\log n = (\log n)/I(\theta_j, \lambda). \tag{4.15}$$

Since $\#\{1 \leq i \leq n : \overline{Y}_i(j) + \sigma(2a_{ni})^{1/2} \geq \lambda\} \leq L_n$, it follows from (4.15) that for $\lambda > \theta_j$

$$\sum_{i=1}^{n} P\left\{\overline{Y}_i(j) + \sigma(2a_{ni})^{1/2} \geq \lambda\right\} \leq (1 + o(1))(\log n)/I(\theta_j, \lambda).$$

With h_i and g_{ni} given by (4.12) and (4.13), we define the allocation rule φ^* as in Section 3. Theorem 3 is therefore applicable to this special case and shows that φ^* provides an asymptotically efficient allocation rule for k normal populations with common known variance σ^2. We note that $\overline{Y}_i(j)$ is the maximum likelihood estimate of θ_j based on $Y_{j1}, \ldots, Y_{ji}$, and that the upper confidence bound $g_{ni}(Y_{j1}, \ldots, Y_{j1})$ can be expressed in terms of generalized likelihood ratios as follows:

$$\begin{aligned} g_{ni}(Y_{j1}, \ldots, Y_{ji}) &= \inf\{\lambda \geq \overline{Y}_i(j) : I(\overline{Y}_i(j), \lambda) \geq a_{ni}\} \\ &= \inf\left\{\lambda \geq \overline{Y}_i(j) : \sup_{\theta} \prod_{t=1}^{i} f(Y_{jt}; \theta) \Big/ \prod_{t=1}^{i} f(Y_{jt}; \lambda) \geq e^{ia_{ni}}\right\}. \end{aligned} \tag{4.16}$$

The upper confidence bounds in the next two examples are also of this general form.

EXAMPLE 2. Let Y_{ji} be independent Bernoulli random variables such that Y_{ji} has density

$$f(y; \theta_j) = \theta_j^y (1 - \theta_j)^{1-y}, \qquad y = 0, 1, \tag{4.17}$$

with respect to the counting measure ν. Here $\mu(\theta) = \theta$, $\Theta = (0, 1)$, and

$$I(\theta, \lambda) = \theta \log(\theta/\lambda) + (1 - \theta)\log\{(1 - \theta)/(1 - \lambda)\}. \tag{4.18}$$

Conditions (1.6), (1.7), and (1.9) are clearly satisfied.

For $j = 1, \ldots, k$, define $\overline{Y}_i(j)$ and h_i as in (4.12), and note that (4.14) still holds. Let a_{ni} ($n = 1, 2, \ldots,\ i = 1, \ldots, n$) be positive constants satisfying (4.10) and such that

$$\lim_{\delta \downarrow 0}\left(\liminf_{n \to \infty} \min\{a_{ni} : i \leq \delta \log n\}\right) = \infty, \tag{4.19a}$$

$$\lim_{d \to \infty}\left(\limsup_{n \to \infty} \max\{a_{ni} : d \log n \leq i \leq n\}\right) = 0, \tag{4.19b}$$

and

$$\lim_{n \to \infty} \max\{|ia_{ni}/\log n - 1| : \delta \log n \leq i \leq d \log n\} = 0 \qquad \forall 0 < \delta < d. \tag{4.19c}$$

(Note that (4.19) is a weaker assumption than (4.11).) Define g_{ni} as in

(4.16), where the parameter λ can only take values in $(0,1)$ and where we now set $\inf \varnothing = 1$. Since $I(\bar{Y}_i(j), \lambda)$ is a convex function in λ with minimum at $\lambda = \bar{Y}_i(j)$, we have the equivalence

$$1 > r \geq g_{ni}(Y_{j1}, \dots, Y_{ji}) \Leftrightarrow 1 > r \geq \bar{Y}_i(j) \text{ and } I(\bar{Y}_i(j), r) \geq a_{ni}. \tag{4.20}$$

Let $1 > \lambda > \theta_j$ and define $L = \sup\{i : \bar{Y}_i(j) \geq \lambda\}$, $L_n = \sup\{1 \leq i \leq n : I(\bar{Y}_i(j), \lambda) < a_{ni}\}$. Then $EL < \infty$ (cf. [1]), and an argument similar to the proof of (4.15) shows that $EL_n \sim (\log n)/I(\theta_j, \lambda)$. From (4.20), it follows that $\#\{1 \leq i \leq n : g_{ni}(Y_{j1}, \dots, Y_{ji}) > \lambda\} \leq L + L_n$, and therefore

$$\sum_{i=1}^{n} P\{g_{ni}(Y_{j1}, \dots, Y_{ji}) > \lambda\} \leq (1 + o(1))(\log n)/I(\theta_j, \lambda).$$

Hence condition (3.2) is satisfied.

We now show that condition (3.1) is also satisfied. Let $\theta_j > r > 0$. Then by (4.19a), we can choose $\delta > 0$ and n_0 such that

$$\sup_{\theta} I(\theta, r) = \max\{|\log r|, |\log(1 - r)|\} < a_{ni}$$
$$\text{for } i \leq \delta \log n \text{ and } n \geq n_0. \tag{4.21}$$

Let $\rho = (\theta_j/r)^r\{(1 - \theta_j)/(1 - r)\}^{1-r}$. We note that $\log \rho = -I(r, \theta_j) < 0$ and that

$$r \geq \bar{Y}_i(j) \Rightarrow \prod_{t=1}^{i} f(Y_{jt}; \theta_j) \Big/ \prod_{t=1}^{i} f(Y_{jt}; r) \leq \rho^i. \tag{4.22}$$

Hence $P\{r \geq \bar{Y}_i(j)\} \leq \rho^i$, so we can choose $d > \delta$ such that

$$P\{r \geq \bar{Y}_i(j) \text{ for some } i \geq d \log n\} = o(n^{-1}).$$

Moreover, from (4.20), (4.21), and (4.22), it follows that

$$\begin{aligned}
&P\{r \geq g_{ni}(Y_{j1}, \dots, Y_{ji}) \text{ for some } i \leq d \log n\} \\
&\quad = P\{r \geq \bar{Y}_i(j) \text{ and } I(\bar{Y}_i(j), r) \geq a_{ni} \text{ for some } \delta \log n \leq i \leq d \log n\} \\
&\quad \leq P\Bigg\{\prod_{t=1}^{i} f(Y_{jt}; \theta_j) \Big/ \prod_{t=1}^{i} f(Y_{jt}; r) \leq \rho^i \text{ and} \\
&\qquad \sup_{0 \leq \theta \leq r} \prod_{t=1}^{i} f(Y_{jt}; \theta) \Big/ \prod_{t=1}^{i} f(Y_{jt}; r) \geq e^{i a_{ni}} \\
&\qquad \text{for some } \delta \log n \leq i \leq d \log n\Bigg\} \\
&\quad = o(n^{-1}), \qquad \text{by (4.19c) and Lemma 2 (ii).}
\end{aligned}$$

With h_i defined in (4.12) and g_{ni} defined in (4.16), the argument above shows that Theorem 3 is applicable, so the rule φ^* provides an asymptotically efficient allocation rule for k Bernoulli populations. In view of (4.20), we do not need the explicit value of $g_{ni}(Y_{j1}, \ldots, Y_{ji})$ to implement the rule φ^*. In fact, the allocation criterion (3.8) can now be rewritten as follows: Letting $\hat{\mu}_n(j) = \overline{Y}_{T_n(j)}(j)$, sample from Π_j at stage $n + 1 = km + j$ only if

$$\hat{\mu}_n(j) \geq \hat{\mu}_n(j_n) \qquad \text{or} \qquad I(\hat{\mu}_n(j), \hat{\mu}_n(j_n)) \leq a_{n, T_n(j)}. \tag{4.23}$$

EXAMPLE 3. Let Y_{ji} be independent Poisson random variables such that Y_{ji} has density

$$f(y; \theta_j) = e^{-\theta_j}\theta_j^y/y!, \qquad y = 0, 1, \ldots$$

with respect to the counting measure ν. Here $\mu(\theta) = \theta$, $\Theta = (0, \infty)$, and

$$I(\theta, \lambda) = \theta \log(\theta/\lambda) - (\theta - \lambda).$$

Conditions (1.6), (1.7), and (1.9) are clearly satisfied. Letting a_{ni} be positive constants satisfying (4.10) and (4.19), and defining h_i as in (4.12) and g_{ni} as in (4.16), we can use the argument of Example 2 to show that Theorem 3 is again applicable, noting that $\sup_{0 \leq \theta \leq r} I(\theta, r) = r$. Moreover, the allocation criterion (3.8) of the rule φ^* of Theorem 3 can be written in the more convenient form (4.23).

In each of the preceding examples the sample means of the k populations are sufficient statistics and are used in the allocation rule φ^*. We now give an example in which no simple sufficient statistics are available and in which sample medians are used instead of sample means. Moreover, in this example we show how the generalized likelihood ratios of Lemma 2(i) can be applied to construct confidence sequences satisfying (3.1).

EXAMPLE 4. Let Y_{ji} be independent double exponential random variables such that Y_{ji} has density

$$f(y; \theta_j) = \tfrac{1}{2}\exp(-|y - \theta_j|), \qquad -\infty < y < \infty. \tag{4.24}$$

Here $\mu(\theta) = \theta$, $\Theta = (-\infty, \infty)$, ν is Lebesgue measure, and $I(\theta, \lambda) = |\theta - \lambda|$. Conditions (1.6), (1.7), and (1.9) are clearly satisfied.

Let b_n be a nondecreasing sequence of positive numbers such that

$$b_n \to \infty \qquad \text{and} \qquad \log b_n = o(\log n) \qquad \text{as } n \to \infty. \tag{4.25}$$

For $j = 1, \ldots, k$ define

$$M_i(j) = \operatorname{med}\{Y_{j1}, \ldots, Y_{ji}\}, \qquad h_i(Y_{j1}, \ldots, Y_{ji}) = M_i(j), \tag{4.26}$$

$$g_{ni}(Y_{j1}, \ldots, Y_{ji}) = \inf\left\{\theta \geq M_i(j) : \sum_{t=1}^{i} |Y_{jt} - \theta| \geq \log nb_n + \sum_{t=1}^{i} |Y_{jt} - M_{t-1}(j)|\right\}. \tag{4.27}$$

Since $u(\theta) = \Sigma_{t=1}^{i} |Y_{jt} - \theta|$ is a piecewise linear and convex function of θ with its minimum at $\theta = M_i(j)$, we have the equivalence

$$\theta \geq g_{ni}(Y_{j1}, \ldots, Y_{ji}) \Leftrightarrow \theta \geq M_i(j) \text{ and } \sum_{t=1}^{i} |Y_{jt} - \theta| \geq \log nb_n + \sum_{t=1}^{i} |Y_{jt} - M_{t-1}(j)|. \tag{4.28}$$

Since for every $\epsilon > 0$,

$$P\{|M_i(j) - \theta_j| > \epsilon\} = O(\rho^i) \qquad \text{for some } 0 < \rho = \rho(\epsilon) < 1, \tag{4.29}$$

it follows that condition (3.5) is satisfied. Moreover, conditions (3.3) and (3.4) are also satisfied. Furthermore, for $r < \theta_j$,

$$\begin{aligned} &P\{r \geq g_{ni}(Y_{j1}, \ldots, Y_{ji}) \text{ for some } i \leq n\} \\ &\quad \leq P\{\theta_j \geq g_{ni}(Y_{j1}, \ldots, Y_{ji}) \text{ for some } i \leq n\} \\ &\quad \leq P\left\{\prod_{t=1}^{i} f(Y_{jt}; M_{t-1}(j)) \Big/ \prod_{t=1}^{i} f(Y_{jt}; \theta_j) \geq nb_n \text{ for some } i \geq 1\right\}, \\ &\qquad\quad \text{by (4.24) and (4.28)}, \\ &\quad \leq (nb_n)^{-1}, \qquad \text{by Lemma 2(i)}, \\ &\quad = o(n^{-1}). \end{aligned}$$

Hence condition (3.1) is satisfied.

Using the results of [1], it can be shown that condition (3.2) is also satisfied. Hence, with h_i and g_{ni} given by (4.26) and (4.27), Theorem 3 can be applied to show that φ^* is an asymptotically efficient allocation rule for double exponential populations. In view of (4.28), the allocation criterion (3.8) can be rewritten as follows: Sample from Π_j at stage $n+1 = km + j$ only if

$$M_{T_n(j)}(j) \geq M_{T_n(j_n)}(j_n) \tag{4.30}$$

or

$$\sum_{i=1}^{T_n(j)} |Y_{ji} - M_{T_n(j_n)}(j_n)| \leq \log nb_n + \sum_{i=1}^{T_n(j)} |Y_{ji} - M_{i-1}(j)|.$$

References

1. Y. S. Chow and T. L. Lai, Some one-sided theorems on the tail distribution of sample sums with applications to the last time and largest excess of boundary crossings, *Trans. Amer. Math. Soc.* **208** (1975), 51–72.
2. H. Reimnitz, Asymptotic near admissibility and asymptotic near optimality in the two armed bandit problem, *Math. Operationsforsch. Statist. Ser. Statist.*, in press.
3. H. Robbins, Some aspects of the sequential design of experiments, *Bull. Amer. Math. Soc.* **55** (1952), 527–535.
4. H. Robbins and D. Siegmund, A class of stopping rules for testing parametric hypotheses, *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **4** (1972), 37–41.