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A Modified Secant Method for Semismooth Equations

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Abstract—A generalization of the secant method to semismooth equations is proposed. Our goal is to obtain similar convergence as Newton's method, but without evaluating any derivative. Convergence analysis and preliminary numerical results are presented. © 2003 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Zerofinding is a classical problem of numerical analysis. We are interested in semismooth one-dimensional equations. In [1], the superlinear convergence of the classical secant method for semismooth equations is proved, and an improvement closely related to Steffensen's methods is presented. However, in order to apply Steffensen's methods in high dimensions, we need to consider operators $F : X \to X$, and this is a limitation.

In this letter, we analyze a modification of the secant method that requires two function evaluations for step. We could consider general semismooth equations

$$F(x) = 0,$$

related with operators, $F : X \to Y$. From the convergence properties and the numerical results, the modified secant method will be a good alternative to the classical methods.

The paper is organized as follows. In Section 2, we introduce the basic ingredients of semismooth equations. Our modified secant method and convergence analysis are studied in Section 3. Finally, some numerical experiments and conclusions are presented in Section 4.

2. THE BASIC FRAMEWORK

In [2], the definition of semismooth functions to nonlinear operators is extended. We say that $F : \mathbb{R}^n \to \mathbb{R}^m$ is semismooth at x if F is locally Lipschitz at x and the following limit:

$$\lim_{\substack{V \in \partial F(x+th'), \\ h' \to h, t \downarrow 0}} \{Vh'\}$$

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exists for any $h \in \mathbb{R}^n$, where ∂F is the generalized Jacobian defined in [3],

$$\partial F(x) = \operatorname{conv} \partial_B F(x),$$

where the B-differential $\partial_B F(x)$ is defined as [4].

Most nonsmooth equations involve semismooth operators at practice [5]. We say that F is strongly semismooth at x if F is semismooth at x and for any $V \in \partial F(x+h)$, $h \to 0$, $Vh - F'(x;h) = O(\|h\|^2)$.

In 1-D, we will denote by $\delta F(x,y)$ the divided differences of the form

$$\delta F(x,y) = \frac{F(x)-F(y)}{x-y}.$$

For the convergence analysis, we will need the following result (it is presented in [1]).

LEMMA 1. *Suppose that F is semismooth at x^* and denote the lateral derivatives of F at x^* by*

$$d^- = -F'(x^*-) \qquad \text{and} \qquad d^+ = F'(x^*+).$$

Then

$$\begin{aligned} d^- - \delta F(u,v) &= o(1), \qquad \forall\, u \uparrow x^*, \quad v \uparrow x^*, \\ d^+ - \delta F(u,v) &= o(1), \qquad \forall\, u \downarrow x^*, \quad v \downarrow x^*. \end{aligned}$$

Moreover, if F is strongly semismooth at x^*, then

$$\begin{aligned} d^- - \delta F(u,v) &= O(|u-x^*| + |v-x^*|), \qquad \forall\, u,v < x^*, \\ d^- - \delta F(u,v) &= O(|u-x^*| + |v-x^*|), \qquad \forall\, u,v > x^*. \end{aligned}$$

3. A MODIFICATION OF THE SECANT METHOD AND CONVERGENCE ANALYSIS

The classical secant method can be written as

$$x_{n+1} = x_n - \delta F(x_n, x_{n-1})^{-1} F(x_n).$$

Our iterative procedure would be considered as a new approach based in a better approximation to the derivative $F'(x_n)$ from x_n and x_{n-1} in each iteration. It takes the following form:

$$x_{n+1} = x_n - \delta F(x_n, \tilde{x}_{n-1})^{-1} F(x_n), \tag{1}$$

where $\tilde{x}_{n-1} = x_{n-1} + \alpha_n(x_n - x_{n-1})$. These parameters α_n will be a control of the good approximation to the derivative. Theoretically, if $\alpha_n \to 1$, then

$$\delta F(x_n, \tilde{x}_{n-1}) \to F'(x_n).$$

The modified secant method needs two evaluations of the function in each iteration. If we consider a complicated function (or operator), this fact can reduce its competitiveness. In this case, the idea is to consider $\alpha_n = 0$ after some first iterations, because then the secant method usually obtains enough good results.

THEOREM 2. *Suppose that F is semismooth at a solution x^* of $F(x) = 0$. If d^- and d^+ are both positive (or negative), then there are two neighborhoods U and V of x^*, $U \subseteq V$, such that for each $x^0 \in U$, Algorithm 1 is well defined and produces a sequence of iterates $\{x_n\}$ such that*

$$x_n \in V, \qquad n = 0, 1, \dots,$$

and $\{x_n\}$ converges to x^* three-step Q-superlinearly. Furthermore, if

$$\beta := \frac{|d^+ - d^-|}{\min\{|d^+|, |d^-|\}} < 1,$$

then $\{x_n\}$ is Q-linearly convergent with Q-factor β. If F is strongly semismooth at x^*, then $\{x_n\}$ converges to x^* three-step Q-quadratically.

PROOF. This proof is based on the following.

Since F is semismooth at x^*, there is a convex neighbourhood W of x^* such that F is Lipschitz continuous on W. There are two convex neighbourhoods U and V such that

$$x^* \in U \subseteq V \subseteq W,$$
$$2\max\{d^+, d^-\} \geq |\delta F(x, y)| \geq 0.5\min\{d^+, d^-\},$$

whenever

$$x, y \in V \qquad \text{and} \qquad (x - x^*)(y - x^*) > 0.$$

Using the above results, the definition of method (1), and Lemma 1, it is not hard to prove that

$$x_n \in V, \qquad n = 0, 1, 2, \dots \qquad \text{and} \qquad |x_{n+3} - x^*| = o(|x_n - x^*|). \tag{2}$$

In fact, when $x_n, x_{n-1} < x^*$ or $x_n, x_{n-1} > x^*$, then

$$|x_{n+1} - x^*| = o(|x_n - x^*|).$$

Equation (2) is only obtained in the cases $x_{n-1} < x^* < x_n$, $x_n < x^* < x_{n-1}$.

Thus, we have proved the three-step Q-superlinear convergence of $\{x_n\}$.

It is similar to the proof of Theorem 3.2 in [1], and we refer to this paper for the details. ∎

However, in practice, there are some advantages to this modified secant method. First, since $\delta F(x_n, \tilde{x}_{n-1})$ is a better approximation to $F'(x_n)$ than $\delta F(x_n, x_{n-1})$, the convergence will be faster (the first iterations will be better). Next, the size of the neighbourhoods can be higher, that is, we can consider worse starting points x_0, as we will see in numerical experiments. Finally, with our modification, usually x_n, $\tilde{x}_{n-1} > x^*$ or x_n, $\tilde{x}_{n-1} < x^*$, and then we could obtain Q-superlinear convergence (or Q-quadratically if F is strongly semismooth).

REMARK 3. We refer to [6,7] for the definitions and notations of Q-order.

As in [1], it is not hard to prove the following theorem.

THEOREM 4. *Suppose that F is semismooth at a solution x^* of $F(x) = 0$. If d^- and d^+ do not vanish and have different signs, then there exist two neighborhoods U and V of x^*, $U \subseteq V$, such that for each $x^0 \in U$, Algorithm 1 is well defined and produces a sequence $\{x_n\}$ such that*

$$x_n \in V, \qquad n = 0, 1, \dots,$$

and $\{x_n\}$ converges to x^* two-step Q-superlinearly. If F is strongly semismooth at x^*, then $\{x_n\}$ converges to x^* two-step Q-quadratically.

4. NUMERICAL EXPERIMENTS AND CONCLUSIONS

In order to show the performance of the modified secant method, we have compared it with the classical secant method. We have tested on several semismooth equations.

In Table 1, we display the iterates for

$$F_1(x) = \begin{cases} x(x+1), & \text{if } x < 0, \\ -2x(x-1), & \text{if } x \geq 0, \end{cases}$$

with $x_0 = 0.1$, $x_1 = 0.05$. In this case, we have $d^- d^+ > 0$, and obviously, the secant method is three-step Q-quadratically convergent, the modified secant method with $\alpha_n = 0.9$ is two-step Q-quadratically convergent, and the modified secant method with $\alpha_n = 1 - 10^{-10}$ is Q-quadratically convergent.

Table 1.

Iter.	Secant	Mod. Secant $\alpha_n = 0.9$	Mod. Secant $\alpha_n = 1 - 10^{-10}$
2	-5.08×10^{-5}	-2.78×10^{-5}	-2.52×10^{-5}
3	-2.51×10^{-5}	-1.35×10^{-5}	6.38×10^{-10}
4	1.27×10^{-9}	2.14×10^{-10}	-4.29×10^{-19}
5	-1.27×10^{-9}	-2.01×10^{-10}	0
6	-4.25×10^{-10}	3.24×10^{-20}	
7	5.42×10^{-19}	-3.24×10^{-20}	
8	-5.42×10^{-19}	0	
9	-1.81×10^{-19}		
10	0		

If we consider as starting points $x_1 = 0.2$, $x_0 = 0.3$, the conclusions are similar, but the new approach is more convenient to use; see Table 2.

Table 2.

Iter.	Secant	Mod. Secant $\alpha_n = 0.9$	Mod. Secant $\alpha_n = 1 - 10^{-10}$
2	-1.20×10^{-1}	-7.12×10^{-2}	-6.66×10^{-2}
3	-4.06×10^{-2}	3.54×10^{-3}	5.13×10^{-3}
4	5.80×10^{-3}	-1.26×10^{-3}	-2.66×10^{-5}
5	-4.80×10^{-3}	9.92×10^{-7}	7.06×10^{-10}
6	-1.67×10^{-3}	-9.77×10^{-7}	-5.18×10^{-19}
7	8.20×10^{-6}	7.63×10^{-13}	0
8	-8.15×10^{-6}	-7.63×10^{-13}	
9	-2.72×10^{-6}	4.65×10^{-25}	
10	2.22×10^{-11}	-4.65×10^{-25}	
11	-2.22×10^{-11}	0	
12	-7.39×10^{-12}		
13	1.64×10^{-22}		
14	-1.64×10^{-22}		
15	-5.47×10^{-23}		
16	0		

In fact, if we put $x_1 = 0.2$, $x_0 = 0.6$, the classical secant method has a problem of convergence. Nevertheless the modified secant method conserves its good properties; see Table 3.

In Table 4, we list the iterates for

$$F_2(x) = \begin{cases} -x(x+1), & \text{if } x < 0, \\ -2x(x-1), & \text{if } x \geq 0, \end{cases}$$

Table 3.

Iter.	Secant	Mod. Secant $\alpha_n = 0.9$	Mod. Secant $\alpha_n = 1 - 10^{-10}$
2	-6.00×10^{-1}	-8.57×10^{-2}	-6.66×10^{-2}
3	-2.57×10^{-1}	5.71×10^{-3}	5.12×10^{-3}
4	1.08×10^{0}	-1.31×10^{-3}	-2.66×10^{-5}
5	1.08×10^{0}	8.06×10^{-7}	7.06×10^{-10}
6	1.07×10^{0}	-7.96×10^{-7}	-5.18×10^{-19}
7	1.07×10^{0}	5.07×10^{-13}	0
8	1.00×10^{0}	-5.07×10^{-13}	
9	1.00×10^{0}	2.06×10^{-25}	
10	1.00×10^{0}	-2.06×10^{-25}	
11	1.00×10^{0}	0	
12	1.00×10^{0}		
13	1.00×10^{0}		
14	1.00×10^{0}		
15	NAN		

with $x_0 = 0.1$, $x_1 = 0.05$. Here $d^- d^+ < 0$, the two-step Q-quadratic convergence is evident for secant and modified secant method with $\alpha_n = 0.9$. But the modification has a faster convergence. The modified secant method with $\alpha_n = 1 - 10^{-10}$ is Q-quadratically convergent.

Table 4.

Iter.	Secant	Mod. Secant $\alpha_n = 0.9$	Mod. Secant $\alpha_n = 1 - 10^{-10}$
2	-5.08×10^{-5}	-2.78×10^{-5}	-2.52×10^{-5}
3	-7.66×10^{-5}	-1.35×10^{-5}	6.38×10^{-10}
4	3.89×10^{-9}	2.01×10^{-10}	-4.29×10^{-19}
5	1.17×10^{-8}	-2.01×10^{-10}	0
6	-4.54×10^{-17}	3.24×10^{-20}	
7	-6.82×10^{-17}	-3.24×10^{-20}	
8	2.46×10^{-32}	0	
9	7.39×10^{-32}		
10	0		

We have presented a modification of the secant method for semismooth equations. We made a complete analysis of convergence for semismooth one-dimensional equations. The new iterative method seems to work very well in our preliminary numerical results, since we have obtained optimal order of convergence. Of course, the generalization of this modification of higher dimensions is similar to the classical secant method [6].

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