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Vague soft sets and their properties[☆]

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ABSTRACT

Soft set theory, proposed by Molodtsov, has been regarded as an effective mathematical tool to deal with uncertainty. However, it is difficult to be used to represent the vagueness of problem parameters. In this paper, we introduce the notion of *vague soft set* which is an extension to the soft set. The basic properties of vague soft sets are presented and discussed.

1. Introduction

Uncertain or imprecise data are inherent and pervasive in many important applications in the areas such as economics, engineering, environment, social science, medical science and business management. Uncertain data in those applications could be caused by data randomness, information incompleteness, limitations of measuring instruments, delayed data updates, etc. Due to the importance of those applications and the rapidly increasing amount of uncertain data collected and accumulated, research on effective and efficient techniques that are dedicated to modeling uncertain data and tackling uncertainties has attracted much interest in recent years and yet remained challenging at large. There have been a great amount of research and applications in the literature concerning some special tools like probability theory, fuzzy set theory [1], rough set theory [2], vague set theory [3], intuitionistic fuzzy set theory [4,5] and interval mathematics [6,7]; each of these theories has its advantages as well as inherent limitations in dealing with uncertainties. One major problem shared by those theories is their incompatibility with the parameterization tools. In 1999, Molodtsov [8] proposed a completely new concept called soft set theory to model uncertainty, which associates a set with a set of parameters and thus is free from the difficulties caused by the aforementioned problem. It has been demonstrated that soft set theory brings about a rich potential for applications in many fields like functions smoothness, Riemann integration, decision making, measurement theory, game theory, etc. [9].

Soft set theory has received much attention since its introduction by Molodtsov. The concept and basic properties of soft set theory are presented in [8,10]. But later in Section 2 of this paper, we will point out that, in [10], the definition of the intersection of two soft sets is not appropriate and some of the properties given are questionable. Some related critiques

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and additional properties of soft sets can be found in [11,12]. To deal with the fuzziness of problem parameters, Roy and Maji [13] proposed the concept of a fuzzy soft set and provided its properties and an application in decision making under an imprecise environment. Chen et al. [14] presented a definition for soft set parameterization reduction and showed an improved application in another decision making problem. Kong et al. [15] further studied the problem of the reduction of soft sets and fuzzy soft sets by introducing a definition for normal parameter reduction. Aktas and Cagman [16] compared the concept of soft sets with the corresponding concepts of fuzzy sets and rough sets, gave a definition for soft groups, and derived their basic properties, adopting the definition of the soft sets in [8].

By definition, a soft set is a parameterized family of subsets of the universal set. In other words, a soft set is a mapping from a set of parameters to the power set of an initial universe set. In the real world, the difficulty is that the objects in the universal set may not precisely satisfy the problem's parameters, which usually represent some attributes, characteristics, or properties of the objects in the universal set. The concept of fuzzy soft sets proposed in [13] partially resolves this difficulty, but falls short in dealing with additional complexity — that is the mapping may be too vague. It is, therefore, desirable to extend soft set theory and fuzzy soft set theory using the concept of a vague set theory.

The theory of vague sets was first proposed by Gau and Buehrer [3]. A vague set is defined by a truth-membership function t_v and a false-membership function f_v, where $t_v(x)$ is a lower bound on the grade of membership of x derived from the evidence for x, and $f_v(x)$ is a lower bound on the negation of x derived from the evidence against x. The values of $t_v(x)$ and $f_v(x)$ are both defined on the closed interval [0, 1] with each point in a basic set X, where $t_v(x)+f_v(x) \leq 1$. Vague set theory is actually an extension of fuzzy set theory and vague sets are regarded as a special case of context-dependent fuzzy sets. The basic concepts of vague set theory and its extensions, as well as some interesting applications can be found in [17–23].

The purpose of this paper is to further extend the concept of soft set theory by introducing the notion of a vague soft set, deriving its basic properties, and illustrating its potential applications. The paper is organized as follows. Section 2 presents the basic concepts and definitions for a soft set and a vague set, then redefines the concept of the intersection of two soft sets and revises the properties given in [10]. Section 3 introduces the notion of a vague soft set and discusses its properties. Concluding remarks and open questions for further investigation are provided in Section 4.

2. Soft sets and vague sets

In this section, for completeness of presentation and convenience of subsequent discussions, we will first review the concepts and properties of soft set theory [8] and vague set theory [3], and then redefine the intersection of two soft sets and revise the properties given in [10].

2.1. Soft sets

Let U be an initial universe, E a set of parameters, $P(U)$ the power set of U, and $A \subseteq E$. Molodtsov [8] defined a soft set as follows:

Definition 2.1. A pair (F, A) is called a soft set over U, where F is a mapping given by

$$F : A \to P(U).$$

This shows that a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ is regarded as the set of ε-approximate elements of the soft set (F, A). For easy reference, we also quote the commonly cited example from [10] here. This example will be revisited repeatedly in later discussion.

Example 2.1 (*[10]*). Consider a soft set (F, E) which describes "attractiveness of houses" that one is considering to purchase. Suppose that there are six houses in the universe U, denoted by $U = \{h_l, h_2, \ldots, h_6\}$ and $E = \{e_l, e_2, e_3, e_4, e_5\}$, where e_i $(i = 1, \ldots, 5)$ stands for the parameters in a word of "expensive", "beautiful", "wooden", "cheap", and "in the green surroundings" respectively. Thus, to define a soft set means to represent expensive houses, beautiful houses and so on.

Consider the mapping F which is "houses $(\cdot)$" where dot $(\cdot)$ is to be filled up by a parameter $e \in E$. For instance, $F(e_l)$ means "houses (expensive)" whose functional-value is the set $\{h \in U, h \text{ is an expensive house}\} = \{h_2, h_4\}$. Suppose we have $F(e_1) = (h_2, h_4)$, $F(e_2) = (h_1, h_3)$, $F(e_3) = (h_3, h_4, h_5)$, $F(e_4) = (h_1, h_3, h_5)$, $F(e_5) = (h_1)$. We can see that the soft set (F, E) is a parameterized family $\{F(e_i),\ i = 1, \ldots, 5\}$ of subsets of the set U and (F, E) can be viewed as consisting of a collection of approximations: $(F, E) = \{$expensive houses $= \{h_2, h_4\}$, beautiful houses $= \{h_l, h_3\}$, wooden houses $= \{h_3, h_4, h_5\}$, cheap houses $= \{h_l, h_3, h_5\}$, houses in the green surroundings $= \{h_1\}\}$.

Definition 2.2 (*[10]*). For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if $A \subseteq B$ and $\forall \varepsilon \in A$, $F(\varepsilon)$ and $G(\varepsilon)$ are identical approximations. This relationship is denoted by $(F, A)\tilde{\subseteq}(G, B)$. Similarly, (F, A) is said to *be* a soft superset *of* (G, B), if (G, B) is a soft subset of (F, A). We denote it by $(F, A)\tilde{\supseteq}(G, B)$.

Definition 2.3 (*[10]*). Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) *is* a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.4 ([10]). Let $E = \{e_1, e_2, \ldots, e_n\}$ be a set of parameters. The not set of E denoted by $\neg E$ is defined by $\neg E = \{\neg e_1, \neg e_2, \ldots, \neg e_n\}$ where $\neg e_i = not\ e_i$.

Definition 2.5 ([10]). The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, \neg A)$ where $F^c : \neg A \to P(U)$ is a mapping given by $F^c(\alpha) = U - F(\neg\alpha), \forall\alpha \in \neg A$.

Definition 2.6 ([10]). If (F, A) and (G, B) are two soft sets then "(F, A) and (G, B)" denoted by $(F, A) \wedge (G, B)$ *is* defined by $(F, A) \wedge (G, B) = (H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cap G(\beta), \forall(\alpha, \beta) \in A \times B$.

Definition 2.7 ([10]). If (F, A) and (G, B) be two soft sets then "(F, A) or (G, B)" denoted by $(F, A) \vee (G, B)$ is defined by $(F, A) \vee (G, B) = (O, A \times B)$, where $O(\alpha, \beta) = F(\alpha) \cup G(\beta),\ \forall(\alpha, \beta) \in A \times B$.

Definition 2.8 ([10]). The union of two soft sets of (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cup B$ and $\forall e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B \\ G(e) & \text{if } e \in B - A \\ F(e) \cup G(e) & \text{if } e \in B \cap A. \end{cases}$$

We denote it by $(F, A)\tilde{\cup}(G, B) = (H, C)$.

Definition 2.9 ([10]). The intersection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cap B$ and $\forall e \in C, H(e) = F(e)$ or $G(e)$, (as both are same set). We denote it by $(F, A)\tilde{\cap}(G, B) = (H, C)$.

Here below, we provide an example to illustrate how the above definition for the intersection of two soft sets by Maji et al. may cause a contradiction and thus requires modification.

Example 2.2. Consider two soft sets (F, A), (G, B), where U is a set of houses; $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$, and A, B are two parameter sets; $A = \{\text{wooden; beautiful}\}$, *and* $B = \{\text{beautiful}\}$. Noticing the ε-approximate elements may differ from person to person, we assume that $F(\text{wooden}) = \{h_1, h_3\}$, $F(\text{beautiful}) = \{h_2, h_4\}$, $G(\text{beautiful}) = \{h_4\}$. Consider the soft set (H, C) as the intersection of two soft sets (F, A) and (G, B) over U. Since "beautiful" $\in A \cap B$, we have $H(\text{beautiful}) = F(\text{beautiful}) = \{h_2, h_4\} \neq \{h_4\} = G(\text{beautiful}) = H(\text{beautiful})$, and this is a contradiction.

The following proposition was provided by Maji et al. in [10]. We quote it here and then will show that the proposition is problematic.

Proposition 2.1 ([10]). *If* (F, A) *and* (G, B) *are two soft sets on* U, *then*

(i) $((F, A)\tilde{\cup}(G, B))^c = (F, A)^c\tilde{\cup}(G, B)^c$.
(ii) $((F, A)\tilde{\cap}(G, B))^c = (F, A)^c\tilde{\cap}(G, B)^c$.

Now we will demonstrate using an example and then provide a theoretical proof that the first part of this proposition by Maji et al. is not correct as a consequence of following the inappropriate definition for the intersection of two soft sets given in [10]. For the same reason, the second part of the proposition also requires reconsideration.

Example 2.3. Reconsider Example 2.2, the union of two soft sets of (F, A) and (G, B) is $(H, C) = \{H\,(\text{wooden}) = \{h_1, h_3\}$, $H\,(\text{beautiful}) = \{h_2, h_4\}\}$. And we have $(H, C)^c = \{H\,(\text{not wooden}) = \{h_2, h_4, h_5, h_6\}; H\,(\text{not beautiful}) = \{h_1, h_3, h_5, h_6\}\}$; $(F, A)^c = \{F\,(\text{not wooden}) = \{h_2, h_4, h_5, h_6\}; F\,(\text{not beautiful}) = \{h_1, h_3, h_5, h_6\}\}$, and $(G, B)^c = \{G\,(\text{not beautiful}) = \{h_1, h_2, h_3, h_5, h_6\}\}$. We denote $(K, D) = (F, A)^c\tilde{\cup}(G, B)^c$, so $(K, D) = \{K\,(\text{not wooden}) = \{h_2, h_4, h_5, h_6\}, K\,(\text{not beautiful}) = \{h_1, h_2, h_3, h_5, h_6\}\}$. This shows obviously that $((F, A)\tilde{\cup}(G, B))^c \neq (F, A)^c\tilde{\cup}(G, B)^c$.

We provide a proposition below to further confirm the Proposition 2.1(i) is not appropriate.

Proposition 2.2. *if* $\alpha \in A \cap B$ *with* $F(\alpha) \neq G(\alpha)$, *then* $((F, A)\tilde{\cup}(G, B))^c \neq (F, A)^c\tilde{\cup}(G, B)^c$.

Proof. Suppose $(H, A \cup B) = (F, A)\tilde{\cup}(G, B)$, $(K, \neg A \cup \neg B) = (F, A)^c\tilde{\cup}(G, B)^c$. Consider $\alpha \in A \cap B$. On one hand, we have $H(\alpha) = F(\alpha) \cup G(\alpha)$, and for $\neg\alpha \in \neg A \cap \neg B$, we have $H^c(\neg\alpha) = U - H(\alpha)$, i.e., $H^c(\neg\alpha) = U - (F(\alpha) \cup G(\alpha))$. Since $F(\alpha), G(\alpha)$ are two subset of U, $H^c(\neg\alpha) = (U - F(\alpha)) \cap (U - G(\alpha))$, i.e., $H^c(\neg\alpha) = F^c(\neg\alpha) \cap G^c(\neg\alpha)$. But on the other hand, $\alpha \in A \cap B$, then $\neg\alpha \in \neg A \cap \neg B$, $K(\neg\alpha) = F^c(\neg\alpha) \cup G^c(\neg\alpha)$. If $\alpha \in A \cap B$ with $F(\alpha) \neq G(\alpha)$, then $F^c(\neg\alpha) \cap G^c(\neg\alpha) \neq F^c(\neg\alpha) \cup G^c(\neg\alpha)$, and $H^c(\neg\alpha) \neq K(\neg\alpha)$. Therefore, if $\exists\alpha \in A \cap B$, $F(\alpha) \neq G(\alpha)$, $((F, A)\tilde{\cup}(G, B))^c \neq (F, A)^c\tilde{\cup}(G, B)^c$. □

Given the above argument, it is necessary to replace Definition 2.9 and redefine the concept for the intersection of two soft sets as follows.

Definition 2.9′. The intersection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$ and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cap G(e), & \text{if } e \in B \cap A. \end{cases}$$

We denote it by $(F, A)\tilde{\cap}(G, B) = (H, C)$.

Based on this new definition of soft set intersection, we can obtain the following proposition.

Proposition 2.3. *If (F, A) and (G, B) are two soft sets on U, then*

(i) $((F, A)\tilde{\cup}(G, B))^c = (F, A)^c\tilde{\cap}(G, B)^c$.
(ii) $((F, A)\tilde{\cap}(G, B))^c = (F, A)^c\tilde{\cup}(G, B)^c$.

Proof. (i) Suppose that $(F, A)\tilde{\cup}(G, B) = (H, A \cup B)$, where

$$H(\alpha) = \begin{cases} F(\alpha), & \text{if } \alpha \in A - B, \\ G(\alpha), & \text{if } \alpha \in B - A, \\ F(\alpha) \cup G(\alpha), & \text{if } \alpha \in A \cap B. \end{cases}$$

Thus, $((F, A)\tilde{\cup}(G, B))^c = (H, A \cup B)^c = (H^c, \neg A \cup \neg B)$.

Therefore, we have $H^c(\neg\alpha) = U - H(\alpha)$, $\forall \neg\alpha \in \neg A \cup \neg B$.

If $\alpha \in A \cap B$, $H(\alpha) = F(\alpha) \cup G(\alpha)$, then $\neg\alpha \in \neg A \cap \neg B$, $H^c(\neg\alpha) = U - H(\alpha)$, i.e., $H^c(\neg\alpha) = U - (F(\alpha) \cup G(\alpha))$. Since $F(\alpha)$, $G(\alpha)$ are two subsets of U, we have $H^c(\neg\alpha) = (U - F(\alpha)) \cap (U - G(\alpha))$.

Therefore,

$$H^c(\neg\alpha) = \begin{cases} F^c(\neg\alpha), & \text{if } \neg\alpha \in \neg A - \neg B, \\ G^c(\neg\alpha), & \text{if } \neg\alpha \in \neg B - \neg A, \\ F^c(\neg\alpha) \cap G^c(\neg\alpha), & \text{if } \neg\alpha \in \neg A \cap \neg B. \end{cases}$$

Again, $(F, A)^c\tilde{\cap}(G, B)^c = (F^c, \neg A)\tilde{\cap}(G^c, \neg B)$, where

$$H(\alpha) = \begin{cases} F(\alpha), & \text{if } \neg\alpha \in \neg A - \neg B, \\ G(\alpha), & \text{if } \neg\alpha \in \neg B - \neg A, \\ F(\alpha) \cap G(\alpha), & \text{if } \neg\alpha \in \neg A \cap \neg B. \end{cases}$$

Similarly, we can prove (ii) of the proposition. □

Proposition 2.4. *If Φ is a null soft set, $\tilde{A}$ an absolute soft set, and (F, A) a soft set on U, then*

(i) $(F, A)\tilde{\cup}(F, A) = (F, A)$.
(ii) $(F, A)\tilde{\cap}(F, A) = (F, A)$.
(iii) $(F, A)\tilde{\cup}\Phi = (F, A)$.
(iv) $(F, A)\tilde{\cap}\Phi = \Phi$.
(v) $(F, A)\tilde{\cup}\tilde{A} = \tilde{A}$.
(vi) $(F, A)\tilde{\cap}\tilde{A} = (F, A)$.

2.2. Vague sets

Let U be an initial universe set, $U = \{u_1, u_2, \ldots, u_n\}$. A vague set over U is characterized by a truth-membership function t_v and a false-membership function f_v, $t_v : U \to [0, 1]$, $f_v : U \to [0, 1]$, where $t_v(u_i)$ is a lower bound on the grade of membership of u_i derived from the evidence for u_i, $f_v(u_i)$ is a lower bound on the negation of u_i derived from the evidence against u_i, and $t_v(u_i)+f_v(u_i) \le 1$. The grade of membership of u_i in the vague set is bounded to a subinterval $[t_v(u_i), 1-f_v(u_i)]$ of $[0, 1]$. The vague value $[t_v(u_i), 1-f_v(u_i)]$ indicates that the exact grade of membership $\mu_v(u_i)$ of u_i may be unknown, but it is bounded by $t_v(u_i) \le \mu_v(u_i) \le 1 - f_v(u_i)$, where $t_v(u_i) + f_v(u_i) \le 1$.

When the universe U is continuous, a vague set A can be written as

$$A = \int_U [t_A(u_i), 1 - f_A(u_i)]/u_i, \quad u_i \in U.$$

When the universe U is discrete, a vague set A can be written as

$$A = \sum_{i=1}^{n} [t_A(u_i), 1 - f_A(u_i)]/u_i, \quad u_i \in U.$$

For a vague set, Gau and Buehrer have introduced the following definitions concerning its operations, which will be useful to understand the subsequent discussion.

Definition 2.10 ([3]). Let x be a vague value, $x = [t_x, 1 - f_x]$, where $t_x \in [0, 1], f_x \in [0, 1]$, and $0 \le t_x \le 1 - f_x \le 1$. If $t_x = 1$ and $f_x = 0$ (i.e., $x = [1, 1]$), then x is called a unit vague value. If $t_x = 0$ and $f_x = 1$ (i.e., $x = [0, 0]$), then x is called a zero vague value.

Definition 2.11 ([3]). Let x and y be two vague values, where $x = [t_x, 1 - f_x]$ and $y = [t_y, 1 - f_y]$. If $t_x = t_y$ and $f_x = f_y$, then vague values x and y are called equal (i.e., $[t_x, 1 - f_x] = [t_y, 1 - f_y]$).

Let A and B be two vague sets of the universe $U = \{u_1, u_2, \dots, u_n\}$, where

$$A = [t_A(u_1), 1 - f_A(u_1)]/u_1 + [t_A(u_2), 1 - f_A(u_2)]/u_2 + \cdots + [t_A(u_n), 1 - f_A(u_n)]/u_n,$$
$$B = [t_B(u_1), 1 - f_B(u_1)]/u_1 + [t_B(u_2), 1 - f_B(u_2)]/u_2 + \cdots + [t_B(u_n), 1 - f_B(u_n)]/u_n.$$

Definition 2.12 ([3]). Let A be a vague set of the universe U. If $\forall u_i \in U$, $t_A\ (u_i) = 1$ and $f_A(u_i) = 0$, then A is called a unit vague set, where $1 \le i \le n$. If $\forall u_i \in U$, $t_A(u_i) = 0$ and $f_A(u_i) = 1$, then A is called a zero vague set, where $1 \le i \le n$.

Definition 2.13 ([3]). The complement of a vague set A is denoted by A^c and is defined by

$$t_{A^c} = f_A,$$
$$1 - f_{A^c} = 1 - t_A.$$

Definition 2.14 ([3]). Let A and B be two vague sets of the universe U. If $\forall u_i \in U$, $[t_A(u_i), 1 - f_A(u_i)] = [t_B(u_i), 1 - f_B(u_i)]$, then the vague set A and B are called equal, where $1 \le i \le n$.

Definition 2.15 ([3]). Let A and B be two vague sets of the universe U. If $\forall u_i \in U$, $t_A(u_i) \le t_B(u_i)$ and $1 - f_A(u_i) \le 1 - f_B(u_i)$, then the vague set A are included by B, denoted by $A \subseteq B$, where $1 \le i \le n$.

Definition 2.16 ([3]). The union of two vague sets A and B is a vague set C, written as $C = A \cup B$, whose truth-membership and false-membership functions are related to those of A and B by

$$t_C = \max(t_A, t_B),$$
$$1 - f_C = \max(1 - f_A, 1 - f_B) = 1 - \min(f_A, f_B).$$

Definition 2.17 ([3]). The intersection of two vague sets A and B is a vague set C, written as $C = A \cap B$, whose truth-membership and false-membership functions are related to those of A and B by

$$t_C = \min(t_A, t_B),$$
$$1 - f_C = \min(1 - f_A, 1 - f_B) = 1 - \max(f_A, f_B).$$

3. Vague soft sets

As discussed in Section 2, a soft set is a mapping from a set of parameters to the power set of a universe set. However, the notion of soft set, as given in its definition, cannot be used to represent the vagueness of the associated parameters. In this section, we introduce the concept of a vague soft set based on soft set theory and vague set theory. The basic properties of a vague soft set will be discussed.

Let U be a universe, E a set of parameters, $V(U)$ the power set of vague sets on U, and $A \subseteq E$. The concept of a vague soft set is given by the following definition.

Definition 3.1. A pair $(\widehat{F}, A)$ is called a *vague soft set* over U, where $\widehat{F}$ is a mapping given by

$$\widehat{F} : A \to V(U).$$

In other words, a vague soft set over U is a parameterized family of vague set of the universe U. For $\varepsilon \in A$, $\mu_{\widehat{F}(\varepsilon)} : U \to [0, 1]^2$ is regarded as the set of ε-approximate elements of the vague soft set $(\widehat{F}, A)$. To illustrate the idea, let us reconsider the house example discussed previously.

Example 3.1. Consider a vague soft set $(\widehat{F}, E)$, where U is a set of six houses under the consideration of a decision maker to purchase, which is denoted by $U = \{h_l, h_2, \dots, h_6\}$, and E is a parameter set, where $E = \{e_l, e_2, e_3, e_4, e_5\} = \{$expensive; beautiful; wooden; cheap; in the green surroundings$\}$. The vague soft set $(\widehat{F}, E)$ describes the "attractiveness of the houses" to this decision maker.

Suppose that

$\widehat{F}(e_1) = ([0.1, 0.2]/h_1, [0.9, 1]/h_2, [0.3, 0.5]/h_3[0.8, 0.9]/h_4[0.2, 0.4]/h_5, [0.4, 0.6]/h_6)$,
$\widehat{F}(e_2) = ([0.9, 1]/h_1, [0.2, 0.7]/h_2, [0.6, 0.9]/h_3[0.2, 0.4]/h_4[0.3, 0.4]/h_5, [0.1, 0.6]/h_6)$,
$\widehat{F}(e_3) = ([0, 0]/h_1, [0, 0]/h_2, [1, 1]/h_3[1, 1]/h_4[1, 1]/h_5, [0, 0]/h_6)$,
$\widehat{F}(e_4) = ([0.8, 0.9]/h_1, [0, 0.1]/h_2, [0.5, 0.7]/h_3[0.1, 0.2]/h_4[0.6, 0.8]/h_5, [0.4, 0.6]/h_6)$,
$\widehat{F}(e_5) = ([0.9, 1]/h_1, [0.2, 0.3]/h_2, [0.1, 0.4]/h_3[0.1, 0.2]/h_4[0.2, 0.4]/h_5, [0.7, 0.9]/h_6)$.

The vague soft set $(\widehat{F}, E)$ is a parameterized family $\{\widehat{F}(e_i),\ i = 1, 2, 3, 4, 5\}$ of vague sets on U, and $(\widehat{F}, E) = \{$expensive houses $= ([0.1, 0.2]/h_1, [0.9, 1]/h_2, [0.3, 0.5]/h_3[0.8, 0.9]/h_4[0.2, 0.4]/h_5, [0.4, 0.6]/h_6)$, beautiful house $= ([0.9, 1]/h_1, [0.2, 0.7]/h_2, [0.6, 0.9]/h_3[0.2, 0.4]/h_4[0.3, 0.4]/h_5, [0.1, 0.6]/h_6), \ldots\}$.

Definition 3.2. For two vague soft sets $(\widehat{F}, A)$ and $(\widehat{G}, B)$ over a universe U, we say that $(\widehat{F}, A)$ is a vague soft *subset* of $(\widehat{G}, B)$, if $A \subseteq B$ and $\forall \varepsilon \in A$, $\widehat{F}(\varepsilon)$ and $\widehat{G}(\varepsilon)$ are identical approximations. This relationship is denoted by $(\widehat{F}, A)\tilde{\subseteq}(\widehat{G}, B)$. Similarly, $(\widehat{F}, A)$ is said to be a vague soft *superset* of $(\widehat{G}, B)$, if $(\widehat{G}, B)$ is a vague soft subset of $(\widehat{F}, A)$. We denote it by $(\widehat{F}, A)\tilde{\supseteq}(\widehat{G}, B)$.

Example 3.2. Let $A = \{e_1, e_2\}$, $B = \{e_1, e_2, e_3\}$, thus $A \subseteq B$. Let $(\widehat{F}, A)$ and $(\widehat{G}, B)$ be two vague soft sets over the same universe U, and

$\widehat{F}(e_1) = ([0.1, 0.2]/h_1, [0.9, 1]/h_2, [0.3, 0.5]/h_3[0.8, 0.9]/h_4[0.2, 0.4]/h_5, [0.4, 0.6]/h_6)$,
$\widehat{F}(e_2) = ([0.9, 1]/h_1, [0.2, 0.7]/h_2, [0.6, 0.9]/h_3[0.2, 0.4]/h_4[0.3, 0.4]/h_5, [0.1, 0.6]/h_6)$,
$\widehat{G}(e_1) = ([0.1, 0.2]/h_1, [0.9, 1]/h_2, [0.3, 0.5]/h_3[0.8, 0.9]/h_4[0.2, 0.4]/h_5, [0.4, 0.6]/h_6)$,
$\widehat{G}(e_2) = ([0.9, 1]/h_1, [0.2, 0.7]/h_2, [0.6, 0.9]/h_3[0.2, 0.4]/h_4[0.3, 0.4]/h_5, [0.1, 0.6]/h_6)$,
$\widehat{G}(e_3) = ([0, 0]/h_1, [0, 0]/h_2, [1, 1]/h_3[1, 1]/h_4[1, 1]/h_5, [0, 0]/h_6)$.

Following the Definition 3.2, we can obtain $(\widehat{F}, A)\tilde{\subseteq}(\widehat{G}, B)$.

Definition 3.3. Two vague soft sets $(\widehat{F}, A)$ and $(\widehat{G}, B)$ over a universe U are said to be vague soft *equal* if $(\widehat{F}, A)$ is a vague soft subset of $(\widehat{G}, B)$ and $(\widehat{G}, B)$ is a vague soft subset of $(\widehat{F}, A)$.

Definition 3.4. Let $E = \{e_1, e_2, \ldots, e_n\}$ be a parameter set. The *not set* of E denoted by $\neg E$ is defined by $\neg E = \{\neg e_1, \neg e_2, \ldots, \neg e_n\}$ where $\neg e_i =$ *not* e_i.

Definition 3.5. The *complement* of' a vague soft set $(\widehat{F}, A)$ is denoted by $(\widehat{F}, A)^c$ and is defined by $(\widehat{F}, A)^c = (\widehat{F}^c, \neg A)$ where $\widehat{F}^c : \neg A \to V(U)$ is a mapping given by $t_{\widehat{F}^c(\alpha)}(x) = f_{\widehat{F}(\neg\alpha)}(x)$, $1 - f_{\widehat{F}^c(\alpha)}(x) = 1 - t_{\widehat{F}(\neg\alpha)}(x)$, $\forall \alpha \in \neg A, x \in U$.

Example 3.3. For Example 3.1, the complement of a vague soft set $(\widehat{F}, E)$ is given below: $(\widehat{F}, E)^c = \{$not expensive houses $= ([0.8, 0.9]/h_1, [0, 0.1]/h_2, [0.5, 0.7]/h_3[0.1, 0.2]/h_4[0.6, 0.8]/h_5, [0.4, 0.6]/h_6)$, not beautiful houses $= ([0, 0.1]/h_1, [0.3, 0.8]/h_2, [0.1, 0.4]/h_3[0.6, 0.8]/h_4[0.6, 0.7]/h_5, [0.4, 0.9]/h_6)$, not wooden houses $= ([1, 1]/h_1, [1, 1]/h_2, [0, 0]/h_3[0, 0]/h_4[0, 0]/h_5, [1, 1]/h_6)$, not cheap houses $= ([0.1, 0.2]/h_1, [0.9, 1]/h_2, [0.3, 0.5]/h_3[0.8, 0.9]/h_4[0.2, 0.4]/h_5, [0.4, 0.6]/h_6)$, not in the green surroundings $= ([0, 0.1]/h_1, [0.7, 0.8]/h_2, [0.6, 0.9]/h_3[0.8, 0.9]/h_4[0.6, 0.8]/h_5, [0.1, 0.3]/h_6)\}$.

Definition 3.6. A vague soft set $(\widehat{F}, A)$ over U is said to be a *null* vague soft set denoted by $\widehat{\Phi}$, if $\forall \varepsilon \in A$, $t_{\widehat{F}(\varepsilon)}(x) = 0$, $1 - f_{\widehat{F}(\varepsilon)}(x) = 0$, $x \in U$.

Definition 3.7. A vague soft set $(\widehat{F}, A)$ over U is said to be an *absolute* vague soft set denoted by $\widehat{\tilde{A}}$, if $\forall \varepsilon \in A$, $t_{\widehat{F}(\varepsilon)}(x) = 1$, $1 - f_{\widehat{F}(\varepsilon)}(x) = 1$, $x \in U$.

Definition 3.8. If $(\widehat{F}, A)$ and $(\widehat{G}, B)$ are two vague soft sets over U, "$(\widehat{F}, A)$ *and* $(\widehat{G}, B)$" denoted by $(\widehat{F}, A) \wedge (\widehat{G}, B)$ is defined by $(\widehat{F}, A) \wedge (\widehat{G}, B) = (\widehat{H}, A \times B)$, where $t_{\widehat{H}(\alpha,\beta)}(x) = \min\{t_{\widehat{F}(\alpha)}(x), t_{\widehat{G}(\beta)}(x)\}$, $1 - f_{\widehat{H}(\alpha,\beta)}(x) = \min\{1 - f_{\widehat{F}(\alpha)}(x), 1 - f_{\widehat{G}(\beta)}(x)\}$, $\forall (\alpha, \beta) \in A \times B, x \in U$.

Definition 3.9. If $(\widehat{F}, A)$ and $(\widehat{G}, B)$ be two vague soft sets over U, "$(\widehat{F}, A)$ *or* $(\widehat{G}, B)$", denoted by $(\widehat{F}, A) \vee (\widehat{G}, B)$, is defined by $(\widehat{F}, A) \vee (\widehat{G}, B) = (\widehat{O}, A \times B)$, where $t_{\widehat{O}(\alpha,\beta)}(x) = \max(t_{\widehat{F}(\alpha)}(x), t_{\widehat{G}(\beta)}(x))$, $1 - f_{\widehat{O}(\alpha,\beta)}(x) = \max(1 - f_{\widehat{F}(\alpha)}(x), 1 - f_{\widehat{G}(\beta)}(x))$, $\forall (\alpha, \beta) \in A \times B, x \in U$.

Proposition 3.1. *If $(\widehat{F}, A)$ and $(\widehat{G}, B)$ are two vague soft sets on U, then*

(i) $((\widehat{F}, A) \vee (\widehat{G}, B))^c = (\widehat{F}, A)^c \wedge (\widehat{G}, B)^c$.
(ii) $((\widehat{F}, A) \wedge (\widehat{G}, B))^c = (\widehat{F}, A)^c \vee (\widehat{G}, B)^c$.

Proof. (i) Suppose that $(\widehat{F}, A) \vee (\widehat{G}, B) = (\widehat{O}, A \times B)$.

Then, we have $((\widehat{F}, A) \vee (\widehat{G}, B))^c = (\widehat{O}, A \times B)^c = (\widehat{O}^c, \neg(A \times B))$. Hence,

$$(\widehat{F}, A)^c \wedge (\widehat{G}, B)^c = (\widehat{F}^c, \neg A) \wedge (\widehat{G}^c, \neg B) = (\widehat{J}, \neg A \times \neg B) = (\widehat{J}, \neg(A \times B)),$$

where $t_{\widehat{J}(\neg\alpha,\neg\beta)}(x) = \min\{t_{\widehat{F}^c(\neg\alpha)}(x), t_{\widehat{G}^c(\neg\beta)}(x)\}$, $1 - f_{\widehat{J}(\neg\alpha,\neg\beta)}(x) = \min\{1 - f_{\widehat{F}^c(\neg\alpha)}(x), 1 - f_{\widehat{G}^c(\neg\beta)}(x)\}$,

$\forall(\neg\alpha, \neg\beta) \in \neg A \times \neg B, x \in U$.

We take $(\neg\alpha, \neg\beta) \in \neg(A \times B)$,

$$\begin{aligned} t_{\widehat{O}(\neg\alpha,\neg\beta)}(x) &= f_{\widehat{O}(\alpha,\beta)}(x) \\ &= 1 - \max\{1 - f_{\widehat{F}(\alpha)}(x), 1 - f_{\widehat{G}(\beta)}(x)\} \\ &= \min\{f_{\widehat{F}(\alpha)}(x), f_{\widehat{G}(\beta)}(x)\} \\ &= \min\{t_{\widehat{F}^c(\neg\alpha)}(x), t_{\widehat{G}^c(\neg\beta)}(x)\} \\ &= t_{J(\neg\alpha,\neg\beta)}(x). \end{aligned}$$

$$\begin{aligned} 1 - f_{\widehat{O}(\neg\alpha,\neg\beta)}(x) &= 1 - t_{\widehat{O}(\alpha,\beta)}(x) \\ &= 1 - \max\{t_{\widehat{F}(\alpha)}(x), t_{\widehat{G}(\beta)}(x)\} \\ &= \min\{1 - t_{\widehat{F}(\alpha)}(x), 1 - t_{\widehat{G}(\beta)}(x)\} \\ &= \min\{1 - f_{\widehat{F}^c(\neg\alpha)}(x), 1 - f_{\widehat{G}^c(\neg\beta)}(x)\} \\ &= 1 - f_{J(\neg\alpha,\neg\beta)}(x). \end{aligned}$$

Hence, $\widehat{O}^c$ and $\widehat{J}$ are the same operator, and the first part of Proposition 3.1(i) has been proved. Similarly, we can prove the second part (ii) of the proposition. □

Given the definitions above, we can easily obtain the following proposition:

Proposition 3.2. *If $(\widehat{F}, A)$, $(\widehat{G}, B)$ and $(\widehat{H}, C)$ are three vague soft sets on U, then*

(i) $(\widehat{F}, A) \vee ((\widehat{G}, B) \vee (\widehat{H}, C)) = ((\widehat{F}, A) \vee (\widehat{G}, B)) \vee (\widehat{H}, C)$.
(ii) $(\widehat{F}, A) \wedge ((\widehat{G}, B) \wedge (\widehat{H}, C)) = ((\widehat{F}, A) \wedge (\widehat{G}, B)) \wedge (\widehat{H}, C)$.

Definition 3.10. The *union* of two vague soft sets of $(\widehat{F}, A)$ and $(\widehat{G}, B)$ over a universe U is a vague soft set $(\widehat{H}, C)$, where $C = A \cup B$ and $\forall e \in C$,

$$t_{\widehat{H}(e)}(x) = \begin{cases} t_{\widehat{F}(e)}(x), & \text{if } e \in A - B, x \in U, \\ t_{\widehat{G}(e)}(x), & \text{if } e \in B - A, x \in U, \\ \max(t_{\widehat{F}(e)}(x), t_{\widehat{G}(e)}(x)), & \text{if } e \in B \cap A, x \in U. \end{cases}$$

$$1 - f_{\widehat{H}(e)}(x) = \begin{cases} 1 - f_{\widehat{F}(e)}(x), & \text{if } e \in A - B, x \in U, \\ 1 - f_{\widehat{G}(e)}(x), & \text{if } e \in B - A, x \in U, \\ \max(1 - f_{\widehat{F}(e)}(x), 1 - f_{\widehat{G}(e)}(x)), & \text{if } e \in B \cap A, x \in U. \end{cases}$$

We denote it by $(\widehat{F}, A)\tilde{\cup}(\widehat{G}, B) = (\widehat{H}, C)$.

Definition 3.11. The *intersection* of two vague soft sets $(\widehat{F}, A)$ and $(\widehat{G}, B)$ over a universe U is a vague soft set $(\widehat{H}, C)$, where $C = A \cup B$ and $\forall e \in C$,

$$t_{H(e)}(x) = \begin{cases} t_{\widehat{F}(e)}(x), & \text{if } e \in A - B, x \in U, \\ t_{\widehat{G}(e)}(x), & \text{if } e \in B - A, x \in U, \\ \min(t_{\widehat{F}(e)}(x), t_{\breve{G}(e)}(x)), & \text{if } e \in B \cap A, x \in U. \end{cases}$$

$$1 - f_{H(e)}(x) = \begin{cases} 1 - f_{\widehat{F}(e)}(x), & \text{if } e \in A - B, x \in U, \\ 1 - f_{\widehat{G}(e)}(x), & \text{if } e \in B - A, x \in U, \\ \min(1 - f_{\widehat{F}(e)}(x), 1 - f_{\breve{G}(e)}(x)), & \text{if } e \in B \cap A, x \in U. \end{cases}$$

We denote it by $(\widehat{F}, A)\tilde{\cap}(\widehat{G}, B) = (\widehat{H}, C)$.

The following propositions can be obtained based on the definitions above introduced:

Proposition 3.3. *If $\widehat{\Phi}$ is a null vague soft set, $\tilde{A}$ an absolute vague soft set, and $(\widehat{F}, A)$ a vague soft set on U, then*

(i) $(\widehat{F}, A)\tilde{\cup}(\widehat{F}, A) = (\widehat{F}, A)$.
(ii) $(\widehat{F}, A)\tilde{\cap}(\widehat{F}, A) = (\widehat{F}, A)$.

(iii) $(\widehat{F}, A)\tilde{\cup}\widehat{\Phi} = (\widehat{F}, A)$.
(iv) $(\widehat{F}, A)\tilde{\cap}\widehat{\Phi} = \widehat{\Phi}$.
(v) $(\widehat{F}, A)\tilde{\cup}\widehat{\widehat{A}} = \widehat{\widehat{A}}$.
(vi) $(\widehat{F}, A)\tilde{\cap}\widehat{\widehat{A}} = (\widehat{F}, A)$.

Proposition 3.4. *If $(\widehat{F}, A)$ and $(\widehat{G}, B)$ are two vague soft sets on U, then*

(i) $((\widehat{F}, A)\tilde{\cup}(\widehat{G}, B))^c = (\widehat{F}, A)^c\tilde{\cap}(\widehat{G}, B)^c$.
(ii) $((\widehat{F}, A)\tilde{\cap}(\widehat{G}, B))^c = (\widehat{F}, A)^c\tilde{\cup}(\widehat{G}, B)^c$.

Proposition 3.5. *If $(\widehat{F}, A)$, $(\widehat{G}, B)$ and $(\widehat{H}, C)$ are three vague soft sets on U, then*

(i) $(\widehat{F}, A)\tilde{\cup}((\widehat{G}, B)\tilde{\cup}(\widehat{H}, C)) = ((\widehat{F}, A)\tilde{\cup}(\widehat{G}, B))\tilde{\cup}(\widehat{H}, C)$.
(ii) $(\widehat{F}, A)\tilde{\cap}((\widehat{G}, B)\tilde{\cap}(\widehat{H}, C)) = ((\widehat{F}, A)\tilde{\cap}(\widehat{G}, B))\tilde{\cap}(\widehat{H}, C)$.
(iii) $(\widehat{F}, A)\tilde{\cup}((\widehat{G}, B)\tilde{\cap}(\widehat{H}, C)) = ((\widehat{F}, A)\tilde{\cup}(\widehat{G}, B))\tilde{\cap}((\widehat{F}, A)\tilde{\cup}(\widehat{H}, C))$.
(iv) $(\widehat{F}, A)\tilde{\cap}((\widehat{G}, B)\tilde{\cup}(\widehat{H}, C)) = ((\widehat{F}, A)\tilde{\cap}(\widehat{G}, B))\tilde{\cup}((\widehat{F}, A)\tilde{\cap}(\widehat{H}, C))$.

4. Conclusions

In this paper, the basic concepts of a soft set are reviewed, and the concept of the intersection of two soft sets is redefined and some propositions given in [10] are revised. We introduce the notion of a vague soft set as an extension to the soft set. The basic properties of vague soft sets are also presented and discussed. This new extension not only provides a significant addition to existing theories for handling uncertainties, but also leads to potential areas of further field research and pertinent applications. To extend our work, further research could be done to study the issues on the relationship between a soft set, a rough set, and a vague soft set, and to combine these theories to deal with uncertainties. It is also desirable to further explore the applications of using the vague soft set approach to solve real world decision making problems.

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