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Algebraic Algebras over Π-Regular Rings*

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We prove here results complementing our work [1b]. In the process of the proofs we extend some classical results from algebraic algebras over fields [5, Theorem 1; 4, Theorem 6.4.3] to algebraic algebras over Π-regular rings (in the sense of Levitski [6]).

Conventions. Throughout the paper Φ is a commutative ring with identity, A is a Φ-ring or an algebra over Φ, $\Phi[X]$, where X is a subset of A, the subalgebra of A that is generated by X.

I. If A is an algebra over Φ then we say that $a \in A$ is *algebraic* provided $\Phi[a]$ ($= \Phi[\{a\}]$) is a finite module (over Φ). We shall call an algebra A *algebraic* if every element a of A is algebraic. Clearly algebraic algebras over fields are algebraic in our sense. It can be verified that $a \in A$ is algebraic if and only if a is a root of some monic polynomial

$$f(t) = t^n + \alpha_{n-1}t^{n-1} + \cdots + \alpha_1 t \in \Phi[t]$$

of degree $n \geqslant 1$ and without constant term. Then Levitski's algebraic elements (defined in [6]) are the same as the present algebraic elements of A. It follows that complete matric algebra Φ_m and group (semigroup) algebras $\Phi(S)$ of a group (semigroup) S which is locally finite [1b, Theorem 2, Corollary 1] are algebraic algebras.

Given an equation in A of the form $f(x) = 0$, where $f = f(t)$ is a monic polynomial over Φ, there is no way in general to pass to an equation of the lower monic form

$$f^*(x) = x^k + \alpha_{k+1}x^{k+1} + \cdots + \alpha_{k+l}x^{k+l} = 0.$$

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An element x satisfying the latter equation is termed *coalgebraic*. Of course, if Φ is a field there is identity between algebraic and coalgebraic elements. The case $\Phi = \mathbb{Z}$ (the integers) shows that the latter equivalence fails in general to be true [1a, Theorem].

Following Levitski [6] we define the *algebraic* (*coalgebraic*) *kernel* $K = K(A)$ $(K^* = K^*(A))$ of an algebra A to be the sum of all algebraic (coalgebraic) ideals of A thought of as algebras. Since all monic (lower-monic) polynomials $f \in \Phi[t]$ of degree (codegree) $\geqslant 1$ and without constant term form a closed system $\mathcal{M}(\mathcal{M}^*)$ under composition and multiplication of polynomials, $K(K^*)$ has the same properties as in the classical cases of algebras over fields or Φ-rings [6]. Thus $K(K^*)$ is the maximum algebraic (coalgebraic) ideal of A and the kernel of $A/K(A/K^*)$ is 0. We prove now a result which will reduce algebraic conditions to the prime algebra case.

LEMMA 1. *If $x \in A$ is algebraic (coalgebraic) in every prime image $\bar{A}$ of the algebra A, then x is algebraic (coalgebraic).*

Proof. Assume by contradiction that x is nonalgebraic (coalgebraic). The set X of all $f(x)$, $f(t)$ any monic (lower-monic) polynomial $f(t) \in \mathcal{M}(\mathcal{M}^*)$, is a multiplicative subset excluding 0. Then there is some prime (algebra) ideal P of A maximal with respect to the property of not intersecting X. As x is algebraic (coalgebraic) in A/P, there exists $f \in \mathcal{M}(\mathcal{M}^*)$ so that $f(x) \in P \cap X$, a contradiction.

An immediate consequence of Lemma 1 is

LEMMA 2. *If the algebraic (coalgebraic) kernel $K(\bar{A})$ $(K^*(\bar{A}))$ is $\neq 0$ for any prime image $\bar{A}$ of an algebra A, then A must be algebraic (coalgebraic).*

If A is a prime algebra over Φ, then the annihilator Λ of the Φ-module A^+ is a prime ideal of the ring Φ. In the presence of Π-regularity of Φ, Φ/Λ is then a field. Because Φ and Φ/Λ have the same action on A, Φ acts on A as a field of operators. Here, in particular, there must be identity between algebraic and coalgebraic elements. Lemma 1 and the latter observation give

LEMMA 3. *If A is an algebra over a Π-regular ring Φ, then there is identity between algebraic and coalgebraic elements of A. In particular, $K = K^*$ is a Π-regular ideal of A.*

In accordance with PI-rings, we shall say that an algebra A satisfies an *identity*[1] (a *local identity*) if there exists some polynomial in noncommuting

[1] An identity (local identity) in the sense of Procesi is sufficient for the present considerations [7, Introduction].

variables (depending on any given finitely generated subalgebra R of A) with coefficients ± 1 vanishing on $A(R)$. We can now prove our first Theorem.

THEOREM 1. *Let A be an algebra over a Π-regular ring Φ satisfying an identity. Assume that any element a of A can be written (up to a power $a^{k(a)}$) as a sum of algebraic or coalgebraic elements. Then A is algebraic and coalgebraic. Consequently A and each subalgebra of A are Π-regular.*

Proof. By the foregoing, we may assume that A is a prime algebra. We turn A into an algebra A' over the field Φ/Λ where Λ is the annihilator of the Φ-module A^+. Here A' satisfies a polynomial identity over $\Phi = \Phi/\Lambda$. Also every element a of A' can be written as a sum of algebraic elements over Φ' (up to a power of a). By Procesi's result [5, Theorem 1], A' is an algebraic algebra. By Lemma 3, $K = K^* = A$ is Π-regular and by Lemma 1, each subalgebra of A is Π-regular. This proves the Theorem.

II. If A is an algebra as in Theorem 1, then A is an algebraic algebra satisfying an identity. In accordance with the classical case one is then interested to know if some finiteness informations can be given in this instance on the finitely generated subalgebras of A. One then introduces "locally finite" ("locally bounded") algebras A to be those algebras A such that any finitely generated subalgebra F is a finite module (an algebraic algebra of bounded degree, i.e., such that the ranks of the Φ-modules $\Phi[a]$, $a \in F$, are bounded). It can be verified that any locally bounded algebra A satisfies a local identity and that any locally finite algebra A is[2] locally bounded, hence is an algebraic algebra satisfying a local identity. Is the converse true? We shall prove that this is the case for Π-regular algebras A over arbitrary commutative rings Φ (with 1) from which result it will be clear that if A is an algebra as in Theorem 1, A must be locally finite. We need the following:

LEMMA 4. *Let A be a finite module over Φ. Then A is locally finite.*

Proof. Let $\{a_1, ..., a_s\}$ be a finite subset of A. By the assumption, $R = \Phi x_1 + \cdots + \Phi x_m$. Write $a_1 = \sum_{j=1}^{m} \alpha_{ij} x_j$, $x_i x_j = \sum_{k=1}^{m} \alpha_{ijk} x_k$. By Hilbert Basis Theorem, if $\Phi_0 = [1; \alpha_{ij}; \alpha_{k1m}]$ is the subring of Φ generated by 1, α_{ij} and by α_{k1m}, then Φ_0 is a Noetherian ring. Then the subset $A_0 = \sum \Phi_0 x_i$ is a subring of the ring A which is a module over Φ_0, hence a Noetherian module. Then $\Phi_0[a_1, ..., a_s] \subseteq A_0$ is a finite submodule of the Φ_0-module A_0. If $\Phi_0[a_1, ..., a_s] = \sum_{i=1}^{S^1} \Phi_0 c_i$, then $\Phi[a_1, ..., a_s] \subseteq \sum_i \Phi c_i \subseteq \Phi[a_1, ..., a_s]$. Therefore $\Phi[a_1, ..., a_s] = \sum_i \Phi c_i$ is a finite submodule of A and A is locally finite.

[2] By a determinant argument similar to [8, Lemma, p. 255] if A is a finite module of rank r over Φ, then it can be proved that for any $a \in A$, the rank of $\Phi[a]$ is at most.

Lemma 4 can be extended as follows:

LEMMA 5. *Let A be an algebra with identity. Let C be the centre of A. If C is algebraic and if A is locally finite over its centre C, then A is locally finite.*

In corollary,

PROPOSITION 1. *The complete matric algebra Φ_m over a Π-regular ring Φ (with 1) is an algebraic algebra of bounded degree, whence Φ_m and each subalgebra of Φ_m are Π-regular.*

PROPOSITION 2. *A (with 1) is locally finite if and only if A is central locally finite and the centre of A is algebraic.*

Going back to the initial question we define now the *locally finite kernel* $L = L(A)$ of an algebra A to be the sum L of all locally finite ideals of the algebra A. As observed by Levitski, L is the greatest locally finite algebra ideal of A and the quotient algebra A/L has zero kernel $L(A/L)$ [6]. Along the lines of the proof of [4, Theorem 6.4.1], it can be verified (without necessarily condition (4) in [6]) that any one-sided locally finite ideal of A is contained in the kernel $L = L(A)$. The latter result can be exploited as in the classical case in order to prove

LEMMA 5. *Let A be an algebraic algebra satisfying an identity. Let $a \in A$ be a square-zero nilpotent. Then the principal left ideal generated by a is contained in the kernel L of A.*

Proof. It is clear that any left ideal $\mathscr{T}$ of A is algebraic and so is any homomorphic image of $\mathscr{T}$ thought of as an algebra. Also any nilpotent or commutative algebraic algebras are locally finite. By recursion on the degree of the identity, $\mathscr{T}/W$ and W are locally finite where W is the left annihilator of W in $= (a \mid$. Then $\mathscr{T}$ is locally finite, whence $L \supseteq \mathscr{T}$.

Here and elsewhere we repeatedly used the following: Any algebra A with a given ideal B is locally finite (algebraic, coalgebraic) if and only if R/B and B are locally finite (algebraic, coalgebraic). We are now in a position to show.

THEOREM 2. *Let A be an algebraic algebra over Φ. If A or Φ are Π-regular, then the three following conditions are equivalent:*

(1) *A is locally finite.*
(2) *A is locally bounded.*
(3) *A satisfies a local identity.*

Proof. By the foregoing it suffices to show that (3) implies (1). Our proof will go by reduction to the case A is nilpotent-free (by Lemma 5). The second reduction is that we may take A with a finite algebra basis, whence A satisfies an identity. Here A is Π-regular (Theorem 1). By a well-known result, A is strongly regular. Assume $A \neq 0$. There exists a primitive ideal P of the algebra A. The algebra A/P is then a division ring. If Λ is the annihilator of the Φ-module A/P, Λ is a maximal ideal of Φ so that Φ acts on A/P as a field of operators. As A/P satisfies an identity over Φ/Λ, A/P is a finite module over Φ [4, Theorem 6.4.3]. Since A has an algebra basis, A can be written as

$$A = \sum_{i=1}^{n} \Phi x_i + P_0 , \tag{i}$$

where $P_0 \subseteq P$ is a finitely generated ideal of A. By strong regularity, P_0 is a unit ideal of unity e. Multiplication by $1 - e$ of both sides of Eq. (i) yield us with an ideal $A(1 - e) = \sum_i \Phi x_i(1 - e)$ which is a finite module. By Lemma 4, $A(1 - e) \subseteq L$. Since the assumptions go over homomorphic images of A, $A = L$.

III. As an application of the present results let us study algebraic semigroup algebras $\Phi(S)$ of a semigroup over a commutative ring Φ with identity. By definition, $S \subseteq \Phi(S)$ is then Φ-independent. Thus a necessary condition that $\Phi(S)$ be algebraic is that S be a *periodic* semigroup (i.e., for any $x \in S$ there is $m(x) > n(x) \geqslant 1$ so that $x^{m(x)} = x^{n(x)}$ [3, p. 20 or; 1a, Introduction].) If we give a ground to the condition that $\Phi(S)$ is algebraic, namely, that $\Phi(S)$ be locally finite we can then characterize this case as follows:

THEOREM 3. *Any semigroup algebra $\Phi(S)$ is locally finite if and only if S is a locally finite semigroup.*

Proof of the "if". By definition [1b, Theorem 2, Corollary 1], if S' is any finitely generated subsemigroup of S, then S' is a finite set. Consider a finite subset X of $\Phi(S)$. For some finite subset X' of S, $\Phi[X] \subseteq \Phi(S')$, where S' is the subsemigroup generated by X' in S. Since S' is finite, $\Phi(S')$ is a finite module over Φ. By Lemma 4, $\Phi(S')$ is locally finite. Therefore, $\Phi[X]$ is a finite submodule. This shows that $\Phi(S)$ is locally finite.

Proof of the "only if". Let S' be a subsemigroup of S that is generated by some finite subset $E'S'$. The semigroup algebra $\Phi(S') = \Phi[E']$ is a finite subalgebra of $\Phi(S)$. By the assumption, $\Phi(S')$ is a finite module over Φ. Let X be a finite generating set of the finite module $\Phi(S')$. Each $x \in X$ is Φ-expressible over S'. It follows that S' possesses a finite subset X' which is a generating set of the module $\Phi(S')$. As S' is independent over Φ, $S' = X'$ is finite. Therefore, S is locally finite.

From Theorem 3 we derive the following corollary.

COROLLARY. *Let Φ be a Π-regular commutative ring with identity. The following three conditions on a semigroup S (with 1) are equivalent.*

(1) *S is locally finite.*
(2) *S is central locally finite and periodic.*
(3) *$\Phi(S)$ is locally bounded.*

If one (and hence all) of the conditions (1)–(3) are satisfied, then $\Phi(s)$ and each of its subalgebras are Π-regular.

Explanation and proof. If S is a semigroup (with 1) with centre C then we say that S is *central locally finite* provided each subsemigroup of S that is generated by C and by a finite subset of S can be written as a finite union of cosets Cy of C. It can be verified that if S is central locally finite, then $\Phi(S)$ is locally finite over its center [3, pp. 4–6, 2a]. By the foregoing, conditions (1)–(3) are then equivalent. The corollary follows immediately.

Remarks. 1. One can give a direct proof of the equivalence between conditions (1) and (2).

2. One can show that if the semigroup algebra $\Phi(S)$ is Π-regular, then Φ must be Π-regular. Is there any information that can be given on the semigroup S?

The corollary suggests the following open question.

Question. Let R be a ring satisfying an identity. Let S be a periodic semigroup imbedded in the multiplicative semigroup of A. Is S locally finite?

Two cases for which the answer to the question is "yes" are the following: (a) S is a periodic group (by Herstein–Procesi's result [7, Theorem 2]); (b) R is a torsion ring [1b, Theorem 2]. A positive answer to this question will give then a semigroup generalization of Burnside's result on a group of matrices. One could exploit this fact (if true) in order to give a proof of Theorem 3 without using the freeness of the basis S of the semigroup algebra $\Phi(S)$ but merely the existence of a local identity in $\Phi(S)$ and the periodicity of S. In addition to the mentioned cases we shall now treat a rather admittedly restrictive case (yet offering an original combination of classical results with the present results), namely, we will require an S to be: (c) a separative periodic semigroup[3] all of whose idempotent elements commuting pairwise such that for any finitely generated subsemigroup S' of S, there is some integer $n = n(S')$ so that $x^n = x^{m(x)}$ for all $x \in S'$ where

[3] In other words, S is a semigroup in which $x = x^{n(x)}$ for all $x \in S$ [2, p. 131].

$m(x) < n$. It is clear that any periodic group G of bounded degree (such that $x^N = 1$ for all $x \in G$ and for some fixed N) is satisfying condition (c). Under the latter condition we shall show that the answer to the Question is again "yes". We need two more lemmas.

LEMMA 6. *Let S be a semigroup of matrices over a field satisfying* (c). *Then S is locally finite.*

Proof. We may assume that $S \subseteq \Phi_m$ is finitely generated, where Φ is an algebraically closed field Φ. In characteristic $c \neq 0$, S is locally finite (by the author's result). In characteristic 0, if S_e is the maximal subgroup of S belonging to some idempotent $e \in S$, then S_e is a periodic group of bounded degree. It follows that $S_e \subseteq e\Phi_m e \approx \Phi_n$. By [4, Theorem 2.3.4], S_e is finite. Let $A = \Phi[S]$ be the linear span of S over Φ. This is a subalgebra of Φ_m of bounded degree. By a Theorem of Jacobson [6, Theorem 12], $A/\text{rad}(A)$ is a finite direct sum of primitive algebraic algebras over Φ. As Φ is algebraically closed, $A/\text{rad}(A) = \bigoplus_{i=1}^{k} \Phi_{n_i}$. Now for each i, S is irreducible in Φ_{n_i}. Since S is of a bounded degree, $\text{Tr}(S)$ in Φ_{n_i} is finite. By [4, Theorem 2.3.3], S is finite modulo $\text{rad}(A)$. Now $\text{rad}(A)$ distinguish between commuting idempotents of A

$$[\text{i.e., } e = e^2, f = f^2, ef = fe, e - f \in \text{rad}(A) \Rightarrow e = f].$$

Thus S has only a finite number of maximal subgroups S_e. Since all these subgroups are finite, S, which is a finite union of maximal subgroups, must be finite.

LEMMA 7. *Let A be an algebra over an algebraically closed field Φ satisfying an identity. Let S be a subsemigroup of A satisfying* (c), *then S is locally finite.*

Proof. We may assume that S is finitely generated whence of bounded degree. Let R be the linear span of S over Φ. This is a finitely generated algebra with an identity. By Theorems 1 and 2, R is locally finite whence of bounded degree over Φ. As in Lemma 6, we may assume that Φ is infinite of characteristic 0. By Jacobson's result $R/\text{rad}(R) = \bigoplus_{i=1}^{m} \Phi_{n_i}$. By Lemma 6, S is a finite union of maximal subgroups S_e. Here $\Phi[S_e]$ R is of bounded degree. Because $\Phi[S_e]$ is an homomorphic image of the group algebra $\Phi(S_e)$, which is regular since S_e is locally finite (by Procesi–Herstein's result [5, Theorem 2]) and since Φ is of characteristic 0 [4, Theorem 2.1.9], $\Phi[S_e]$ is regular, whence semisimple. By Jacobson's result again, S_e is a finite group. Therefore S is finite.

PROPOSITION 3. *Let A be an algebra over any commutative ring Φ with identity and without divisors of zero. Let S be any semigroup of elements x of A*

which are algebraic over Φ. If, in addition, S satisfies (c), *then A' is locally finite over Φ, where $A' = \Phi[S]$.*

Proof. We will treat the case (sufficient in the present considerations) where $\Phi = \mathbb{Z}$ but it is clear that the general proof can be extended word for word from the latter case. We may assume that S is finitely generated and we first prove that $A' = \mathbb{Z}[S]$ (which is merely a ring) is algebraic over the integers $\mathbb{Z}$. By Lemma 2, it suffices to show this for a prime ring A. In characteristic $c \neq 0$, A is certainly algebraic [1b, Theorem 1]. In characteristic $c = 0$, R is torsion free. Then R, hence S, can be imbedded in an algebra B over the complex numbers satisfying an identity. By Lemma 7, S is locally finite. It follows that $R' = \mathbb{Z}[S]$ is algebraic over the integers (see Theorem 3 and the proof). All in all, we have shown that R' is algebraic over $\mathbb{Z}$. Let $K = K_{\mathbb{Z}}(R')$ be the locally finite kernel of R' as an algebra over $\mathbb{Z}$. Let $R'' = R'/K$. If T is the torsion-part of R'', then T is algebraic and torsion. By the author's result, T is locally finite over $\mathbb{Z}$. As the kernel of R'' is 0, $T = 0$ and R'' is torsion free. Then the image S'' of S in R'' is a locally finite semigroup (by Lemma 7). Then $R'' = \mathbb{Z}[S'']$ is locally finite. Therefore, $K = R'$ and R' is locally finite.

We are now in a position to prove

THEOREM 4. *Let R be a ring satisfying a local identity. Let S be a subsemigroup of R satisfying* (c). *Then S is locally finite.*

Proof. We may assume that S is finitely generated and that S generates R as a ring. By the assumption, R is satisfying an identity. By Proposition 4, R^+ is a finitely generated group. If T is the torsion part of R, then T is a finite subset. It follows that S is locally finite if and only if S is locally finite in R/T which is torsion free. If $\bar{S}$ is the image of S in R/T, then $\bar{S}$ can be imbedded in $R/T \otimes_{\mathbb{Z}} \mathbb{C}$ which is an algebra over $\mathbb{C}$ (the complex numbers) satisfying an identity. By Lemma 7, $\bar{S}$, hence S, are locally finite.

We end the paper by two remarks.

Remark 1. The following strengthening of Theorem 3 has been pointed out by the referee: *Any algebraic semigroup algebra is locally finite.* Thus as in the classical case (of group algebras), to say that a semigroup algebra is algebraic is equivalent with saying that the underlying semigroup is locally finite.

Remark 2. Theorem 2 answers in a particular case a question raised by C. Procesi, namely, is an algebraic algebra over a commutative ring with 1 a locally finite algebra if (and only if) the algebra satisfies an identity locally? The author learned from Dr. Procesi that this question can be answered by "yes" in case the underlying ring is the integers (or a more general type).

Needless to say, this result and Theorem 3 are independent from each other. The author would like to mention, finally, that Procesi et al. are studying in forthcoming papers Procesi's question and the author's question (Part III, p. 10).

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