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Resolvability in graphs and the metric dimension of a graph

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Abstract

For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of vertices in a connected graph G and a vertex v of G, the metric representation of v with respect to W is the k-vector $r(v \mid W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. The set W is a resolving set for G if $r(u \mid W) = r(v \mid W)$ implies that $u = v$ for all pairs u, v of vertices of G. The metric dimension $\dim(G)$ of G is the minimum cardinality of a resolving set for G. Bounds on $\dim(G)$ are presented in terms of the order and the diameter of G. All connected graphs of order n having dimension $1, n-2$, or $n-1$ are determined. A new proof for the dimension of a tree is also presented. From this result sharp bounds on the metric dimension of unicyclic graphs are established. It is shown that $\dim(H) \leqslant \dim(H \times K_2) \leqslant \dim(H) + 1$ for every connected graph H. Moreover, it is shown that for every positive real number ε, there exists a connected graph G and a connected induced subgraph H of G such that $\dim(G)/\dim(H) < \varepsilon$.

Keywords: Distance; Metric dimension; Basis

1. Introduction

The structure of a chemical compound is frequently viewed as a set of functional groups arrayed on a substructure. From a graph-theoretic perspective, the structure is a labeled graph where the vertex and edge labels specify the atom and bond types, respectively. From this perspective, the functional groups and substructure are simply subgraphs of the labeled graph representation. By changing the set of functional groups

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and/or permuting their positions, a collection of compounds is essentially defined that are characterized by the substructure common to them.

Traditionally, these "positions" simply reflect uniquely defined atoms (vertices) of the substructure (common subgraph). These positions seldom form a minimum set in the following sense. Let W be an ordered subset, of cardinality k say, of the vertex set $V(G)$ of the common subgraph G, and let v be a vertex in $V(G)$. We can associate with v an ordered k-tuple that gives the distances from v to each of the vertices in W. Then the smallest such set W for which every two distinct vertices have distinct ordered k-tuples forms a minimum dimensional representation of the positions definable on the common subgraph. In this context, W is referred to as a resolving set relative to $V(G)$.

Under the traditional view, we can determine whether any two compounds in the collection share the same functional group at a particular position. This comparative statement plays a critical role in drug discovery whenever it is to be determined whether the features of a compound are responsible for its pharmacological activity. However, the statement, as it stands, is applicable only to compounds sharing the common subgraph G. By redefining "position" to mean the value of an ordered k-tuple based on a resolving set, the statement can be extended to broader collections of compounds. For example, suppose that G is a subgraph of a graph H, and W is also a resolving set of $V(H)$. Even though G and H define two distinct collections with positions specified by the same k-tuple, the comparative statement remains valid if one of the compounds comes from the G-collection and the other comes from the H-collection. This ability to extend the domain of such comparative statements motivates our interest in resolving sets. In this paper we focus on the problem of determining the minimum cardinality of a resolving set in a graph G relative to $V(G)$.

2. Resolvable sets in graphs

The ideas discussed in Section 1 suggest some mathematical concepts, which we now describe. We denote the standard distance between two vertices u and v in a connected graph G by $d(u,v)$. By an ordered set of vertices, we mean a set $W=\{w_1,w_2,\ldots,w_k\}$ on which the ordering $(w_1,w_2,\ldots,w_k)$ has been imposed. For an ordered subset $W=\{w_1,w_2,\ldots,w_k\}$ of $V(G)$, we refer to the k-vector (ordered k-tuple)

$$r(v\mid W)=(d(v,w_1),d(v,w_2),\ldots,d(v,w_k))$$

as the *(metric) representation of* v *with respect to* W. The set W is called a *resolving set for* G if $r(u\mid W)=r(v\mid W)$ implies that $u=v$ for all u, $v\in V(G)$. Hence if W is a resolving set of cardinality k for a graph G of order n, then the set $\{r(v\mid W)\mid v\in V(G)\}$ consists of n distinct k-vectors. A resolving set of minimum cardinality for a graph G is called a *minimum resolving set.*

For example, consider the graph G of Fig. 1. The set $W_1=\{v_1,v_3\}$ is not a resolving set for G since $r(v_2\mid W_1)=(1,1)=r(v_4\mid W_1)$. On the other hand, $W_2=\{v_1,v_2,v_3\}$ is a

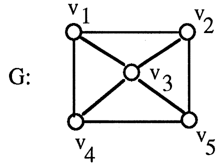

Fig. 1.

resolving set for G since the representations for the vertices of G with respect to W_2 are

$$r(v_1 \mid W_2) = (0,1,1), \quad r(v_2 \mid W_2) = (1,0,1), \quad r(v_3 \mid W_2) = (1,1,0).$$
$$r(v_4 \mid W_2) = (1,2,1), \quad r(v_5 \mid W_2) = (2,1,1),$$

However, W_2 is not a minimum resolving set since $W_3 = \{v_1, v_2\}$ is also a resolving set. Since no single vertex constitutes a resolving set for G, it follows that W_3 is a minimum resolving set.

We now return to our discussion of the classification problem in chemistry. A fundamental problem in the study of chemical structures is to determine ways to represent a set of chemical compounds such that distinct compounds have distinct representations. One way to accomplish this is first to associate a graph with each compound (which can be done quite naturally) and define an integer-valued metric on the set of corresponding graphs such that the distance between some pairs of graphs is 1. Next a metagraph G is constructed whose vertices are these graphs, where two vertices are adjacent in G if and only if the distance between their corresponding graphs is 1. Then the representations of the vertices of G are computed with respect to some minimum resolving set of G, thereby providing representations of the compounds as well.

The idea of resolving sets and minimum resolving sets has appeared in the literature previously. In [4] and later in [5], Slater introduced the concept of a resolving set for a connected graph G under the term *locating set*. He referred to a minimum resolving set as a *reference set* for G. He called the cardinality of a minimum resolving set (reference set) the *location number* of G. Slater described the usefulness of these ideas when working with sonar and loran stations. Independently, Harary and Melter [2] discovered these concepts as well but used the term *metric dimension*, rather than location number. We adopt the terminology of Harary and Melter. Consequently, the *metric dimension* or, more simply, the *dimension* $\dim(G)$ of a connected graph G is the cardinality of a minimum resolving set. Because of the suggestiveness of this terminology to linear algebra, we also refer to a minimum resolving set as a *basis* for G. Hence, the vertices of G have distinct representations with respect to the basis vertices. For the graph G of Fig. 1, $\dim(G) = 2$ and $\{v_1, v_2\}$ is a basis for G. It was noted in [1, p. 204] that determining the metric dimension of a graph is an NP-complete problem.

We next describe the problem of finding the metric dimension and a basis for a graph in terms of an integer programming problem. Let G be a connected graph of order n with $V(G)=\{v_1,v_2,\ldots,v_n\}$ and let $D=[d_{ij}]$ be the distance matrix of G, that is, $d_{ij}=d(v_i,v_j)$ for $1\leqslant i,\ j\leqslant n$. For $x_i\in\{0,1\}$ for $1\leqslant i\leqslant n$, define the function F by

$$F(x_1,x_2,\ldots,x_n)=x_1+x_2+\cdots+x_n.$$

Minimizing F subject to the $\binom{n}{2}$ constraints

$$|d_{i1}-d_{j1}|x_1+|d_{i2}-d_{j2}|x_2+\cdots+|d_{in}-d_{jn}|x_n>0 \quad \text{for } 1\leqslant i<j\leqslant n$$

is equivalent to finding a basis in the sense that if $x_1',x_2',\ldots,x_n'$ is a set of values for which F attains its minimum, then $W=\{v_i \mid x_i'=1\}$ is a basis for G, and, conversely, if $W=\{v_{i1},v_{i2},\ldots,v_{in}\}$ is a basis for G and if we define

$$x_s'=\begin{cases}1 & \text{if } s=i_j \text{ for some } j\ (1\leqslant j\leqslant k)\\ 0 & \text{otherwise}\end{cases}$$

then $F(x_1',x_2',\ldots,x_n')$ is a minimum subject to the given constraints.

Notice, for each connected graph G and each ordered set $W=\{w_1,w_2,\ldots,w_k\}$ of vertices of G, that the ith coordinate of $r(w_i \mid W)$ is 0 and that the ith coordinate of all other vertex representations is positive. Thus, certainly $r(u \mid W)=r(v \mid W)$ implies that $u=v$ for $u\in W$. Therefore, when testing whether an ordered subset W of $V(G)$ is a resolving set for G, we need only be concerned with the vertices of $V(G)-W$. Consequently, $V(G)$ and $V(G)-\{v\}$ are resolving sets for every nontrivial connected graph G and every vertex v of G. As a result, if G is a connected graph of order $n\geqslant 2$, then $1\leqslant\dim(G)\leqslant n-1$. On the other hand, if we know the diameter of G, then we can obtain an improved upper bound in general for $\dim(G)$, as well as a lower bound. For positive integers d and n, we define $f(n,d)$ to be the least positive integer k for which $k+d^k\geqslant n$.

Theorem 1. *If G is a connected graph of order $n\geqslant 2$ and diameter d, then*

$$f(n,d)\leqslant\dim(G)\leqslant n-d.$$

Proof. First, we establish the upper bound. Let u and v be vertices of G for which $d(u,v)=d$, and let $u=v_0,v_1,\ldots,v_d=v$ be a u–v path of length d. Let $W=V(G)-\{v_1,v_2,\ldots,v_d\}$. Since $d(u,v_i)=i$ for $1\leqslant i\leqslant d$ and $u\in W$, it follows that W is a resolving set of cardinality $n-d$ for G. Thus, $\dim(G)\leqslant n-d$.

Next, we consider the lower bound. Let B be a basis for G of dimension k. Since each representation of a vertex of $V(G)-B$ is a k-vector, every coordinate of which is a positive integer not exceeding d, and all $n-k$ representations are distinct, it follows that $d^k\geqslant n-k$. Hence, $f(n,d)\leqslant k=\dim(G)$. □

The inequality given in the upper bound of Theorem 1 can be strict. For example, the tree T_1 of Fig. 2 has order $n=8$ and diameter $d=4$, but $B_1=\{v_1,v_8\}$ is a basis for T_1 and so $\dim(T_1)=2$ and $n-d=4$. On the other hand, the tree T_2 of Fig. 2 shows that the upper bound in Theorem 1 can be sharp since T_2 also has order $n=8$ and diameter $d=4$, while $B_2=\{w_1,w_6,w_7,w_8\}$ is a basis and so $\dim(T_2)=4$.

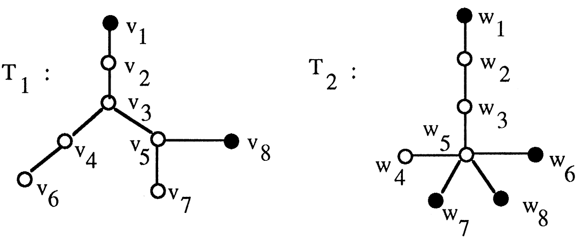

Fig. 2.

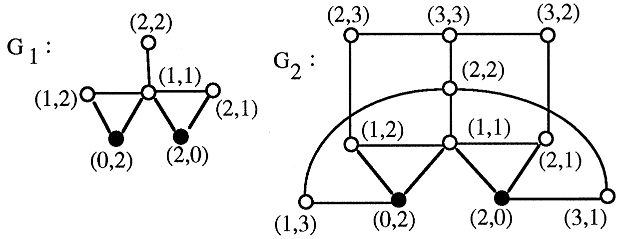

Fig. 3.

The lower bound in Theorem 1 can be attained for graphs of diameter 2 or 3. For example, the graph G_1 of Fig. 3 has order 6, diameter 2, and dimension $f(6,2)=2$. Also, the graph G_2 of Fig. 3 has order 11, diameter 3, and dimension $f(11,3)=2$. The vertices of both graphs in Fig. 3 are lableled with their representations.

On the other hand, if $d=\dim(G)\geqslant 4$ and $\dim(G)\geqslant 2$, then the lower bound given in Theorem 1 cannot be sharp. To see this, suppose, to the contrary, that there exists a graph G of order n, diameter d, and $\dim(G)=k$ such that $k+d^k=n$. Let W be a basis for G. Then all k-tuples of positive integers not exceeding d must appear as a representation of some vertex of G with respect to W. However, since the k-vector $(1,1,\ldots)$ appears, it follows that $d(w_1,w_2)\leqslant 2$. For some vertex v of $G, r(v\mid W) =(1,4,\ldots)$. Thus, $d(v,w_1)=1$, but

$$d(v,w_2)=4>1+2\geqslant d(v,w_1)+d(w_1,w_2),$$

which is impossible; so no such graph exists.

If $G=P_n$, then by Theorem 1, $\dim(G)=1$ and either end-vertex of G constitutes a basis. Indeed, as we next show, paths are the only graphs of dimension 1.

Theorem 2. *A connected graph G of order n has dimension* 1 *if and only if $G=P_n$.*

Proof. We have already noted that if $G=P_n$, then $\dim(G)=1$. For the converse, assume that G is a connected graph with $\dim(G)=1$ and basis $W=\{w\}$. For each vertex v of G, $r(v\mid W)=d(v,w)$ is a nonnegative integer less than n. Since the representations of the vertices of G with respect to W are distinct, there exists a vertex u of G such that $d(u,w)=n-1$. Consequently, the diameter of G is $n-1$, which implies that $G=P_n$. □

On the other hand, consider $G=K_n$, $n\geqslant 2$, and let W be a basis for G. If $u\notin W$, then every coordinate of $r(u\,|\,W)$ is 1. Therefore, every resolving set for G must contain all but one vertex of G, so $\dim(K_n)=n-1$. By Theorem 1, if G is a connected graph that is not complete, then $\dim(G)\leqslant n-2$. Hence, we have the following result.

Theorem 3. *A connected graph G of order $n\geqslant 2$ has dimension $n-1$ if and only if $G=K_n$.*

3. Graphs with dimension $n-2$

In Theorem 3, we characterized those graphs of order n having dimension $n-1$. Indeed, these are precisely the complete graphs. In this section, we classify those graphs of order n with dimension $n-2$. For graphs G and H we use $G\cup H$ to denote the disjoint union of G and H and $G+H$ to denote the graph obtained from the disjoint union of G and H by joining every vertex of G with every vertex of H.

Theorem 4. *Let G be a connected graph of order $n\geqslant 4$. Then $\dim(G)=n-2$ if and only if $G=K_{s,t}$ $(s,t\geqslant 1), G=K_s+\overline{K_t}$ $(s\geqslant 1, t\geqslant 2)$, or $G=K_s+(K_1\cup K_t)$ $(s,t\geqslant 1)$.*

Proof. Each of the graphs mentioned in the statement of the theorem has dimension $n-2$. To see this, note that if the vertices of a graph are partitioned as $V_1\cup V_2\cup\cdots\cup V_p$ where either V_i is independent and its vertices have identical open neighborhoods or V_i induces a clique and its vertices have identical closed neighborhoods, then the dimension is at least $(\,|V_1|-1)+(\,|V_2|-1)+\cdots+(\,|V_p|-1)$.

For the converse, assume that G is a connected graph of order $n\geqslant 4$ such that $\dim(G)=n-2$. By Theorems 1 and 3, it follows that G has diameter 2. If G is bipartite, then since the diameter of G is 2, $G=K_{s,t}$ for some integers s, $t\geqslant 1$. Hence, we may assume that G is not bipartite. Therefore, G contains an odd cycle. Let C_r be a smallest odd cycle in G. We claim that $r=3$. Certainly, C_r is an induced cycle of G. If G contains an induced k-cycle $v_1,v_2,\ldots,v_k,v_1$, where $k\geqslant 5$, then $W=V(G)-\{v_2,v_3,v_4\}$ is a resolving set of cardinality $n-3$, for if we let $w_1=v_1$ and $w_2=v_5$, then $r(v_2\,|\,W)=(1,s,\ldots), r(v_3\,|\,W)=(2,2,\ldots)$, and $r(v_4\,|\,W)=(t,1,\ldots)$ where $s,t\geqslant 2$. Hence, $\dim(G)\leqslant n-3$, which is a contradiction. Thus G has no induced cycle of length $k>5$ and so $r=3$ and G contains a triangle.

Let Y be the vertex set of a maximum clique of G. Since G contains a triangle, $|\,Y\,|\geqslant 3$. Let $U=V(G)-Y$. Since G is not complete, $|\,U\,|\geqslant 1$. If $|\,U\,|=1$, then $G=K_s+(K_1\cup K_t)$ for some integers s and t. Now, $s\geqslant 1$ since G is connected and $t\geqslant 1$ since G is not complete. From these observations, we may assume that $|\,U\,|\geqslant 2$.

First, we show that U is an independent set of vertices. Suppose, to the contrary, that U is not independent. Then U contains two adjacent vertices u and w. Because of the defining property of Y, there exist $v\in Y$ such that $uv\notin E(G)$ and $v'\in Y$ such that $wv'\notin E(G)$, where v and v' are not necessarily distinct. We consider two cases.

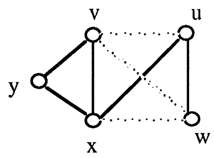

Fig. 4.

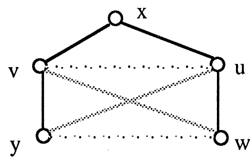

Fig. 5.

Case 1: *There exists a vertex* $v \in Y$ *such that* $uv, wv \notin E(G)$. We now consider two subcases.

Subcase 1.1: *There exists a vertex* $x \in Y$ *that is adjacent to exactly one of* u *and* w, *say* u. Since $|Y| \geqslant 3$, there exists a vertex $y \in Y$ that is distinct from v and x. Thus G contains the subgraph shown in Fig. 4, where dotted lines indicate that the given edge is not present.

Let $W = V(G) - \{u, w, y\}$. Letting $w_1 = v$ and $w_2 = x$, we have

$$r(u \mid W) = (2, 1, \ldots),$$

$$r(w \mid W) = (2, 2, \ldots),$$

$$r(y \mid W) = (1, 1, \ldots).$$

So W is a resolving set of cardinality $n - 3$, which is a contradiction.

Subcase 1.2: *Every vertex of* Y *is adjacent to either both* u *and* w *or to neither* u *nor* w. If u and w are adjacent to every vertex in $Y - \{v\}$, then the vertices of $(Y - \{v\}) \cup \{u, w\}$ are pairwise adjacent, contradicting the defining property of Y. Thus, there exists a vertex $y \in Y$ such that y is distinct from v, and y is adjacent to neither u nor w.

Since the diameter of G is 2, there is a vertex x of G that is adjacent to both u and v. Thus, G contains the subgraph shown in Fig. 5, where dotted lines indicate that the given edges are not in G.

Let $W = V(G) - \{x, y, w\}$ and label $w_1 = v$ and $w_2 = u$. Then

$$r(x \mid W) = (1, 1, \ldots),$$

$$r(y \mid W) = (1, 2, \ldots),$$

$$r(w \mid W) = (2, 1, \ldots).$$

Thus, W is a resolving set of cardinality $n - 3$, producing a contradiction.

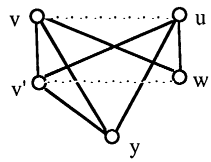

Fig. 6.

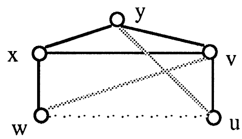

Fig. 7.

Case 2: *For each vertex* v of Y, v *is adjacent to at least one of* u *and* w. Because Y is the vertex set of a maximum clique, there exist vertices $v, v' \in Y$ such that $uv, wv' \notin E(G)$. Necessarily, $vw, v'u \in E(G)$. Since $|Y| \geqslant 3$, there exists a vertex y in Y distinct from v and v'. Now, at least one of the edges yu and yw must be present in G, say yu. Thus, G contains the subgraph shown in Fig. 6, where again dotted edges indicate that the given edge is not in G.

Let $W = V(G) - \{u, w, y\}$ and label $w_1 = v$ and $w_2 = v'$. Then

$$r(u \mid W) = (2, 1, \ldots),$$

$$r(w \mid W) = (1, 2, \ldots),$$

$$r(y \mid W) = (1, 1, \ldots).$$

Again, W is a resolving set of cardinality $n - 3$, which is a contradiction. Thus, as claimed, U is independent.

Next, we claim that $N(u) = N(w)$ for all $u, w \in U$. Let u and w be any two vertices of U. Suppose that $uv \in E(G)$ for some vertex v of G. Necessarily, $v \in Y$. We show that $wv \in E(G)$. Assume, to the contrary, that $wv \notin E(G)$. Since Y is the vertex set of a maximum clique, there exists $y \in Y$ such that $uy \notin E(G)$. Since G is connected and U is independent, w is adjacent to some vertex of Y. If w is adjacent only to y, then since w and y are not adjacent to u, $d(w, u) = 3$, which contradicts the fact that the diameter of G is 2. Thus there exists a vertex x in Y distinct from y such that $wx \in E(G)$. Therefore, G contains the subgraph shown in Fig. 7, where again dotted edges are not in G.

Let $W = V(G) - \{u, w, x\}$ and label $w_1 = v$ and $w_2 = y$. Then

$$r(u \mid W) = (1, 2, \ldots),$$

$$r(w \mid W) = (2, \ldots),$$

$$r(x \mid W) = (1, 1, \ldots).$$

Thus, W is a resolving set of cardinality $n - 3$, producing a contradiction.

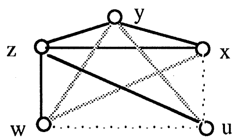

Fig. 8.

Therefore, $V(G) = Y \cup U$, where Y induces a clique, U is independent, $|Y| \geqslant 3$, $|U| \geqslant 2$, and $N(u) = N(w)$ for all $u, w \in U$.

Next, we claim that for $u \in U$, there is at most one vertex of Y not contained in $N(u)$. Suppose, to the contrary, that there are two vertices $x, y \in Y$ not in $N(u)$. Let w be a vertex of U that is distinct from u. Therefore, $N(w) = N(u)$. Since G is connected, there exists $z \in Y$ such that $z \in N(u) = N(w)$. Thus, G contains the subgraph shown in Fig. 8, where dotted edges are not edges of G.

Let $W = V(G) - \{y, w, z\}$ and label $w_1 = x$ and $w_2 = u$. Then

$$r(y \mid W) = (1, 2, \ldots),$$

$$r(w \mid W) = (2, 2, \ldots),$$

$$r(z \mid W) = (1, 1, \ldots).$$

So W is a resolving set of cardinality $n - 3$, producing a contradiction.

Now, $N(u) = Y$ or $N(u) = Y - \{v\}$ for some $v \in Y$. If $N(u) = Y$, then $G = K_s + \overline{K_t}$ for $s = |Y| \geqslant 3$ and $t = |U| \geqslant 2$. If $N(u) = Y - \{v\}$, then $G = K_s + (K_1 \cup \overline{K_t})$, where $V(K_1) = \{v\}, s = |Y| - 1 \geqslant 2$, and $t = |U| \geqslant 2$. However, $K_s + (K_1 \cup \overline{K_t}) = K_s + \overline{K_{t+1}}$. In either case, G is the join of a complete graph and an empty graph. $\square$

4. The dimensions of trees and unicylic graphs

In Theorem 2, we noted that the dimension of the path P_n ($n \geqslant 2$) is 1. By Theorem 4, the dimension of the star $K_{1,n-1}$ ($n \geqslant 3$) of order n is $n - 2$. Indeed, for every tree T of order $n \geqslant 3$, $1 \leqslant \dim(T) \leqslant n - 2$. In this section, we present a formula for the dimension of trees that are not paths as well as bounds for the dimension of unicyclic graphs. The formula for trees has also been established by Slater [4] and Harary and Melter [2]. The proof we include here uses ideas different to those used in these two papers. First, a few definitions are in order.

A vertex of degree at least 3 in a graph G will be called a *major vertex* of G. Any end-vertex u of G is said to be a *terminal vertex of a major vertex* v of G if $d(u, v) < d(u, w)$ for every other major vertex w of G. The *terminal degree* $\text{ter}(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree. Let $\sigma(G)$ denote the sum of the terminal degrees of the major vertices of G, and let $\text{ex}(G)$ denote the number of exterior major vertices of G.

The following lemma which holds for general graphs will be used in the proofs of the next two theorems.

Lemma 1. *If G is a graph, then* $\dim(G) \geqslant \sigma(G) - \mathrm{ex}(G)$.

Proof. Let W be any resolving set and let v be an exterior major vertex of G. Let $k = \mathrm{ter}(v)$ and let $u_1, u_2, \ldots, u_k$ denote the terminal vertices of v. Thus the branch of G at v containing u_i $(1 \leqslant i \leqslant k)$ is a v–u_i path Q_i. We claim that W contains at least one vertex from each of the paths $Q_i - v$ $(1 \leqslant i \leqslant k)$ with at most one exception. Suppose, to the contrary, that two of these paths contain no vertex of W, say $Q_1 - v$ and $Q_2 - v$. Let u_1' and u_2' be the vertices adjacent to v on Q_1 and Q_2, respectively. Since neither $Q_1 - v$ nor $Q_2 - v$ contains a vertex of W, it follows that $r(u_1' \mid W) = r(u_2' \mid W)$, contradicting the fact that W is a resolving set. Thus, as claimed, W contains at least one vertex from each of the paths $Q_i - v$ $(1 \leqslant i \leqslant k)$ with at most one exception. Consequently, $\dim(G) \geqslant \sigma(G) - \mathrm{ex}(G)$. □

Theorem 5. *If T is a tree that is not a path, then*

$$\dim(T) = \sigma(T) - \mathrm{ex}(T).$$

Proof. By Lemma 1, $\dim(T) \geqslant \sigma(T) - \mathrm{ex}(T)$.

We now verify the reverse inequality. We construct a set W of vertices of T by placing every terminal vertex, except one, of each exterior major vertex of T in W. We claim that W is a resolving set of T. In order to see this, let u be an arbitrary vertex of T. We consider two cases.

Case 1: *Suppose that there is some exterior major vertex w of T and a terminal vertex x of w such that u lies on the w–x path of T.* We now consider two subcases.

Subcase 1.1: *Suppose that $x \in W$.* Let v be any vertex of T different from u. If v lies on the u–x path of T, then $d(v,x) < d(u,x)$; otherwise, $d(u,x) < d(v,x)$. In either case, $r(u \mid W) \neq r(v \mid W)$.

Subcase 1.2: *Suppose that $x \notin W$.* Let v be any vertex of T different from u. If there is some vertex y in W such that either v lies on the u–y path of T or u lies on the v–y path, then $d(v,y) < d(u,y)$ or $d(u,y) < d(v,y)$, respectively. In either case, $r(u \mid W) \neq r(v \mid W)$. Thus, we may assume that every path between v and a vertex of W does not contain u and that every path between u and a vertex of W does not contain v. Necessarily, then, there exists an exterior major vertex w' and a terminal vertex x' of w' such that v lies on the w'–x' path of T and $v \neq w'$. Moreover, $x' \notin W$ and $u \neq w$. Note that $w \neq w'$, for otherwise either x or x' belongs to W.

Since the degrees of both w and w' are at least 3, there exists a branch B at w that contains neither w' nor x and there is a branch B' at w' that contains neither w nor x'. Necessarily, both B and B' must contain a vertex of W. Let z and z' be vertices of W belonging to B and B', respectively. If $d(u,z') \neq d(v,z')$, then $r(u \mid W) \neq r(v \mid W)$; so we may assume that $d(u,z') = d(v,z')$. In this case, $d(u,z) < d(v,z)$, implying that $r(u \mid W) \neq r(v \mid W)$.

Case 2: *Suppose, for every exterior major vertex w of T and every terminal vertex x of w, that u does not lie on the w–x path of T.* Then there are at least two branches at u, say B and B', each of which contains some exterior major vertex of terminal degree at least 2. Therefore, each of B and B' contains a vertex of W. Let z and z' be vertices of W in B and B', respectively. Let v be a vertex of T distinct from u. If v belongs to B, then the v–z' path of T contains u; so $d(u,z') < d(v,z')$ and $r(u \mid W) \neq r(v \mid W)$. If v does not belong to B, then the v–z path of T contains u. Hence $d(u,z) < d(v,z)$ and so $r(u \mid W) \neq r(v \mid W)$.

Hence W is a resolving set of T and $\dim(T) \leqslant |W| = \sigma(T) - \mathrm{ex}(T)$. □

It is often of interest to know how the value of a graphical parameter is affected when a small change is made in a graph. In this context, we answer the question in the case of dimension when a single edge is added to a tree. We show, in fact, that the dimension can increase by at most 1 or decrease by at most 2 and that all values in this range are attainable. Since the proof of this result is similar to that of Theorem 5, we only provide a detailed outline of its proof.

Theorem 6. *If T is a tree of order at least 3 and e is an edge of $\bar{T}$, then*

$$\dim(T) - 2 \leqslant \dim(T + e) \leqslant \dim(T) + 1.$$

Proof. Since $\sigma(T + e) \geqslant \sigma(T) - 2$ and $\mathrm{ex}(T + e) \leqslant \mathrm{ex}(T)$, it follows that

$$\sigma(T + e) - \mathrm{ex}(T + e) \geqslant \sigma(T) - \mathrm{ex}(T) - 2 = \dim(T) - 2.$$

By Lemma 1, $\dim(T + e) \geqslant \sigma(T + e) - \mathrm{ex}(T + e)$. From this it follows that $\dim(T + e) \geqslant \dim(T) - 2$.

It remains to show then that $\dim(T + e) \leqslant \dim(T) + 1$. Let W be a set of vertices of $T + e$ that contains for each exterior major vertex v of $T + e$, all terminal vertices v with one exception. Let C denote the unique cycle of $T + e$. We consider four cases.

Case 1: *C contains at least three major vertices, each of which has a branch containing a vertex of W.* In this case W is a resolving set for $T + e$ and so $\dim(T + e) \leqslant |W| \leqslant \dim(T)$.

Case 2: *C contains exactly two major vertices v and w, each of which has a branch containing a vertex of W.* Let u be a vertex of C distinct from v and w. In this case, $W \cup \{u\}$ is a resolving set for $T + e$, so $\dim(T + e) \leqslant |W| + 1 \leqslant \dim(T) + 1$.

Case 3: *C contains exactly one major vertex v that has a branch containing a vertex of W.* Let H be the branch of $T + e$ that contains C. Except possibly for v, every vertex of H has degree at least 3 in $T + e$. Since at least one vertex of H has terminal degree in T one greater than its terminal degree in $T + e$, it follows that $\sigma(T) - \mathrm{ex}(T) = |W| + 1$. Let W' denote the set obtained by adding to W two vertices of C that are distinct from v. Then W' is a resolving set for $T + e$ and

$$\dim(T + e) \leqslant |W'| = |W| + 2 \leqslant \sigma(T) - \mathrm{ex}(T) + 1 = \dim(T) + 1.$$

Case 4: *There is no major vertex belonging to C that has a branch containing a vertex of W.* Thus $W=\emptyset$. In this case, T is a caterpillar with maximum degree at most 3. If T is not a path, then $\dim(T)=2$ and any three vertices of C constitute a resolving set for $T+e$. Hence $\dim(T+e)\leqslant\dim(T)+1$. On the other hand, if T is a path, then $\dim(T)=1$. Let W' be the set consisting of the two end-vertices of T. Then W' is a resolving set for $T+e$ for every possible choice of e and so $\dim(T+e)\leqslant\dim(T)+1$. □

By Theorem 6, if T is a tree of order at least 3 and e is an edge of $\bar{T}$, then $\dim(T+e)=\dim(T)+k$ for some $k\in\{-2,-1,0,1\}$. We show that for each such k, there exists a tree T and an edge e of $\bar{T}$ such that $\dim(T+e)=\dim(T)+k$.

(1) $k=-2$. Let T be the tree obtained by subdividing the central edge of a double star having two major vertices u and v. Let u_1 be an end-vertex adjacent to u and v_1 an end-vertex adjacent with v. Then $\dim(T+u_1v_1)=\dim(T)-2$.

(2) $k=-1$. Let T be the tree define in (1). Then $\dim(T+uv_1)=\dim(T)-1$.

(3) $k=0$. Let T be a star containing end-vertices x and y. Then $\dim(T+xy)=\dim(T)$.

(4) $k=1$. Let T be a path containing nonadjacent vertices w and z. Then $\dim(T+wz)=\dim(T)+1$.

When an edge is added to a tree the result is of course a unicyclic graph G (a connected graph containing exactly one cycle). Using different parameters and a different approach, Poisson and Zhang [3] established upper and lower bounds for dim(G) and showed that a range of four values is possible, which is consistent with Theorem 6.

5. The dimension of the product $H\times K_2$

In this section we show, for each connected graph H, that the dimension of the cartesian product $H\times K_2$ is either $\dim(H)$ or $\dim(H)+1$; where the cartesian product of two graphs G and H is defined as the graph with vertex set $V(G)\times V(H)$ and edge set $\{(u,v)(x,y)\,|\,u=x \text{ and } vy\in E(H) \text{ or } v=y \text{ and } ux\in E(G)\}$.

Theorem 7. *For every connected graph H,*

$$\dim(H)\leqslant\dim(H\times K_2)\leqslant\dim(H)+1.$$

Proof. First, we establish the upper bound. Let $G=H\times K_2$ where H_1 and H_2 are the two copies of H in G. Let W be a basis for H and let $W_1=\{w_1,w_2,\ldots,w_k\}$ and $W_2=\{u_1,u_2,\ldots,u_k\}$ be the bases of H_1 and H_2, respectively, corresponding to W. We claim that $U=W_1\cup\{u_1\}$ is a resolving set for G.

Let x and y be vertices of G such that $r(x\,|\,U)=r(y\,|\,U)$. We show that $x=y$. This is certainly the case if x or y belongs to U. Thus we may assume that $x,y\notin U$. We consider three cases.

Case 1: *Both x and y belong to H_1.* Then $d_G(x,w_i)=d_{H_1}(x,w_i)$ and $d_G(y,w_i)=d_{H_1}(y,w_i)$ for $i=1,2,\ldots,k$. Suppose that $x\neq y$. Since W is a resolving set for H_1, it follows that $d_{H_1}(x,w_i) \neq d_{H_1}(y,w_i)$ for some i $(1 \leqslant i \leqslant k)$. Hence, $d_G(x,w_i) \neq d_G(y,w_i)$, which contradicts the fact that $r(x \mid U) = r(y \mid U)$. Therefore, $x = y$.

Case 2: *Either $x \in V(H_1)$ and $y \in V(H_2)$, or $x \in V(H_2)$ and $y \in V(H_1)$, say the former*. In this case, $d_G(x,u_1) = d_G(x,w_1) + 1$ and $d_G(y,u_1) = d_G(y,w_1) - 1$. Thus, either $d_G(x,u_1) \neq d_G(y,u_1)$ or $d_G(x,w_1) \neq d_G(y,w_1)$, contradicting the fact that $r(x \mid U) = r(y \mid U)$. Thus $x = y$.

Case 3: *Both x and y belong to H_2*. First, suppose that at least one of x and y belongs to W, say $x \in W$. Then $x = u_i$ for some i $(1 \leqslant i \leqslant k)$, so $d_G(x,w_i) = 1$. Since $r(x \mid U) = r(y \mid U)$, it follows that $d_G(y,w_i) = 1$. However, the only vertex in H_2 adjacent to w_i is u_i; so $y = u_i = x$.

For $x, y \notin U$, let x' and y' be the vertices in H_1 corresponding to x and y, respectively. As in Case 1, if $x' \neq y'$, then $d_G(x',w_i) \neq d_G(y',w_i)$ for some i $(1 \leqslant i \leqslant k)$. Now, $d_G(x,w_i) = d_G(x',w_i) + 1$ and $d_G(y,w_i) = d_G(y',w_i) + 1$; so $d_G(x',w_i) = d_G(y',w_i)$, which implies that $x' = y'$ and, consequently, that $x = y$.

Now, we establish the lower bound stated in the theorem. Again, let $G = H \times K_2$, where H_1 and H_2 are the two copies of H in G. Let V_i be the vertex set of H_i for $i = 1,2$. Thus $V(G) = V_1 \cup V_2$. Let W be a basis for G, with $W_1 = W \cap V_1$ and $W_2 = W \cap V_2$. Let $U_1 \subseteq V(H_1)$ be the union of W_1 and the set W_2' consisting of those vertices of V_1 corresponding to W_2. Thus,

$$|U_1| = |W_1 \cup W_2'| \leqslant |W_1| + |W_2'| = |W|.$$

We claim that U_1 is a resolving set for H_1.

Let u and v be distinct vertices of H_1. We show that $r(u \mid U_1) \neq r(v \mid U_1)$. If either u or v belongs to W_1, then this is certainly the case. Otherwise, there exists a vertex $w \in W$ such that $d_G(u,w) \neq d_G(v,w)$. If $w \in V_1$, then $d_{H_1}(u,w) = d_G(u,w) \neq d_G(v,w) = d_{H_1}(v,w)$. If $w \in V_2$, then let w' be the vertex of U_1 corresponding to w. So $d_{H_1}(u,w') = d_G(u,w) - 1 \neq d_G(v,w) - 1 = d_{H_1}(v,w')$. In either case, $r(u \mid U_1) \neq r(v \mid U_1)$. □

It is possible to have equality in either bound in Theorem 7. For example, if $H = K_3$, then $\dim(H) = \dim(H \times K_2) = 2$; while if $H = C_4$, then $\dim(H) = 2$ and $\dim(H \times K_2) = 3$. Referring to this last example we see that a simple inductive argument yields $\dim(Q_n) \leqslant n$ for $n \geqslant 2$. We know, in fact, that $\dim(Q_n) = n$ for $2 \leqslant n \leqslant 4$.

Of course, H is an induced subgraph of $H \times K_2$, and Theorem 7 states that

$$\frac{\dim(H \times K_2)}{\dim(H)} \geqslant 1.$$

By choosing the integer m to be arbitrarily large and by letting $G = K_{1,m}$ and $H = K_2$, we see that we can make the ratio $\dim(G)/\dim(H)$ as large as we wish, where H is an induced subgraph of G. Although this may not be surprising, it may be unexpected that, in fact we can make $\dim(G)/\dim(H)$ as small as we wish, where H is an induced subgraph of G. We now verify the truth of this statement.

Let $n \geqslant 3$ be an integer. Label the vertices of the star $K_{1,2^{n+1}}$ with $v_0, v_1, v_2, \ldots, v_{2^n}, v_1', v_2', \ldots, v_{2^n}'$, where v_0 is the central vertex. Then we add two new vertices x and x' and

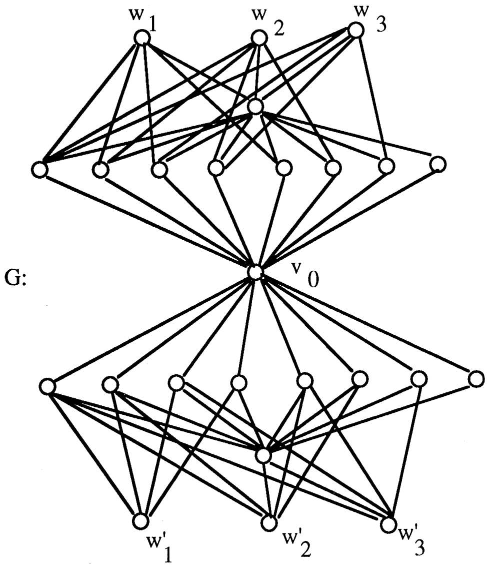

Fig. 9.

the 2^{n+1} edges xv_i and $x'v_i'$ for $1 \leqslant i \leqslant 2^n$. Next we add two sets $W = \{w_1, w_2, \ldots, w_n\}$ and $W' = \{w_1', w_2', \ldots, w_n'\}$ of vertices, together with the edges $w_i x$ and $w_i' x'$ for $1 \leqslant i \leqslant n$. Finally we add edges between W and $\{v_1, v_2, \ldots, v_{2^n}\}$ so that each of the 2^n possible n-tuples of 1's and 2's appears exactly once as $r(v_i \mid W)$ for $1 \leqslant i \leqslant 2^n$. Similarly, edges are added between W' and $\{v_1', v_2', \ldots, v_{2^n}'\}$ so that the representations $r(v_i' \mid W')$ are distinct for $1 \leqslant i \leqslant 2^n$. Denote the resulting graph by G. The graph G for $n = 3$ is shown in Fig. 9.

We claim that $W \cup W'$ is a resolving set for G. By construction, $r(v_i \mid W \cup W') = r(v_j \mid W \cup W')$ implies that $i = j$, and $r(v_i' \mid W \cup W') = r(v_j' \mid W \cup W')$ implies that $i = j$. Observe that

$$
\begin{aligned}
&r(x \mid W \cup W') = (1, 1, \ldots, 1, 4, 4, \ldots, 4),\\
&r(v_i \mid W \cup W') = (-, -, \ldots, -, 3, 3, \ldots, 3), \quad 1 \leqslant i \leqslant 2^n,\\
&r(v_0 \mid W \cup W') = (2, 2, \ldots, 2, 2, 2, \ldots, 2),\\
&r(v_i' \mid W \cup W') = (3, 3, \ldots, 3, -, -, \ldots, -), \quad 1 \leqslant i \leqslant 2^n,\\
&r(x' \mid W \cup W') = (4, 4, \ldots, 4, 1, 1, \ldots, 1).
\end{aligned}
$$

Thus, $W \cup W'$ is a resolving set of cardinality $2n$ for G. Since G contains $H = K_{1,2^{n+1}}$ as an induced subgraph, it follows that

$$\frac{\dim(G)}{\dim(H)} \leqslant \frac{2n}{2^{n+1} - 1}.$$

Also, because

$$\lim_{n \to \infty} \frac{2n}{2^{n+1} - 1} = 0$$

there exists a graph G and an induced subgraph H of G such that $\dim(G)/\dim(H)$ is arbitrarily small. We summarize this below.

Theorem 8. *For every* $\varepsilon > 0$, *there exists a connected graph G and a connected induced subgraph H of G such that* $\dim(G)/\dim(H) < \varepsilon$.

References

[1] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, New York, 1979.
[2] F. Harary, R.A. Melter, On the metric dimension of a graph, Ars Combin. 2 (1976) 191–195.
[3] C. Poisson, P. Zhang, The dimension of unicyclic graphs, J. Combin. Math. Combin. Comput., accepted.
[4] P.J. Slater, Leaves of trees, Congr. Numer. 14 (1975) 549–559.
[5] P.J. Slater, Dominating and reference sets in a graph, J. Math. Phys. Sci. 22 (1988) 445–455.