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Equilibrium points, stability and numerical solutions of fractional-order predator–prey and rabies models

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Abstract

In this paper we are concerned with the fractional-order predator–prey model and the fractional-order rabies model. Existence and uniqueness of solutions are proved. The stability of equilibrium points are studied. Numerical solutions of these models are given. An example is given where the equilibrium point is a centre for the integer order system but locally asymptotically stable for its fractional-order counterpart.

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1. Introduction

Biology is a rich source for mathematical ideas. We argue that fractional-order differential equations [18] are, at least, as stable as their integer order counterpart. The relation between memory and fractional mathematics is pointed out. Then in Sections 2–5 sufficient conditions for the local asymptotic stability of some biologically inspired, fractional non-autonomous equations are derived. An example is given whose internal solution is a centre, in the case of integer order, and numerical solutions indicate that it is stable in the case of fractional order.

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Definition 1. A complex adaptive system consists of inhomogeneous, interacting adaptive agents.

Definition 2. An emergent property of a CAS is a property of the system as a whole which does not exist at the individual elements (agents) level. Typical examples are the brain, the immune system, the economy, social systems, ecology, insects swarm, etc.

Therefore to understand a complex system one has to study the system as a whole and not to decompose it into its constituents. This totalistic approach is against the standard reductionist one, which tries to decompose any system to its constituents and hopes that by understanding the elements one can understand the whole system.

Recently [19] it has become apparent that fractional equations solve some of the above mentioned problems for the PDE approach. To see this consider the following evolution equation:

$$df/dt = -\lambda^2 \int_0^t k(t - t') f(t') dt'.$$

If the system has no memory then $k(t - t') = \delta(t - t')$. If the system has an ideal memory then

$$k(t - t') = \begin{cases} 1 & \text{if } t > t', \\ 0 & \text{if } t' > t. \end{cases}$$

Using Laplace transform, $L[f] = 1$ if there is no memory and $L[f] = 1/s$ for perfect memory hence the case of non-ideal memory is expected to be given by $L[f] = 1/s^\alpha$, $0 < \alpha < 1$. In this case the above system becomes

$$df/dt = -\lambda^2 \int_0^t (t - t')^{\alpha-1} f(t') dt',$$

$$f(t) = f_0 E_{\alpha+1}(-\lambda^2 t^{\alpha+1}),$$

where $E_\alpha(z) = \sum_{k=0}^\infty z^k / \Gamma(\alpha k + 1)$ is Mittag-Leffler function.

It is also argued that there is a relevance between fractals and fractional differentiation [19]. Moreover since it is known that systems with memory are typically more stable than their memoryless counterpart we expect the following conclusion:

“Fractional order differential equations are, at least, as stable as their integer order counterpart.”

In the following sections both analytical and numerical results will be given which supports this conclusion.

Now we give the definition of fractional-order integration and fractional-order differentiation:

Definition 3. The fractional integral of order $\beta \in R^+$ of the function $f(t)$, $t > 0$, is defined by

$$I^\beta f(t) = \int_0^t \frac{(t - s)^{\beta-1}}{\Gamma(\beta)} f(s) ds \tag{1}$$

and the fractional derivative of order $\alpha \in (n - 1, n)$ of $f(t)$, $t > 0$, is defined by

$$D_*^\alpha f(t) = I^{n-\alpha} D^n f(t), \quad D_* = \frac{d}{dt}. \tag{2}$$

The following properties are some of the main ones of the fractional derivatives and integrals (see [9–15,17]).

Let $\beta, \gamma \in R^+$ and $\alpha \in (0, 1)$. Then

- (i) $I_a^\beta : L^1 \rightarrow L^1$, and if $f(x) \in L^1$, then $I_a^\gamma I_a^\beta f(x) = I_a^{\gamma+\beta} f(x)$.
- (ii) $\lim_{\beta \rightarrow n} I_a^\beta f(x) = I_a^n f(x)$ uniformly on $[a, b]$, $n = 1, 2, 3, \dots$, where $I_a^1 f(x) = \int_a^x f(s) ds$.
- (iii) $\lim_{\beta \rightarrow 0} I_a^\beta f(x) = f(x)$ weakly.
- (iv) If $f(x)$ is absolutely continuous on $[a, b]$, then $\lim_{\alpha \rightarrow 1} D_*^\alpha f(x) = \frac{df(x)}{dx}$.
- (v) If $f(x) = k \neq 0$, k is a constant, then $D_*^\alpha k = 0$.

The following lemma can be easily proved (see [14]).

Lemma 1. *Let $\beta \in (0, 1)$ if $f \in C[0, T]$, then $I^\beta f(t)|_{t=0} = 0$.*

2. Existence and uniqueness

Consider the fractional-order Lotka–Volterra predator–prey system

$$D_*^\alpha x_1(t) = x_1(t)(r - ax_1(t) - bx_2(t)), \quad t \in (0, T], \tag{3}$$

$$D_*^\alpha x_2(t) = x_2(t)(-d + cx_1(t)), \quad t \in (0, T], \tag{4}$$

with the initial values

$$x_1(t)|_{t=0} = x_1(0) \quad \text{and} \quad x_2(t)|_{t=0} = x_2(0), \tag{5}$$

where $0 < \alpha \leq 1$, $x_1 \geq 0, x_2 \geq 0$ are prey and predator densities, respectively, and all constants r, a, b, c and d are positive.

Lemma 2. *The initial value problem (3)–(5) can be written in the form*

$$D_*^\alpha X(t) = A_1 X(t) - x_1(t)A_2 X(t), \quad t \in (0, T] \text{ and } X(0) = X_0, \tag{6}$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A_1 = \begin{bmatrix} r & 0 \\ 0 & -d \end{bmatrix}, \quad A_2 = \begin{bmatrix} a & b \\ 0 & -c \end{bmatrix}, \quad \text{and} \quad X_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

Definition 4. Let $C^*[0, T]$ be the class of continuous column vector $X(t)$ whose components $x_1, x_2 \in C[0, T]$, the class of continuous functions on the interval $[0, T]$. The norm of $X \in C^*[0, T]$ is given by

$$\|X\| = \sum_{i=1}^2 \sup_t |x_i(t)|.$$

Definition 5. By a solution of the initial value problem (6) we mean a column vector $X \in C^*[0, T]$. This vector satisfies the system (6).

Now we have the following existence theorem.

Theorem 1. *The initial value problem (6) has a unique solution.*

Proof. The proof follows from Theorems 2.1 and 2.2 of [6]. \square

3. Equilibrium points and their asymptotic stability

Let $\alpha \in (0, 1]$ and consider the system

$$\begin{aligned} D_*^\alpha x_1(t) &= f_1(x_1, x_2), \\ D_*^\alpha x_2(t) &= f_2(x_1, x_2), \end{aligned} \tag{7}$$

with the initial values

$$x_1(0) = x_{01} \quad \text{and} \quad x_2(0) = x_{02}. \tag{8}$$

To evaluate the equilibrium points, let

$$D_*^\alpha x_i(t) = 0 \quad \Rightarrow \quad f_i(x_1^{\text{eq}}, x_2^{\text{eq}}) = 0, \quad i = 1, 2,$$

from which we can get the equilibrium points $x_1^{\text{eq}}, x_2^{\text{eq}}$.

To evaluate the asymptotic stability, let

$$x_i(t) = x_i^{\text{eq}} + \varepsilon_i(t),$$

then

$$D_*^\alpha (x_i^{\text{eq}} + \varepsilon_i) = f_i(x_1^{\text{eq}} + \varepsilon_1, x_2^{\text{eq}} + \varepsilon_2)$$

which implies that

$$D_*^\alpha \varepsilon_i(t) = f_i(x_1^{\text{eq}} + \varepsilon_1, x_2^{\text{eq}} + \varepsilon_2)$$

but

$$\begin{aligned} f_i(x_1^{\text{eq}} + \varepsilon_1, x_2^{\text{eq}} + \varepsilon_2) &\simeq f_i(x_1^{\text{eq}}, x_2^{\text{eq}}) + \left. \frac{\partial f_i}{\partial x_1} \right|_{\text{eq}} \varepsilon_1 + \left. \frac{\partial f_i}{\partial x_2} \right|_{\text{eq}} \varepsilon_2 + \dots \\ \Rightarrow f_i(x_1^{\text{eq}} + \varepsilon_1, x_2^{\text{eq}} + \varepsilon_2) &\simeq \left. \frac{\partial f_i}{\partial x_1} \right|_{\text{eq}} \varepsilon_1 + \left. \frac{\partial f_i}{\partial x_2} \right|_{\text{eq}} \varepsilon_2, \end{aligned}$$

where $f_i(x_1^{\text{eq}}, x_2^{\text{eq}}) = 0$, then

$$D_*^\alpha \varepsilon_i(t) \simeq \left. \frac{\partial f_i}{\partial x_1} \right|_{\text{eq}} \varepsilon_1 + \left. \frac{\partial f_i}{\partial x_2} \right|_{\text{eq}} \varepsilon_2$$

and we obtain the system

$$D_*^\alpha \varepsilon = A\varepsilon \tag{9}$$

with the initial values

$$\varepsilon_1(0) = x_1(0) - x_1^{\text{eq}} \quad \text{and} \quad \varepsilon_2(0) = x_2(0) - x_2^{\text{eq}}, \tag{10}$$

where

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and

$$a_{ij} = \left. \frac{\partial f_i}{\partial x_j} \right|_{\text{eq}}, \quad i, j = 1, 2.$$

We have

$$B^{-1}AB = C,$$

where C is a diagonal matrix of A given by

$$C = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

where λ_1 and λ_2 are the eigenvalues of A and B is the eigenvectors of A , then

$$AB = BC, \quad A = BCB^{-1},$$

which implies that

$$D_*^\alpha \varepsilon = (BCB^{-1})\varepsilon, \quad D_*^\alpha (B^{-1}\varepsilon) = C(B^{-1}\varepsilon),$$

then

$$D_*^\alpha \eta = C\eta, \quad \eta = B^{-1}\varepsilon, \quad (11)$$

where

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix},$$

i.e.

$$D_*^\alpha \eta_1 = \lambda_1 \eta_1, \quad (12)$$

$$D_*^\alpha \eta_2 = \lambda_2 \eta_2, \quad (13)$$

the solutions of Eqs. (12)–(13) are given by Mittag-Leffler functions (see [10])

$$\eta_1(t) = \sum_{n=0}^{\infty} \frac{(\lambda_1)^n t^{n\alpha}}{\Gamma(n\alpha + 1)} \eta_1(0) = E_\alpha(\lambda_1 t^\alpha) \eta_1(0), \quad (14)$$

$$\eta_2(t) = \sum_{n=0}^{\infty} \frac{(\lambda_2)^n t^{n\alpha}}{\Gamma(n\alpha + 1)} \eta_2(0) = E_\alpha(\lambda_2 t^\alpha) \eta_2(0). \quad (15)$$

Using the result of Matignon [16] then if

$$|\arg(\lambda_1)| > \alpha\pi/2 \quad \text{and} \quad |\arg(\lambda_2)| > \alpha\pi/2$$

then $\eta_1(t)$, $\eta_2(t)$ are decreasing and then $\varepsilon_1(t)$, $\varepsilon_2(t)$ are decreasing.

So the equilibrium point $(x_1^{\text{eq}}, x_2^{\text{eq}})$ is locally asymptotically stable if both the eigenvalues of the matrix A are negative ($|\arg(\lambda_1)| > \alpha\pi/2$, $|\arg(\lambda_2)| > \alpha\pi/2$). This confirms our statement in Section 1 that fractional-order differential equations are, at least, as stable as their integer order counterpart.

4. Fractional-order Lotka–Volterra predator–prey model

Consider the fractional-order Lotka–Volterra predator–prey system

$$D_*^\alpha x_1(t) = x_1(t)(r - ax_1(t) - bx_2(t)), \tag{16}$$

$$D_*^\alpha x_2(t) = x_2(t)(-d + cx_1(t)). \tag{17}$$

To evaluate the equilibrium points, let

$$D_*^\alpha x_i(t) = 0, \quad i = 1, 2,$$

then $(x_1^{eq}, x_2^{eq}) = (0, 0), (\frac{r}{a}, 0), (\frac{d}{c}, \frac{cr-ad}{cb})$, are the equilibrium points.

For $(x_1^{eq}, x_2^{eq}) = (0, 0)$ we find that

$$A = \begin{bmatrix} r & 0 \\ 0 & -d \end{bmatrix},$$

its eigenvalues are

$$\lambda_1 = r > 0, \quad \lambda_2 = -d < 0.$$

Hence the equilibrium point $(x_1^{eq}, x_2^{eq}) = (0, 0)$ is unstable.

For $(x_1^{eq}, x_2^{eq}) = (\frac{r}{a}, 0)$ we find that

$$A = \begin{bmatrix} -r & -\frac{br}{a} \\ 0 & \frac{cr}{a} - d \end{bmatrix},$$

its eigenvalues are

$$\lambda_1 = -r < 0, \quad \lambda_2 = \frac{cr}{a} - d < 0 \quad \text{if } cr < ad.$$

Hence the equilibrium point $(x_1^{eq}, x_2^{eq}) = (\frac{r}{a}, 0)$ is locally asymptotically stable if $cr < ad$.

For $(x_1^{eq}, x_2^{eq}) = (\frac{d}{c}, \frac{cr-ad}{cb})$ we find that

$$A = \begin{bmatrix} -\frac{ad}{c} & -\frac{bd}{c} \\ \frac{cr-ad}{b} & 0 \end{bmatrix},$$

its eigenvalues are

$$\lambda_1 = \frac{-ad + \sqrt{a^2d^2 - 4cd(cr - ad)}}{2c},$$

$$\lambda_2 = \frac{-ad - \sqrt{a^2d^2 - 4cd(cr - ad)}}{2c}.$$

A sufficient condition for the local asymptotic stability of the equilibrium point $(x_1^{eq}, x_2^{eq}) = (\frac{d}{c}, \frac{cr-ad}{cb})$ is $|\arg(\lambda_1)| > \alpha\pi/2, |\arg(\lambda_2)| > \alpha\pi/2$.

In the special case $a = 0$ it is known that the internal equilibrium point is a centre ($\arg(\lambda_1) = \pi/2, \arg(\lambda_2) = -\pi/2$) for the integer order system ($\alpha = 1$). In the fractional case $0 < \alpha < 1$ the internal equilibrium point is locally asymptotically stable. The numerical simulations in the next section will support this result.

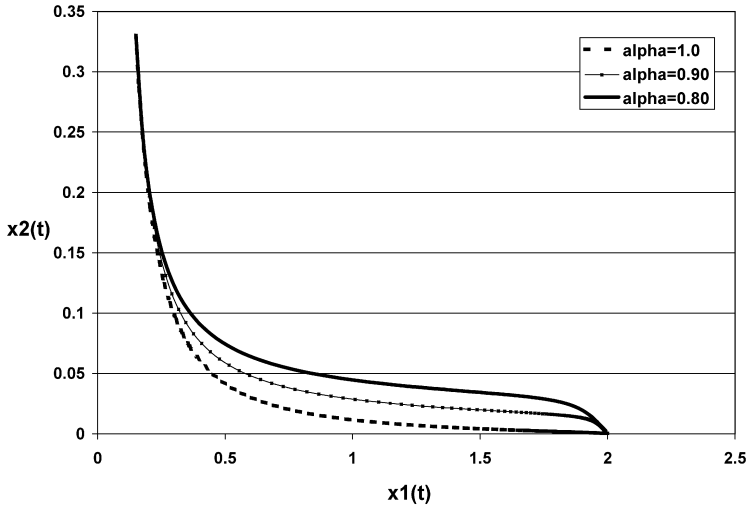


Fig. 1.

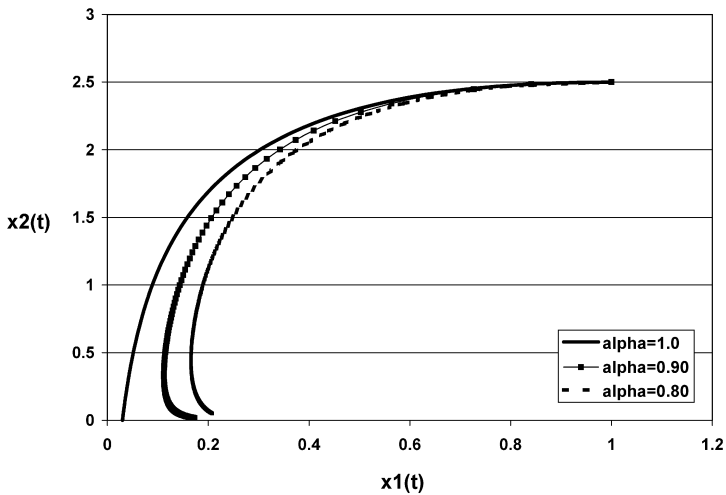


Fig. 2.

5. Fractional-order rabies model

In Eqs. (16) and (17) when we take $r = 0, a = 0, c = b$, we obtain the fractional-order rabies model

$$D_*^\alpha x_1(t) = -bx_1x_2, \tag{18}$$

$$D_*^\alpha x_2(t) = bx_1x_2 - dx_2, \tag{19}$$

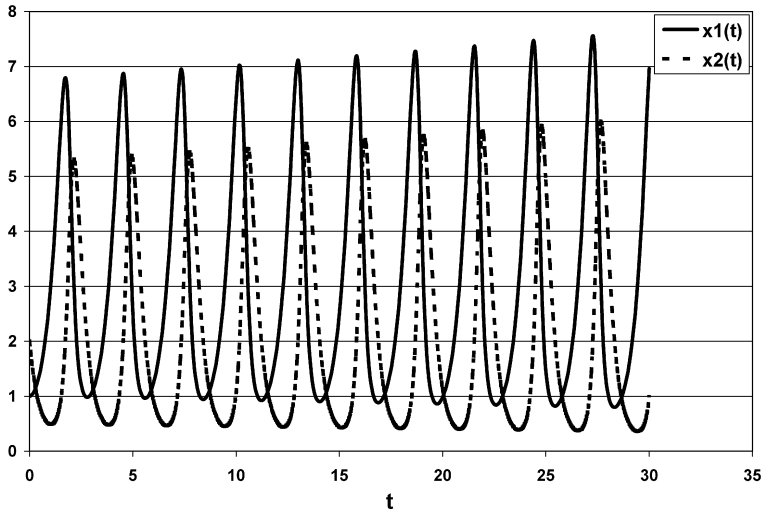


Fig. 3.

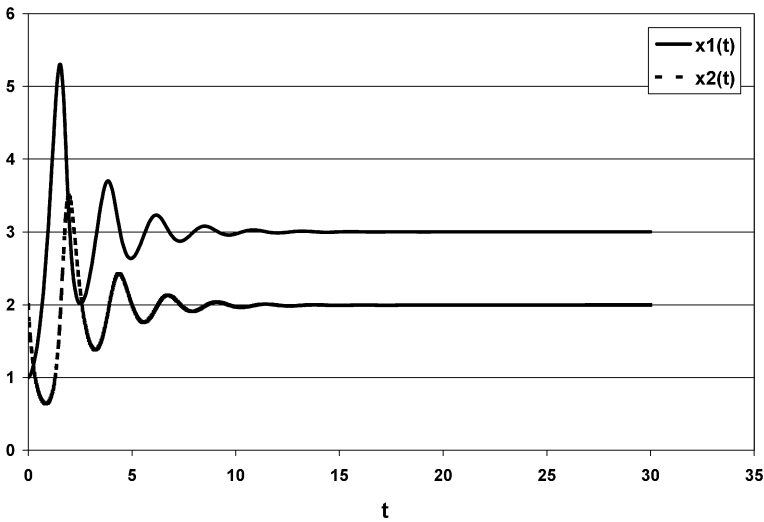


Fig. 4.

where $0 < \alpha \leq 1$, $x_1 \geq 0$, $x_2 \geq 0$ are healthy and infected foxes, respectively, and all constants b, d are positive and $(x_1^{\text{eq}}, x_2^{\text{eq}}) = (0, 0)$, $(\frac{d}{b}, 0)$ are the equilibrium points. For $(x_1^{\text{eq}}, x_2^{\text{eq}}) = (0, 0)$ we find that

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -d \end{bmatrix},$$

its eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = -d.$$

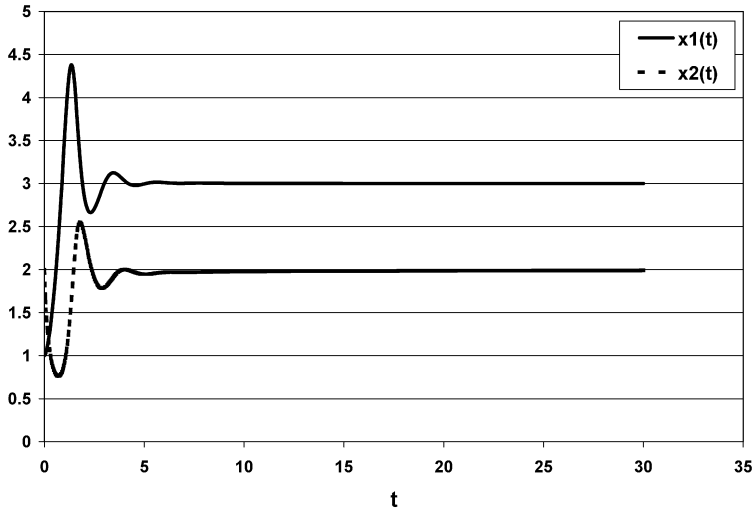


Fig. 5.

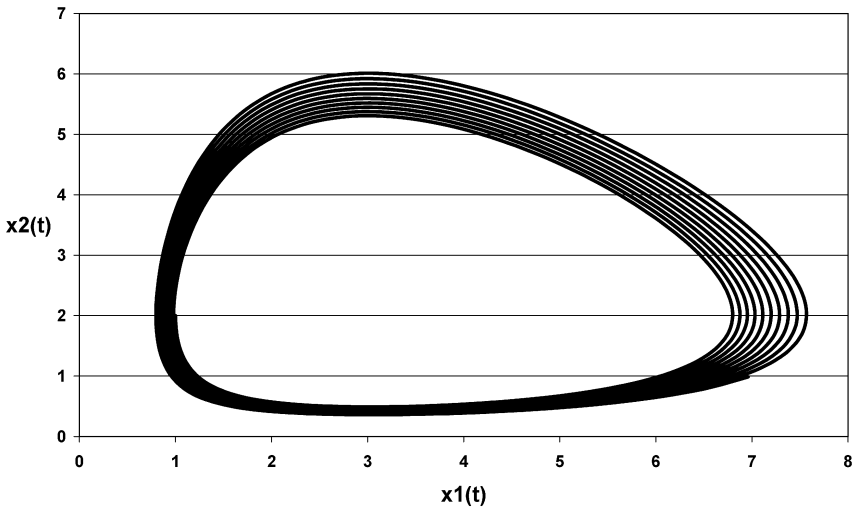


Fig. 6.

For $(x_1^{eq}, x_2^{eq}) = (\frac{d}{b}, 0)$ we find that

$$A = \begin{bmatrix} 0 & -d \\ 0 & 0 \end{bmatrix},$$

its eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = 0.$$

Hence the equilibrium is a centre.

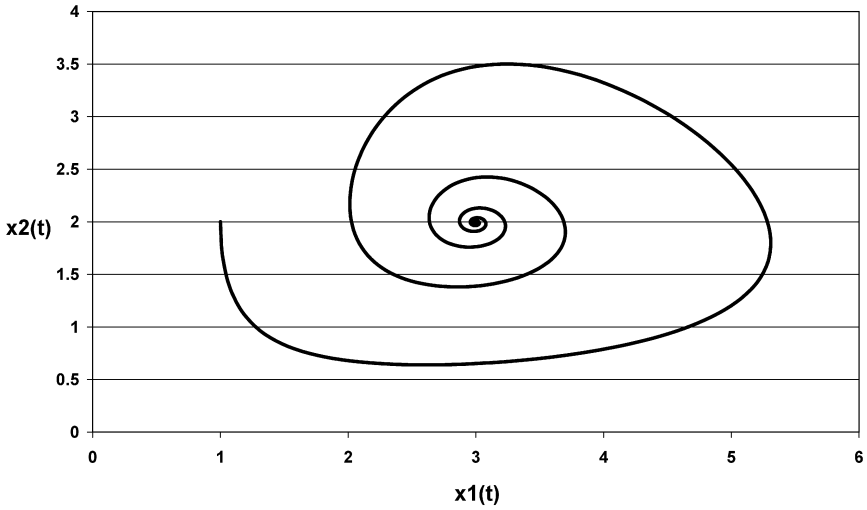


Fig. 7.

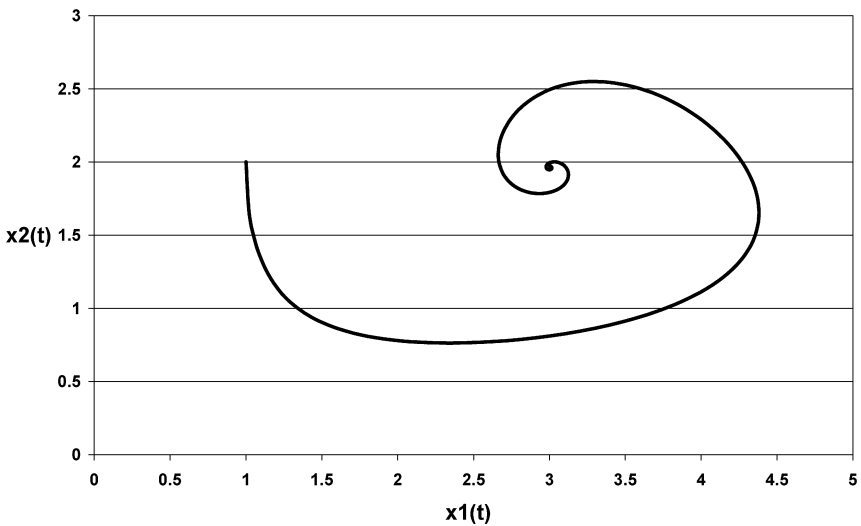


Fig. 8.

6. Numerical methods and results

An Adams-type predictor–corrector method has been introduced in [2,3] and investigated further in [1,4–9,13]. In this paper we use an Adams-type predictor–corrector method for the numerical solution of fractional integral equation.

The key to the derivation of the method is to replace the original problem (3)–(5) by an equivalent fractional integral equation

$$X(t) = X_0 + I^\alpha (A_1 X(t) - x_1(t) A_2 X(t)) \tag{20}$$

and then apply the *PECE* (Predict, Evaluate, Correct, Evaluate) method.

The approximate solutions displayed in Figs. 1–8 for the step size 0.05 and different $0 < \alpha \leq 1$. In Fig. 1 we take $b = 1$, $c = 1$, $r = 2$, $d = 3$, $a = 1$, $x_1(0) = 0.15$ and $x_2(0) = 0.33$. In Fig. 2 we take $r = 0$, $a = 0$, $b = 1$, $d = 1$, $x_1(0) = 1$ and $x_2(0) = 2.5$. In Figs. 3–8 we take $a = 0$, $b = 1$, $c = 1$, $r = 2$, $d = 3$, $x_1(0) = 1$ and $x_2(0) = 2$. In Figs. 3 and 6 we take $\alpha = 1$. In Figs. 4 and 7 we take $\alpha = 0.9$. In Figs. 5 and 8 we take $\alpha = 0.8$.

7. Conclusion

Existence and uniqueness of solutions of fractional order systems have been studied. We have argued that fractional-order differential equations are, at least, as stable as their integer order counterpart. We studied equilibrium points, existence, uniqueness, stability, numerical solution, of Lotka–Volterra predator–prey system and used numerical solutions to show that although the internal solution for integer order case is only a centre, it is stable for its fractional-order counterpart.

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