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Hamiltonian iterated line graphs

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Abstract

The n-iterated line graph of a graph G is $L^n(G) = L(L^{n-1}(G))$, where $L^1(G)$ denotes the line graph $L(G)$ of G, and $L^{n-1}(G)$ is assumed to be nonempty. Harary and Nash-Williams characterized those graphs G for which $L(G)$ is hamiltonian. In this paper, we will give a characterization of those graphs G for which $L^n(G)$ is hamiltonian, for each $n \geqslant 2$. This is not a simple consequence of Harary and Nash-Williams' result. As an application, we show two methods for determining the hamiltonian index of a graph and enhance various results on the hamiltonian index known earlier.

MSC: 05C45; 05C35

Keywords: Iterated line graph; Hamiltonian index; Split block; Contraction of graphs; Complexity

1. Introduction

The graphs considered in this paper are finite undirected graphs and are allowed to have multiple edges but no loops. We follow the notation of Bondy and Murty [3], unless otherwise stated.

All results in this paper are related to the well-studied concept of the line graph operation on graphs. The *line graph* $L(G)$ of a graph G has $E(G)$ as its vertex set and two vertices are adjacent in $L(G)$ if and only if they are adjacent as edges in G.

Harary and Nash-Williams characterized those graphs G for which $L(G)$ is hamiltonian.

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Theorem 1 (Harary and Nash-Williams [11]). *Let G be a connected graph with at least three edges. Then $L(G)$ is hamiltonian if and only if G has a closed trail T such that each edge of G is incident with at least one vertex of T.*

It follows from Theorem 1 that the line graph of a hamiltonian graph is hamiltonian, while the converse is not true in general. The following corollary is also immediate.

Corollary 2. *Let G be a graph with at least 3 edges. If G has a spanning closed trail, then $L(G)$ is hamiltonian.*

Theorem 1 has been used by many authors to investigate the cyclic properties of line graphs. In fact, the paper [11] in which they presented Theorem 1, has been cited from the year 1995 to 1998 in at least 12 published papers that are covered by the CompuMath Citation Index. If one thinks about Theorem 1, Corollary 2, and the line graph operation more carefully, it becomes natural to believe that for most graphs, after applying the line graph operation iteratively a finite number of times, the resulting graph will become hamiltonian. Two natural questions then can be raised.

(1) For which graphs is this indeed the case?
(2) If this is the case for a graph G, what is the smallest number of iterations that will yield a hamiltonian graph?

In order to investigate this kind of questions, Chartrand [8] considered the n-iterated line graph $L^n(G)$ of G and introduced the *hamiltonian index of a graph,* denoted by $h(G)$, i.e., the minimum number n such that $L^n(G)$ is hamiltonian. Here the *n-iterated line graph* $L^n(G)$ of a graph is defined to be $L(L^{n-1}(G))$, where $L^1(G)$ denotes the line graph $L(G)$ of G, and $L^{n-1}(G)$ is assumed to have a nonempty edge set. In fact, he also gave another proof of Theorem 1. He showed that for any graph G other than a path, the hamiltonian index of G exists. With the aid of Theorem 1, Chartrand and Wall [9] determined the hamiltonian index of a tree other than a path, and showed that if G is connected and has a cycle of length l, then $h(G)\leqslant|V(G)|-l$. They also showed that $h(G)\leqslant 2$ for any connected graph G with minimum degree $\delta(G)\geqslant 3$. Kapoor and Stewart [12] determined $h(G)$ for a graph G that is homeomorphic to $K_{2,n}$, for $n\geqslant 3$.

Catlin [6] developed a reduction method to investigate *supereulerian graphs*, i.e., graphs that have a spanning closed trail. For a connected subgraph H of G, let G/H denote the graph obtained from G by contracting H to a single vertex and deleting any resulting loops. A graph H is called *collapsible* if for every even subset $S\subseteq V(H)$, there is a subgraph T of H such that $H-E(T)$ is connected and the set of odd degree vertices of T is S.

Theorem 3 (Catlin [6]). *Let H be a collapsible subgraph of G. Then G is supereulerian if and only if G/H is supereulerian.*

After Catlin introduced this reduction method, many results about hamiltonian line graphs have been derived; for surveys see [5,10]. Zhan [19] used Catlin's method and

Theorem 1 to prove that every 7-connected line graph is hamiltonian. An interesting conjecture related to this result, that was posed by Thomassen [16], is still open and reads as follows: Every 4-connected line graph is hamiltonian.

Catlin's reduction method was also used to investigate the hamiltonian index of a graph. Lai [13] and Catlin et al. [7] used Catlin's method to give some upper bounds on $h(G)$ that are related to so-called *branches*; we will come back to this later. Saražin [15] used Catlin's method to show that the hamiltonian index of a simple graph G other than a path, is at most $|V(G)| - \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G.

Theorem 1 is a good tool for investigating cyclic properties of line graphs. However, when one uses it to investigate the (hamiltonian) cycles in the n-iterated line graph of a graph, closed trails in its $(n-1)$-iterated line graph should be considered. Since it is not convenient to examine $(n-1)$-iterated line graphs when $n \geqslant 2$, this leads to a natural question: for any integer $n \geqslant 2$, does there exist a characterization of those graphs G for which $L^n(G)$ is hamiltonian? This was also mentioned in [4]. The answer is affirmative. We will give such a characterization in Section 3. As its application, in Section 4 we will examine the hamiltonian index of a graph and give two methods for determining it. One of them resembles Catlin's reduction method. We also present some new upper bounds on the hamiltonian index in Section 5. Our results enhance various results on the hamiltonian index known earlier.

2. More terminology and notation

Throughout the paper we will use the following notation and terminology. The multigraph of order 2 with two edges will be called 2-*cycle* and denoted by C_2. Let H be a subgraph of a graph $G=(V,E)$. Then $V(H)$ and $E(H)$ denote the sets of vertices and edges of H, respectively, and $\bar{E}(H)$ denotes the set of all edges of G that are incident with vertices of H. If $u \in V(H)$, then $E_H(u)$ denotes the set of all edges of H that are incident with u, and $d_H(u)=|E_H(u)|$ is the *degree* of u in H. A graph H is called a *circuit* if it is connected and every vertex has an even degree. Note that by this definition (the trivial subgraph induced by) a single vertex is also a circuit.

Define $V_i(H)=\{v \in V(H):\ d_H(v)=i\}$ and $W(H)=V(H) \setminus V_2(H)$. A *branch* in G is a nontrivial path with ends in $W(G)$ and with internal vertices, if any, that have degree 2 (and thus are not in $W(G)$). We denote by $B(G)$ the set of branches of G. Define $B_1(G)=\{b \in B(G):\ V(b) \cap V_1(G) \neq \emptyset\}$.

The *distance* $d_H(G_1,G_2)$ between two subgraphs G_1 and G_2 of H is defined to be $\min\{d_H(v_1,v_2):\ v_1 \in V(G_1)$ and $v_2 \in V(G_2)\}$, where $d_H(v_1,v_2)$ denotes the number of edges of a shortest path between v_1 and v_2 in H.

Finally, $EU_k(G)$ denotes the set of those subgraphs H of a graph G that satisfy the following conditions:

(I) $d_H(x) \equiv 0 \pmod 2$ for every $x \in V(H)$;

(II) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

(III) $d_G(H_1, H-H_1) \leqslant k-1$ for every subgraph H_1 of H;

(IV) $|E(b)| \leqslant k+1$ for every branch $b \in B(G)$ with $E(b) \cap E(H) = \emptyset$;
(V) $|E(b)| \leqslant k$ for every branch $b \in B_1(G)$.

$EU_n(G)$ will play an important role in our main result, which is Theorem 6.

3. Characterization of graphs with iterated line graphs that are hamiltonian

Our aim in this section is to give a characterization of graphs with iterated line graphs that are hamiltonian. Our main result, Theorem 6, is a direct consequence of Theorems 4 and 5.

We start with a close relationship between $EU_k(L(G))$ and $EU_{k+1}(G)$, the proof of which will be postponed.

Theorem 4. *Let G be a connected graph and $k \geqslant 1$ be an integer. Then $EU_k(L(G)) \neq \emptyset$ if and only if $EU_{k+1}(G) \neq \emptyset$.*

We will use Theorem 1 to characterize graphs with 2-iterated line graphs that are hamiltonian. The proof of this will also be postponed.

Theorem 5. *Let G be a connected graph with at least three edges. Then $L^2(G)$ is hamiltonian if and only if $EU_2(G) \neq \emptyset$.*

Using Theorems 4 and 5, one easily derives the following main result by induction.

Theorem 6. *Let G be a connected graph with at least three edges and $n \geqslant 2$. Then $L^n(G)$ is hamiltonian if and only if $EU_n(G) \neq \emptyset$.*

Comparing Theorem 1 with Theorem 6, one might think that $L(G)$ is hamiltonian if and only if $EU_1(G)$ is nonempty. Unfortunately, this is not true because every subgraph in $EU_1(G)$ should satisfy (II). For example, Fig. 1 shows that w is a vertex of degree 4 but does not belong to the unique circuit $C = G_0 - w$ such that $\bar{E}(C) = E(G_0)$. Hence $EU_1(G_0)$ is empty, but $L(G_0)$ is hamiltonian, by Theorem 1. The following theorem is a consequence of Theorem 6.

Theorem 7. *For $n \geqslant 2$, $L^n(G)$ is hamiltonian if and only if there exists exactly one component G_1 of G such that $EU_n(G_1) \neq \emptyset$, and any other component of G is a path of length at most $n-1$.*

In order to prove Theorems 4 and 5, we first present some auxiliary results. We omit the proof of the following lemma since it is a slight modification of the proof of Theorem 1 [11]. We first introduce a notation related to Lemma 8. For any subgraph C of $L(G)$, by $S(G,C)$ we denote the collection of circuits H of G, such that $L(G[\bar{E}(H)])$ contains C, and C contains all elements of $E(H)$. Here and throughout, $G[S]$ denotes the subgraph of G induced by S, where $S \subseteq V(G)$ or $S \subseteq E(G)$.

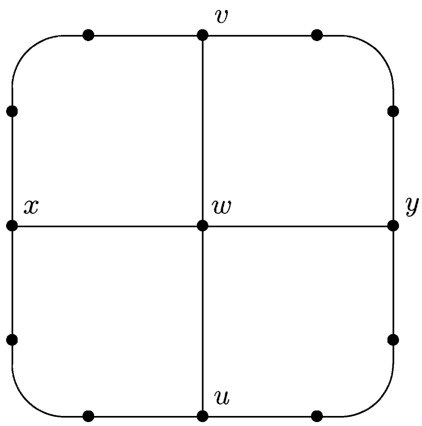

Fig. 1. A graph G_0 with $wu, wv, wx, wy \in E(G_0)$.

Lemma 8. A. *If C is a cycle of $L(G)$ with $|E(C)| \geqslant 3$, then $S(G,C)$ is nonempty.*
B. *If G has a circuit H such that $\bar{E}(H)$ has at least three edges, then $L(G)$ has a cycle C with $V(C) = \bar{E}(H)$.*

The following lemma is known.

Lemma 9 (Beineke [1]). *$K_{1,3}$ is not an induced subgraph of the line graph of any graph.*

Lemma 10. *Let $b = u_1u_2 \cdots u_s$ $(s \geqslant 3)$ be a path of G and $e_i = u_iu_{i+1}$. Then $b \in B(G)$ if and only if $b' = e_1e_2 \cdots e_{s-1} \in B(L(G))$.*

Proof. $b = u_1u_2 \cdots u_s = G[\{e_1, e_2, \ldots, e_{s-1}\}] \in B(G) \Leftrightarrow u_1, u_s \in W(G)$ and $d_G(u_i) = 2$ for $i \in \{2, 3, \ldots, s-1\} \Leftrightarrow e_1, e_{s-1} \in W(L(G))$ and $d_{L(G)}(e_i) = 2$ for $i \in \{2, 3, \ldots, s-2\}$ $\Leftrightarrow b' = e_1e_2 \cdots e_{s-1} \in B(L(G))$. □

Lemma 11. *Let H be a subgraph of G in $EU_k(G)$ with a minimum number of components. Then there exist no multiple edges in $\bar{E}(H_1) \cap \bar{E}(H_2)$ for any two components H_1 and H_2 of H.*

Proof. Otherwise there would exist two components H_1, H_2 of H and edges e_1, e_2 in $\bar{E}(H_1) \cap \bar{E}(H_2)$ with the same set of endvertices. One can easily check that $H' = H + \{e_1, e_2\} \in EU_k(G)$, which is a contradiction because H' contains fewer components than H. □

A *eulerian subgraph* of G is a circuit which contains at least one cycle of length at least 3.

Lemma 12. *Let G be a connected graph and C be a eulerian subgraph of the line graph $L(G)$. Then there exists a subgraph H of G with*

(1) $d_H(x) \equiv 0 \pmod 2$ *for every* $x \in V(H)$;
(2) $d_G(x) \geqslant 3$ *for every vertex* $x \in V(G)$ *with* $d_H(x) = 0$;
(3) *for any two components* H^0, H^{00} *of* H, *there exists a sequence of components* $H^0 = H_1, H_2, \ldots, H_s = H^{00}$ *of* H *such that* $d_G(H_i, H_{i+1}) \leqslant 1$ *for* $i \in \{1, 2, \ldots, s-1\}$;
(4) $L(G[\bar{E}(H)])$ *contains* C, *and* C *contains all elements of* $E(H)$.

Proof. Since C is a eulerian subgraph of $L(G)$ and $L(G)$ is a simple graph, we can let $C_1, C_2, \ldots, C_m$ be the edge-disjoint cycles with $C = \bigcup_{i=1}^m C_i$.

By Lemma 8A, we can find m subgraphs $F_1, F_2, \ldots, F_m$ of G such that $F_i \in S(G, C_i)$ for $i \in \{1, 2, \ldots, m\}$. Hence, there exist m_i edge-disjoint cycles $D_{i,1}, D_{i,2}, \ldots, D_{i,m_i}$ (possibly, for $m_i = 1$, $D_{i,1}$ might be a single vertex) such that $F_i = \bigcup_{j=1}^{m_i} D_{i,j}$. Define

$$H' = \bigcup_{i=1}^{m} \bigcup_{j=1}^{m_i} D_{i,j}.$$

For any $e \in E(H')$, let

$$r_{H'}(e) = \left| \left\{ C' \colon e \in E(C') \text{ and } C' \in \bigcup_{i=1}^{m} \bigcup_{j=1}^{m_i} \{D_{i,j}\} \right\} \right|.$$

We construct a subgraph H of G from H' as follows:

$$V(H) = V(H') \quad \text{and} \quad E(H) = E(H') \backslash \{e \in E(H') \colon r_{H'}(e) \equiv 0 \pmod 2\}.$$

Next we will prove that H satisfies (1) to (4).

For an $x \in V(D_{i,j})$, the cycle $D_{i,j}$ is counted exactly twice in $\sum_{e \in E_{H'}(x)} r_{H'}(e)$ which is therefore an even number. If we denote $E_i(x) = \{e \in E_{H'}(x) \colon r_{H'}(e) \equiv i \pmod 2\}$ $(i = 0, 1)$, then

$$\sum_{e \in E_{H'}(x)} r_{H'}(e) = \sum_{e \in E_0(x)} r_{H'}(e) + \sum_{e \in E_1(x)} r_{H'}(e)$$

and so $\sum_{e \in E_1(x)} r_{H'}(e)$ is even, which implies that $d_H(x) = |E_1(x)|$ is even. Thus (1) holds.

Obviously $d_G(w) \geqslant 2$ for all $w \in V(H)$. If there were a $w \in V(H)$ with $d_G(w) = 2$ and $d_H(w) = 0$, then we would have two cycles D', D'' in the set $\{D_{i,j}\}$ such that $e_1, e_2 \in E(D') \cap E(D'')$, where e_1, e_2 are the two edges incident to w. But then there would exist two cycles C_p, C_q having the edge $e_1 e_2$ in common in the line graph $L(G)$, contrary to the choice of the cycles C_i. This proves (2).

Since $H' = \bigcup_{i=1}^m F_i$ and $F_i \in S(G, C_i), L(G[\bar{E}(H')])$ contains $C = \bigcup_{i=1}^m C_i$ which contains all elements of $E(H')$. However, $V(H) = V(H')$ implies that $\bar{E}(H) = \bar{E}(H')$, hence $E(H) \subseteq V(C) \subseteq \bar{E}(H)$ and (4) holds.

Suppose that H has a subgraph H^* with $d_G(H^*, H - H^*) \geqslant 2$. Then $\bar{E}(H^*) \cap \bar{E}(H - H^*)$ would be empty and C disconnected. This contradiction shows that (3) is true for H, too, which completes the proof of Lemma 12. □

Now we can present the proofs of Theorems 4 and 5.

Proof of Theorem 4. Supposing that $EU_{k+1}(G) \neq \emptyset$, we choose an $H \in EU_{k+1}(G)$ with a minimum number of components which we denote by $C_1, \ldots, C_t$.

By Lemma 8B, we can find a cycle C_i' of $L(G)$ with $V(C_i') = \bar{E}(C_i)(i = 1, \ldots, t)$. Hence C_i' is a cycle of $L(G)$ with length at least 3 since $H \in EU_{k+1}(G)$. Let $H' = \bigcup_{i=1}^{t} C_i'$. We will prove that $H' \in EU_k(L(G))$.

Since $\bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$ and $V(H') = \bigcup_{i=1}^{t} \bar{E}(C_i)$,

$$\bigcup_{i=3}^{\Delta(L(G))} V_i(L(G)) \subseteq V(H').$$

Since $d_G(C_i, C_j) \geqslant 1$, by Lemma 11, $E(C_i') \cap E(C_j') = \emptyset$ for $\{i, j\} \subseteq \{1, 2, \ldots, t\}$ with $i \neq j$, which implies that H' satisfies (I).

Obviously H' contains no isolated vertex by definition of H, hence H' satisfies (II).

Take an arbitrary $T \subseteq \{1, \ldots, t\}$. By the choice of H, it follows that $d_G(\bigcup_{i \in T} C_i, H - \bigcup_{i \in T} C_i) \leqslant k$. Let $P = xu_1 \cdots u_s y$ be a shortest path from $\bigcup_{i \in T} C_i$ to $H - \bigcup_{i \in T} C_i$, where $x \in V(\bigcup_{i \in T} C_i)$, $y \in V(H - \bigcup_{i \in T} C_i)$ and $s \leqslant k - 1$. Evidently, $L(P)$ is a path from $\bigcup_{i \in T} C_i'$ to $H' - \bigcup_{i \in T} C_i'$ with length s, thus $d_{L(G)}(\bigcup_{i \in T} C_i', H' - \bigcup_{i \in T} C_i') \leqslant k - 1$, which implies that (III) holds for H'.

Since H satisfies (IV) and (V), using Lemma 10 one can easily check that H' satisfies (IV) and (V).

Conversely, suppose $EU_k(L(G)) \neq \emptyset$. Let H be a subgraph of $L(G)$ in $EU_k(L(G))$ with a minimum number of isolated vertices. Then H contains no isolated vertices. For, suppose $C_1 = \{e_0\}$ is an isolated vertex of H, then by (II), $d_{L(G)}(e_0) \geqslant 3$ and by Lemma 9, there exist $e_1, e_2 \in N_{L(G)}(e_0)$ such that $e_1 e_2 \in E(L(G))$. Now we construct a subgraph H_0 of $L(G)$ as follows.

$$H_0 = \begin{cases} H + \{e_0e_1, e_1e_2, e_2e_0\} & \text{if } e_1e_2 \notin E(H), \\ H + \{e_0e_1, e_0e_2\} - \{e_1e_2\} & \text{if } e_1e_2 \in E(H). \end{cases}$$

Obviously $H_0 \in EU_k(L(G))$ has fewer isolated vertices than H has, a contradiction.

Let $H_1, H_2, \ldots, H_m$ be the components of H. Since $H \in EU_k(L(G))$ and H contains no isolated vertices, H_i is a eulerian subgraph of $L(G)$ for $i \in \{1, 2, \ldots, m\}$. Hence for any $H_i (i \in \{1, 2, \ldots, m\})$, by Lemma 12, there exists a subgraph C_i of G satisfying (1) to (4). Set

$$C = \left(\bigcup_{i=3}^{\Delta(G)} V_i(G) \right) \cup \left(\bigcup_{i=1}^{m} C_i \right).$$

We will prove that $C \in EU_{k+1}(G)$.

Since $V(H_i)\cap V(H_j)=\emptyset$ for $\{i,j\}\subseteq\{1,2,\ldots,m\}$ with $i\neq j$, $E(C_i)\cap E(C_j)=\emptyset$. It follows that $d_C(x)\equiv 0 \pmod 2$ for every $x\in V(C)$, which implies that C satisfies (I). Since C_i satisfies (2), $d_G(x)\geqslant 3$ for every $x\in V(C)$ with $d_C(x)=0$. Thus (II) holds.

Since $\bigcup_{i=3}^{\Delta(L(G))} V_i(L(G))\subseteq V(H)$, $d_G(x,G[V(C)\backslash\{x\}])\leqslant k$ for every vertex x in C with $d_C(x)=0$. Take an arbitrary $T\subseteq\{1,2,\ldots,m\}$. By the choice of H, it follows that $d_{L(G)}(\bigcup_{i\in T}H_i, H-\bigcup_{i\in T}H_i)\leqslant k-1$. Let $P=e_1e_2\cdots e_s$ be a shortest path from $\bigcup_{i\in T}H_i$ to $H-\bigcup_{i\in T}H_i$, where $e_1\in V(\bigcup_{i\in T}H_i)\subseteq\bar{E}(\bigcup_{i\in T}C_i)$ and $e_s\in V(H-\bigcup_{i\in T}H_i)\subseteq\bar{E}(C-\bigcup_{i\in T}C_i)$, and $s\leqslant k$. Since e_t and e_{t+1} are two adjacent edges in G for each $t\in\{1,2,\ldots,s-1\}$, it follows that $G[\{e_1,e_2,\ldots,e_s\}]$ is connected. Hence

$$d_G\left(\bigcup_{i\in T}C_i, C-\bigcup_{i\in T}C_i\right)\leqslant|E(G[\{e_1,e_2,\ldots,e_s\}])|\leqslant s\leqslant k,$$

which implies that C satisfies (III) by Lemma 12.

Since H satisfies (III) to (V), using Lemma 10 one can easily check that C satisfies (IV) and (V). It follows that $C\in EU_{k+1}(G)$. □

Proof of Theorem 5. Supposing that $EU_2(G)\neq\emptyset$, we choose an $H\in EU_2(G)$ with a minimum number of components that are denoted by $H_1,H_2,\ldots,H_t$.

Since $H\in EU_2(G)$, $|\bar{E}(H_i)|\geqslant 3$ for $i\in\{1,2,\ldots,t\}$. Hence, by Lemma 8B, we can find a cycle C_i of $L(G)$ with length at least 3 such that $V(C_i)=\bar{E}(H_i)$, for $i\in\{1,2,\ldots,t\}$. Let

$$C=\bigcup_{i=1}^{t}C_i.$$

By Lemma 11, $C_1,C_2,\ldots,C_t$ are t edge-disjoint cycles in $L(G)$. Hence C is a subgraph of $L(G)$ satisfying (I). Since $d_G(H_1,H-H_1)\leqslant 1$ for any subgraph H_1 of H, C is a connected subgraph of $L(G)$. By Lemma 10 and since $H\in EU_2(G)$, any branch $b\in B(L(G))$ with $E(b)\cap E(C)=\emptyset$ has length at most 2 and any branch in $B_1(L(G))$ has length at most 1. Since H satisfies (II),

$$\bigcup_{i=3}^{\Delta(L(G))}V_i(L(G))\subseteq V(C).$$

Hence $\bar{E}(C)=E(L(G))$ which implies that $L^2(G)$ is hamiltonian by Theorem 1.

Conversely, suppose that $L^2(G)$ is hamiltonian. By Theorem 1, there exists a circuit C of $L(G)$ such that $E(L(G))=\bar{E}(C)$. Select a C with a maximum number of vertices of degree at least 3. Then

Claim 1. $\bigcup_{i=3}^{\Delta(L(G))}V_i(L(G))\subseteq V(C)$.

Proof. Otherwise let $e_0\in(\bigcup_{i=3}^{\Delta(L(G))}V_i(L(G)))\backslash V(C)$. By Lemma 9, there exist two vertices $e_1,e_2\in N_{L(G)}(e_0)$ such that $e_1e_2\in E(L(G))$. Since $\bar{E}(C)=E(L(G))$ and $e_0\notin V(C)$,

$\{e_1,e_2\}\subseteq V(C)$. Now we construct a subgraph C_0 of $L(G)$ as follows,

$$C_0=\begin{cases} C+\{e_0e_1,e_0e_2\}-\{e_1e_2\} & \text{if } e_1e_2\in E(C),\\ C+\{e_0e_1,e_0e_2,e_1e_2\} & \text{if } e_1e_2\notin E(C).\end{cases}$$

Obviously C_0 is a circuit such that $E(L(G))=\bar{E}(C_0)$, but C_0 contradicts the maximality of C. This completes the proof of Claim 1. □

Hence C is a eulerian subgraph of $L(G)$ since $L(G)$ is a simple graph. By Lemma 12, G has a subgraph H satisfying (1) to (4).

Claim 2. *$d_G(x,H)\leqslant 1$ for any $x\in\bigcup_{i=3}^{\Delta(G)}V_i(G)$.*

Proof. If G is either a star or a cycle, then the conclusion holds. If G is neither a star nor a cycle, then $E_G(x)\cap(\bigcup_{i=3}^{\Delta(L(G))}V_i(L(G)))\neq\emptyset$ for every vertex x in $\bigcup_{i=3}^{\Delta(G)}V_i(G)$. Hence by Claim 1 and (4), there exists an edge e_x such that

$$e_x\in E_G(x)\cap\left(\bigcup_{i=3}^{\Delta(L(G))}V_i(L(G))\right)\subseteq V(C)\subseteq\bar{E}(H).$$

This implies that e_x has an endvertex in H. This completes the proof Claim 2.

We will prove that $H'=H\cup(\bigcup_{i=3}^{\Delta(G)}V_i(G))\in EU_2(G)$. Claim 2 and property (3) of H imply that $d_G(H_1',H'-H_1')\leqslant 1$ for every subgraph H_1' of H', thus H' satisfies (III). It follows from Lemma 10 and $\bar{E}(C)=E(L(G))$ that $|E(b)|\leqslant 3$ for $b\in B(G)$ with $E(b)\cap E(H)=\emptyset$ and $|E(b)|\leqslant 2$ for $b\in B_1(G)$. Hence $H'\in EU_2(G)$. □

4. Methods for determining the hamiltonian index of a graph

In this section, we will give two methods for determining the hamiltonian index of a graph.

Define

$CB(G)=\{b\in B(G)$: *any edge of b is a cut edge of G*$\}$ and

$CB_1(G)=B_1(G)$.

One can easily see that $CB(G)\backslash CB_1(G)$ is the set of bridge-paths of G and $CB_1(G)$ is the set of its end-paths (see [14]).

As in [8], if $L^0(G)$ stands for G, then we define the hamiltonian index $h(G)$ of a graph G to be

$$h(G)=\min\{n:\ L^n(G)\ \textit{is hamiltonian}\}.$$

Since the hamiltonian index does not exist for paths and 2-cycles, we will exclude them in the rest of this section. Thus, G will always stand for a connected graph other than a path or a 2-cycle in this section.

4.1. Split blocks of a graph

Define $k(G)=0$ if G is 2-connected; $k(G)=1$ if G is not 2-connected and $CB(G)=\emptyset$; $k(G)=\max\{\max\{|E(b)|+1: b\in CB(G)\setminus CB_1(G)\}, \max\{|E(b)|: b\in CB_1(G)\}\}$, otherwise.

Chartrand and Wall obtained the hamiltonian index of a tree.

Theorem 13 (Chartrand and Wall [9]). *Let T be a tree. Then*

$$h(T)=k(T).$$

A *block* of a graph G is a maximal connected subgraph which contains no cut vertex of itself. A block of G is called an *acyclic block* if it is a single edge of G and a *cyclic block* otherwise. Recently, Saražin generalized the above result as follows:

Theorem 14 (Saražin [14]). *If every cyclic block of G is hamiltonian, then*

$$h(G)=k(G).$$

In this section, we will characterize those graphs G for which $h(G)=k(G)$. To do this, for each cyclic block B of G, we construct a *split block* SB from B as follows:

(a) split each vertex $x\in V_2(B)\cap(\bigcup_{i=3}^{\Delta(G)} V_i(G))$ into a triangle $x_1x_2x_3$ in SB;
(b) replace the two edges ux and vx (say) in $E(B)$ by ux_1 and vx_2 in $E(SB)$.

This construction is illustrated in Fig. 2.

Let G' denote the resulting graph obtained by performing (a) and (b). Define $S(G')=\{F'\subseteq G'$: F' has no vertices of odd degree, and if a triangle created by performing (a) has a vertex in F', then all vertices of the triangle are in F' and have degree two in $F'\}$. Then there exists a one-to-one correspondence Φ between any subgraph F' in $S(G')$ and the subgraph with even degrees, $F=\Phi(F')$, of G, which is obtained by contracting all triangles in F' created in step (a).

Let $SB_1,SB_2,\ldots,SB_t$ be all split blocks of G. For two branches $b_1\in B(G)$ and $b_2\in\bigcup_{i=1}^{t} B(SB_i)$, we say $b_1=b_2$ if the internal vertices of b_1 and b_2 coincide, and if the endvertices either coincide, or the endvertices of b_2 belong to triangles obtained from endvertices of b_1 via construction of split blocks.

The following lemma is immediate.

Lemma 15. *Let $SB_1,SB_2,\ldots,SB_t$ be all split blocks of G. Then*

$$B(G)\setminus CB(G)=\bigcup_{i=1}^{t}(B(SB_i)\setminus B_2(SB_i)),$$

where $B_2(SB_i)$ is the set of branches of SB_i of length 2 that are contained in triangles resulting from the construction of split blocks.

The following lemma is necessary for our proof.

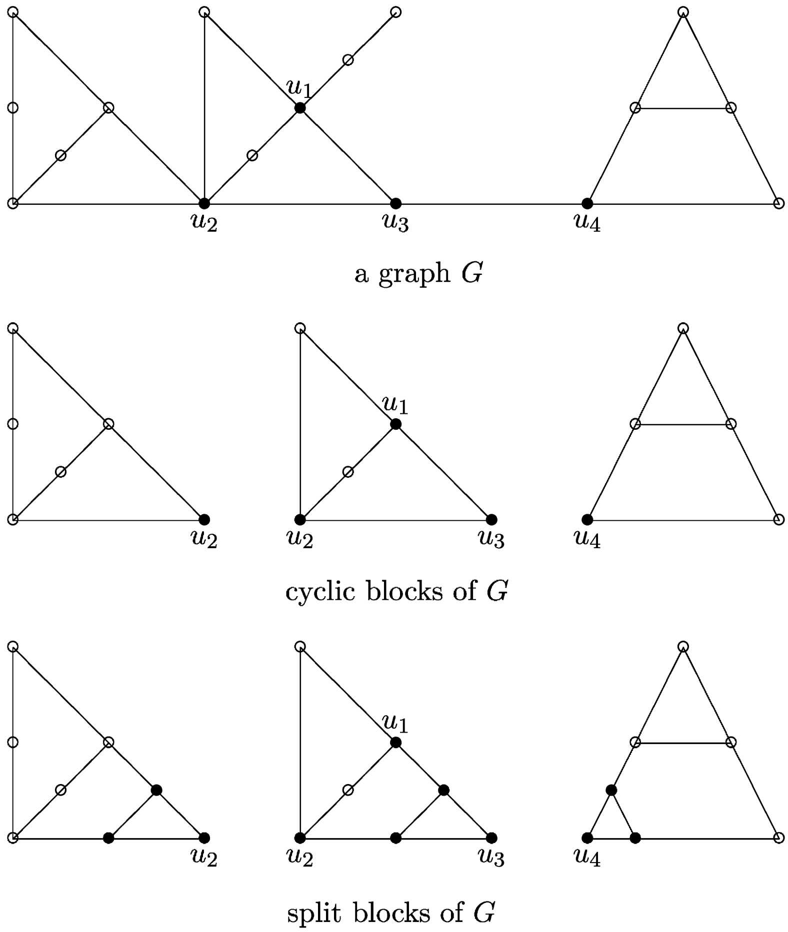

Fig. 2. Splitting a graph.

Lemma 16. *Let G be a graph with $h(G) \geqslant 2$ and let H be a subgraph in $EU_{h(G)}(G)$. For $F \subseteq H$, if p is a path from F to $H - F$ such that $|E(p)| \geqslant 2$ and the internal vertices of p are not in $V(H)$, then $p \in B(G)$.*

Proof. This follows from H satisfying (I), (II) and $|E(p)| \geqslant 2$. □

Lemma 17. *Let G be a connected graph and let $SB_1, SB_2, \ldots, SB_t$ be all split blocks of G. Then*

$$h(G) \geqslant \max\{h(SB_1), h(SB_2), \ldots, h(SB_t), k(G)\}.$$

Proof. Clearly $h(G) \geqslant k(G)$. It remains to prove that $h(G) \geqslant h(SB_i)$ for any $i \in \{1, 2, \ldots, t\}$. If $h(G) = 0$, then G itself is a single block and the lemma follows. If

$h(G)=1$, then $k(G)\leqslant 1$. Hence the lemma follows from Theorem 1. Next we assume that $h(G)\geqslant 2$, which implies that there exists a subgraph H in $EU_{h(G)}(G)$ by Theorem 6. Obviously H is a union of subgraphs in different blocks, i.e., $H=\bigcup_{i=1}^{t} H_i$, where $H_i \in B_i$. Let

$$H_i' = \Phi^{-1}(H_i).$$

We will prove that $H_i' \in EU_{h(G)}(SB_i)$. Clearly H_i' satisfies (I) and (II). By Lemma 15, H_i' satisfies (IV) and (V). It remains to show that H_i' satisfies (III), i.e., $d_G(F', H_i' - F') \leqslant h(G) - 1$ for each subgraph $F' \subseteq H_i'$. If this were not true, there would exist an H_i' with a subgraph F' such that $d_G(F', H_i' - F') \geqslant h(G) \geqslant 2$. It follows from (II) and the definition of H_i' that any shortest path from F' to $H_i' - F'$ is in $B(SB_i)\backslash B_2(SB_i)$. By Lemma 15, p is in $B(G)\backslash CB(G)$. Let $F=\Phi(F')$. Since any path from F' to $H_i' - F'$ is also a path from F to $H_i - F$, p is a shortest such path. Hence $|E(p)|\geqslant h(G)$. On the other hand, since $H \in EU_{h(G)}(G)$, $|E(p)| \leqslant h(G)-1$, which is a contradiction. This implies that H_i' satisfies (III) for each $i\in\{1,2,\ldots,t\}$. Therefore $H_i' \in EU_{h(G)}(SB_i)$, and it follows that $h(SB_i)\leqslant h(G)$ by Theorem 6. □

Now we can state our main results of this section.

Theorem 18. *Let G be a connected graph and let $SB_1, SB_2, \ldots, SB_t$ be all split blocks of G. Then*

$$h(G) = \max\{h(SB_1), h(SB_2), \ldots, h(SB_t), k(G)\}.$$

Proof. Let

$$m(G) = \max\{h(SB_1), h(SB_2), \ldots, h(SB_t), k(G)\}.$$

By Lemma 17, we only need to prove that $h(G)\leqslant m(G)$. If $m(G)=0$, which implies that $k(G)=0$, then G has only one split block of itself. Thus the theorem follows. If $m(G)=1$, which implies that $k(G)\leqslant 1$, then the theorem follows by Theorem 1 and Lemma 17. So we only need to consider the case that $m(G)\geqslant 2$.

By Theorem 6, for any $i\in\{1,2,\ldots,t\}$, there exists a subgraph H_i' such that $H_i' \in EU_{m(G)}(SB_i)$ and H_i' contains all vertices in triangles created by performing (a). Since H_i' satisfies (I), $H_i' \in S(G')$. Let

$$H = \bigcup_{i=1}^{t} \Phi(H_i').$$

We will prove that $H \in EU_{m(G)}(G)$. Since $E(H_i')\cap E(H_j') = \emptyset$ for $\{i,j\}\subseteq\{1,2,\ldots,t\}$ with $i\neq j$, H satisfies (I). Obviously H satisfies (II). Using Lemma 15, we obtain that H satisfies (IV) and (V).

It remains to prove that $d_G(F, H-F)\leqslant m(G)-1$ for any subgraph $F\subseteq H$. If this were not the case, then there would exist a subgraph F of H with $d_G(F,H-F)\geqslant m(G)\geqslant 2$. It follows from Lemma 16 and the definition of $k(G)$ that any shortest path p of G from

F to $H-F$ is in $B(G)\backslash CB(G)$. By Lemma 15, p is in $\bigcup_{i=1}^{t}(B(SB_i)\backslash B_2(SB_i))$. Without loss of generality, we may assume that p is in $B(SB_1)\backslash B_2(SB_1)$. Let $H_1=\Phi(H_1')$ and $F'=\Phi^{-1}(F\cap H_1)$. Since every path from F' to $H_1'-F'$ is also a path from F to H_1-F, p is a shortest such path. Hence $|E(p)|\geqslant m(G)$. On the other hand, by $H_1'\in EU_{m(G)}(SB_1)$, $|E(p)|\leqslant m(G)-1$, which is a contradiction. This implies that H satisfies (III). So $H\in EU_{m(G)}(G)$, implying that $h(G)\leqslant m(G)$ by Theorem 6. □

We conclude this section with a characterization of graphs G for which $h(G)=k(G)$.

Corollary 19. *Let G be a connected graph and let $SB_1,SB_2,\ldots,SB_t$ be all the split blocks of G. Then $h(G)=k(G)$ if and only if $h(SB_i)\leqslant k(G)$ for $i\in\{1,2,\ldots,t\}$.*

Remark. It is not difficult to determine $k(G)$ of a graph G. By Theorem 18, we can determine the hamiltonian index of a graph by first determining the hamiltonian indices of its split blocks. Since each split block of a connected graph is 2-connected, we only need to consider graphs of connectivity at least two.

4.2. The contraction of a graph

Catlin [6] developed a reduction method for determining whether a graph G has a spanning circuit. Using this reduction method and Theorem 1, several authors obtained good bounds for $h(G)$ (see [7,10,13,15]). Here we give a similar reduction method for determining $h(G)$ of graphs G with $h(G)\geqslant 4$.

For $\{b_1,b_2,\ldots,b_m\}\subseteq B(G)$ with $|E(b_i)|\geqslant 2$ for each $i\in\{1,2,\ldots,m\}$, the contraction of G is defined to be a graph, denoted by $G//\{b_1,b_2,\ldots,b_m\}$, which is obtained from G by contracting an edge of b_i, i.e., replacing b_i by a new branch of length $|E(b_i)|-1$, for each $i\in\{1,2,\ldots,m\}$.

Theorem 20. *Let G be a connected graph and let $b_1,b_2,\ldots,b_m$ be all branches of length at least 2 in G. If $h(G)\geqslant 4$, then*

$$(*)\quad h(G)=h(G//\{b_1,b_2,\ldots,b_m\})+1.$$

Proof. Let $G'=G//\{b_1,b_2,\ldots,b_m\}$. Clearly $h(G')\leqslant h(G)$, by Theorem 6. If $h(G')\leqslant 1$, then there exists a connected subgraph H' in which every vertex has even degree such that $E(G')=\bar{E}(H')$, by Theorem 1. Let $b_1',b_2',\ldots,b_m'$ be the branches of G' corresponding to the branches $b_1,b_2,\ldots,b_m$, respectively. Let H'' be the subgraph of G obtained from H' by replacing $b_1',b_2',\ldots,b_m'$ by $b_1,b_2,\ldots,b_m$, respectively. By H we denote the subgraph with

$$V(H)=V(H'')\cup\left(\bigcup_{i=3}^{\Delta(G)}V_i(G)\right)\quad\text{and}\quad E(H)=E(H'').$$

One can easily check that $H\in EU_3(G)$. Hence $h(G)\leqslant 3$ by Theorem 6, a contradiction implying that $h(G')\geqslant 2$. It follows from Theorem 6 and $h(G)\geqslant h(G')\geqslant 2$ that

$EU_{h(G)}(G) \neq \emptyset$ and $EU_{h(G')}(G') \neq \emptyset$. Take any subgraph $H \in EU_{h(G)}(G)$ and let H' be the subgraph of G' corresponding to H. It follows from Lemma 16 and the definition of G' that $H' \in EU_{h(G)-1}(G')$. Hence $h(G') \leqslant h(G) - 1$ by Theorem 6. Similarly, take any subgraph $H' \in EU_{h(G')}(G')$ and let H be the subgraph of G corresponding to H'. It follows from Lemma 16 and the definition of G' that $H \in EU_{h(G')+1}(G)$. Hence $h(G) \leqslant h(G') + 1$ by Theorem 6. Thus $(*)$ is true. □

Remark. The condition in Theorem 20 is best possible in the following sense: there exists a family of graphs with hamiltonian index 3 for which $(*)$ does not hold. Let $C = u_1 u_2 \cdots u_{3s} \cdots u_t$ be a cycle of length at least $t, t \geqslant 3s+1 \geqslant 13$, and let w, v_1, v_2, v_3 be four vertices not belonging to C. Let G_0 be the graph with $V(G_0) = V(C) \cup \{w, v_1, v_2, v_3\}$ and $E(G_0) = E(C) \cup \{wv_1, v_1u_s, wv_2, v_2u_{2s}, wv_3, v_3u_{3s}\}$. One can easily check that $h(G_0) = 3$ but that its contraction has hamiltonian index 1, which implies that $(*)$ does not necessarily hold for a graph with hamiltonian index 3.

The complexity of determining the hamiltonian index (not exceeding 1) of a graph is NP-complete [2]. So far, we do not know how difficult it is to determine the hamiltonian index (exceeding 1) of a graph. However we conjecture that this is polynomial. By Theorem 20, we only need to consider the complexity of determining whether the hamiltonian index is 2 or 3.

5. Upper bounds for the hamiltonian index of a graph

In this section, we will give some upper bounds on the hamiltonian index of a graph. For every connected graph G with $\Delta(G) \geqslant 3$, define

$$B_0(G) = \{b \in B(G)\colon\ G[V(b)] \textit{ is a cycle of } G\}$$

and

$$k = \max\{|E(b)|\colon\ b \in B(G) \backslash B_0(G)\}.$$

Now for each $b \in B_0(G)$, denote by $C(b)$ the cycle induced by $V(b)$. We take a subgraph H of G with

$$V(H) = \left(\bigcup_{b \in B_0(G)} V(b) \right) \cup \left(\bigcup_{i=3}^{\Delta(G)} V_i(G) \right)$$

and

$$E(H) = \bigcup_{b \in B_0(G)} (E(b) \backslash \{e\colon\ |\{b\colon\ e \in C(b)\}| \equiv 0 \pmod 2\}).$$

It is easily seen that $H \in EU_{k+1}(G)$. Hence we obtain the next result.

Theorem 21. *Let G be a connected graph that is not a path. Then*

$$h(G) \leqslant \max\{|E(b)|: b \in B(G) \backslash B_0(G)\} + 1.$$

In order to show that the upper bound in Theorem 21 is sharp, we construct a graph G_0 as follows: Let p be a path of length $k, k \geqslant 1$, and let C_1, C_2 be two cycles. G_0 is obtained by identifying the two end-vertices of p with two vertices of C_1 and C_2, respectively. By Theorem 6, $L^{k+1}(G_0)$ is hamiltonian but $L^k(G_0)$ is not.

We will present some corollaries of Theorem 21. Corollary 22 is in fact stronger than the result in [13].

Corollary 22. *Let G be a simple connected graph that is not a path. Then*

$$h(G) \leqslant \max\{|E(b)|: b \in B(G) \backslash B_0(G)\} + 1.$$

Corollary 23 (Chartrand and Wall [9]). *If G is a connected graph such that* $\delta(G) \geqslant 3$, *then*

$$h(G) \leqslant 2.$$

Next, we give a simple proof of the following known result.

Theorem 24 (Saražin [15]). *If G is a connected simple graph with* $\Delta(G) \geqslant 3$, *then*

$$h(G) \leqslant |V(G)| - \Delta(G).$$

Proof. Let w be a vertex of G with $d_G(w) = \Delta(G)$.

First we define H' as follows:

$$V(H') = \bigcup_{i=3}^{\Delta(G)} V_i(G)$$

and

$$E(H') = \emptyset.$$

Now let $H = H' \cup H''$, where H'' is a maximal circuit of G through w (i.e., there is no circuit K such that $K \neq H''$ and K contains H'').

Since G is a connected simple graph, it follows that $H \in EU_{|V(G)|-\Delta(G)}(G)$. Hence by Theorem 6, $h(G) \leqslant |V(G)| - \Delta(G)$. $\square$

Note that the graph in Theorem 24 must be simple, which is not mentioned in [15]. Recently, with regard to Theorem 6, the first author [17] has proved that the hamiltonian index $h(G)$ of a graph G is less than the *diameter* of G, i.e., $\max\{d_G(u,v): u, v \in V(G)\}$, which improves the bound in Theorem 24 because $d(G) - 1 \leqslant |V(G)| - \Delta(G)$ [18].

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