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Computing the asymptotes for a real plane algebraic curve

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Abstract

The purpose of this paper is to present an algorithm for computing all the asymptotes of a real plane algebraic curve. By this algorithm, all the asymptotes of a real plane algebraic curve may be represented via polynomial real root isolation.

Keywords: Real plane algebraic curve; Infinite branch; Asymptote; Sturm sequence; Transfer principle

0. Introduction

As an important kind of algebraic varieties, algebraic curves were discussed in many monographs, e.g. Ref. [13]. In Refs. [4,5,7], the topology of real plane algebraic curves was effectively determined. The concept of asymptotes also is very important in the study of real plane algebraic curves. The asymptote of some branch of a real plane algebraic curve reflects the status of this branch at the points with sufficiently large coordinates.

For the sake of precision, we need to give the concept of asymptotes of a real plane algebraic curve. In the sequel, for a line ℓ and a point P in $\mathbb{R}^2$, write $d(P,\ell)$ for the distance of P from ℓ. Moreover, by $P(a,b)$ denote the point P with coordinates (a,b) in $\mathbb{R}^2$.

Definition. Let $\mathcal{C}$ be a real plane algebraic curve in the real plane $\mathbb{R}^2$, and $\mathcal{B}$ an infinitely elongating branch of $\mathcal{C}$. A line ℓ in $\mathbb{R}^2$ is called the asymptote of $\mathcal{B}$, if for every positive $\epsilon \in \mathbb{R}$, there

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exists a positive $\Delta \in \mathbb{R}$ such that $d(P, \ell) < \epsilon$ for all points $P(a, b)$ on $\mathcal{B}$ with $a^2 + b^2 > \Delta$. In this case, we also say that ℓ is an asymptote of $\mathcal{C}$.

If an infinitely elongating branch $\mathcal{B}$ can be defined by some explicit equation of the form $y = \phi(x)$ (or $x = \psi(y)$), where ϕ (or ψ) is a continuous function on an infinite interval, it is easy to decide whether or not $\mathcal{B}$ possesses an asymptote by investigating the existences of the limits of certain functions when $x \to \infty$ (or $y \to \infty$). Moreover, if the limits of these functions when $x \to \infty$ (or $y \to \infty$) can be effectively computed, we may obtain the equation of the asymptote of $\mathcal{B}$. However, if this branch $\mathcal{B}$ is implicitly defined and its equation cannot be converted into an explicit form, both the decision and the computation of the asymptote of $\mathcal{B}$ require certain tricks.

Let $\mathcal{C}$ be a real plane algebraic curve defined by the equation $f(x, y) = 0$, where $f(x, y)$ is a non-constant polynomial in $\mathbb{R}[x, y]$, the ring of bivariate polynomials over the field $\mathbb{R}$ of real numbers. For the computations of the asymptotes of $\mathcal{C}$, there are the following possible misunderstandings:

- Vertical asymptotes correspond to $x = a$ where a is any real root of the leading coefficient of f as a polynomial over $\mathbb{R}[x]$ in one variable y.
- Horizontal asymptotes correspond to $y = b$ where b is any real root of the leading coefficient of f as a polynomial over $\mathbb{R}[y]$ in one variable x.
- The slope of the oblique asymptotes correspond to the real ratios, which are the real zeros of the highest total degree homogeneous part of $f(x, y)$.

These misunderstandings may be cleared up by the following

Example. Let $\mathcal{C}$ be a curve defined by the equation $f(x, y) = 0$, where $f(x, y) = x^4y^2 + x^2y^4 - x^2y^2 + x^2 + y^2 - 1$.

Since $x^4y^2 + x^2y^4 - x^2y^2 + x^2 + y^2 - 1 = (x^2y^2 + 1)(x^2 + y^2 - 1)$, the curve $\mathcal{C}$ is actually a circle. Hence, C has not any (vertical or horizontal) asymptote. However, the leading coefficient of f as a polynomial over $\mathbb{R}[x]$ in one variable y is x^2, and it has a real root $x = 0$. Moreover, the leading coefficient of f as a polynomial over $\mathbb{R}[y]$ in one variable x is y^2, and it has a real root $y = 0$. By the rotation of axes: $x = \frac{\sqrt{2}}{2}(x' + y')$, $y = \frac{\sqrt{2}}{2}(x' - y')$, the equation of the circle $\mathcal{C}$ is changed into $g(x', y') = 0$, where $g(x', y') = \frac{1}{4}x'^6 - \frac{1}{4}x'^4y'^2 - \frac{1}{4}x'^2y'^4 + \frac{1}{4}y'^6 - \frac{1}{4}x'^4 + \frac{1}{2}x'^2y'^2 - \frac{1}{4}y'^4 + x'^2 + y'^2 - 1$. Of course, the circle $\mathcal{C}$ has not any (oblique) asymptote. However, the highest total degree homogeneous part of $g(x', y')$ is $\frac{1}{4}x'^6 - \frac{1}{4}x'^4y'^2 - \frac{1}{4}x'^2y'^4 + \frac{1}{4}y'^6$, and $(1, 1)$ is one of its real zeros.

In this paper, we will present an effective method for deciding and computing the asymptotes of a real plane algebraic curve with implicit equation $f(x, y) = 0$, where $f(x, y) \in \mathbb{R}[x, y]$. By this method, for an algebraic curve $\mathcal{C}$ in $\mathbb{R}^2$, we may decide whether or not a branch of $\mathcal{C}$ has an asymptote, compute all the asymptotes of $\mathcal{C}$, and determine those branches whose asymptotes are the same.

1. Preliminaries

Before establishing the main results, we need some preliminaries. First, we extend the field $\mathbb{R}$ of real numbers to an ordered field containing an infinitely large element η.

Let η be an indeterminate over $\mathbb{R}$. Then the ordering $\leqslant$ of $\mathbb{R}$ can be uniquely extended to an ordering of the field $\mathbb{R}(\eta)$, denoted still by $\leqslant$, such that η is positive and infinitely large over $\mathbb{R}$. Obviously, for a non-zero element $\frac{g}{h} \in \mathbb{R}(\eta)$ with $g, h \in \mathbb{R}[\eta]$, $\frac{g}{h} < 0$, if and only if the leading coefficient of gh is negative as an univariate polynomial in η over $\mathbb{R}$. Thereby, for every non-zero $g \in \mathbb{R}[\eta]$, we have $\mathrm{sign}_{\mathbb{R}(\eta)}(g) = \mathrm{sign}(\mathrm{lcoeff}(g, \eta))$, where $\mathrm{lcoeff}(g, \eta)$ stands for the leading coefficient of g as an univariate polynomial in η over $\mathbb{R}$, and $\mathrm{sign}_{\mathbb{R}(\eta)}(g)$, $\mathrm{sign}(\mathrm{lcoeff}(g, \eta))$ are the signs of g, $\mathrm{lcoeff}(g, \eta)$ with respect to the orderings of $\mathbb{R}(\eta)$, $\mathbb{R}$ respectively.

Denote by R the real closure of $(\mathbb{R}(\eta), \leqslant)$, and the only ordering of R is still denoted by $\leqslant$. Then $\mathbb{R} \subset R$. In the sequel, for a non-zero element α in R, write $\mathrm{sign}_R(\alpha)$ for the sign of α with respect to the only ordering $\leqslant$ of R. Moreover, construct the two subsets of R as follows:

$$A := \{z \in R \mid \text{ for some positive number } d \in \mathbb{R},\ -d \leqslant z \leqslant d\},$$

$$I := \{z \in R \mid \text{ for every positive number } d \in \mathbb{R},\ -d \leqslant z \leqslant d\}.$$

Obviously, I consists of all elements in R infinitesimal over $\mathbb{R}$. By the structure of the ordering $\leqslant$, we have $\mathbb{R} \subset A$, $\eta^{-1} \in I$, and but $\eta \notin A$. By the familiar facts on real valuations (see Proposition 1.3 in [8] or the relevant theorems in §5 of [9]), A is a real valuation ring of R, and I is the only maximal ideal of A. Moreover, (A, I) is compatible with $\leqslant$; in other words, both A and I are convex in R with respect to $\leqslant$. In the sequel, every element in A is called bounded (over $\mathbb{R}$), but every element in $R \setminus A$ is called unbounded (over $\mathbb{R}$).

Let π be the real place associated with the valuation ring A. Then π is a mapping of R into $\mathbb{R} \cup \{\infty\}$ satisfying the following conditions:

(1.1) The restricted mapping $\pi|_A$ is an $\mathbb{R}$-homomorphism of A onto $\mathbb{R}$ such that I is exactly the kernel of $\pi|_A$.
(1.2) For any $\alpha, \beta \in A$, $\alpha \leqslant \beta$ implies $\pi(\alpha) \leqslant \pi(\beta)$.

By condition (1.1), we have $\pi(g(\eta^{-1})) = g(0)$ for every $g \in \mathbb{R}[z]$, and $\alpha - \pi(\alpha) \in I$ for all $\alpha \in A$.

In the sequel, we adopt the usual symbols as follows: $]a, b[$ (or $[a, b]$) stands for the open (or closed) interval in $\mathbb{R}$ with endpoints a, b, and but $]a, b[_R$ (or $[a, b]_R$) stands for the open (or closed) interval in R with endpoints a, b. Of course, (finite or infinite) half open-closed intervals may be similarly denoted.

Now let $\theta \in R \setminus A$ be a positive element. Then θ is infinitely large over $\mathbb{R}$. Clearly, θ is transcendental over $\mathbb{R}$. Thereby, for an indeterminate z over R, every element in $\mathbb{R}[\theta][z]$ may be considered as a polynomial in θ over $\mathbb{R}[z]$. For a non-zero $\Phi(z) \in \mathbb{R}[\theta][z]$, denote by $\mathrm{lcoeff}(\Phi(z), \theta)$ the leading coefficient of $\Phi(z)$ as a polynomial in θ over $\mathbb{R}[z]$. Obviously, $\mathrm{lcoeff}(\Phi(z), \theta) \in \mathbb{R}[z]$.

Lemma 1.1. *Let $\theta \in R \setminus A$ be a positive element, and let $\Phi(z)$ be a non-zero element in $\mathbb{R}[\theta][z]$. If $B \in \mathbb{R}$ is an upper bound for every real root of* $\mathrm{lcoeff}(\Phi(z), \theta)$*, i.e. $|e| < B$ for every real root e of* $\mathrm{lcoeff}(\Phi(z), \theta)$*, then every bounded root of $\Phi(z)$ in R belongs to $]-B, B[_R$.*

Proof. Suppose that the lemma above is false. Then there is at least one bounded root α of $\Phi(z)$ in R such that either $\alpha \leqslant -B$ or $\alpha \geqslant B$. Put

$$\Phi(z) = a_0(z)\theta^d + a_1(z)\theta^{d-1} + \cdots + a_d(z),$$

where $a_0(z) = \mathrm{lcoeff}(\Phi(z), \theta)$, and $a_i(z) \in \mathbb{R}[z]$, $i = 1, \ldots, d$.

By the equality $\Phi(\alpha) = 0$, we have

$$a_0(\alpha) + a_1(\alpha)\theta^{-1} + \cdots + a_d(\alpha)\theta^{-d} = 0.$$

Observe that $\theta^{-1} \in I$ and $\alpha \in A$. Then $\pi(\theta^{-1}) = 0$, and $\pi(\alpha) \in \mathbb{R}$. So we have

$$a_0\big(\pi(\alpha)\big) = \pi\left(a_0(\alpha) + a_1(\alpha)\theta^{-1} + \cdots + a_d(\alpha)\theta^{-d}\right) = 0.$$

This implies that $\pi(\alpha)$ is a real root of $\mathrm{lcoeff}(\Phi(z), \theta)$. By condition (1.2), we have either $\pi(\alpha) \leqslant \pi(-B) = -B$ or $\pi(\alpha) \geqslant \pi(B) = B$; this contradicts the hypothesis of Lemma 1.1. The proof is complete. $\square$

Remark. According to the proof of Lemma 1.1, it is easy to see that $\pi(\alpha)$ is a real root of $\mathrm{lcoeff}(\Phi(z), \theta)$ for every bounded root α of $\Phi(z)$ in R, where $\Phi(z)$ is as in Lemma 1.1.

Now, we may establish the following

Proposition 1.2. *Let $\Phi(z)$ be as in Lemma* 1.1. *Then we may effectively compute the numbers of bounded and unbounded roots of $\Phi(z)$ in R.*

Proof. According to Lemma 1.1, we implement the following effective computations:

Step 1. By Euclidean division, compute such a Sturm sequence of $\Phi(z)$ as follows:

$$\Phi_0 = \phi(z), \quad \Phi_1 = \frac{\partial \Phi(z)}{\partial z}, \quad \ldots, \quad \Phi_m,$$

where $\Phi_i \in \mathbb{R}[\theta][z]$, $i = 0, \ldots, m$.

Step 2. Extract the leading coefficient $u_i(z)$ of Φ_i as a polynomial in θ and the leading coefficient $v_i(\theta)$ of Φ_i as a polynomial in z, $i = 0, \ldots, m$. Note: $u_i(z) \in \mathbb{R}[z]$ but $v_i(\theta) \in \mathbb{R}[\theta]$ for all i.

Step 3. Extract the leading coefficient a_i of $u_i(z)$ and the leading coefficient b_i of $v_i(\theta)$, $i = 0, \ldots, m$.

Step 4. Write $\deg(u_i; z)$, $\deg(\Phi_i; z)$ respectively for the degrees of u_i, Φ_i as polynomials in z over R, $i = 0, \ldots, m$, and compute the numbers V_1, V_2, V_3 and V_4 of sign variations in the lists $[(-1)^{\deg(u_i;z)} a_i \mid i = 0, \ldots, m]$, $[a_i \mid i = 0, \ldots, m]$, $[(-1)^{\deg(\Phi_i;z)} b_i \mid i = 0, \ldots, m]$ and $[b_i \mid i = 0, \ldots, m]$ respectively.

Then, we have the following claims:

(1) The number of bounded roots of $\Phi(z)$ in R is $V_1 - V_2$;
(2) The number of unbounded roots of $\Phi(z)$ in R is $V_2 + V_3 - V_1 - V_4$.

Indeed, by Sturm's Theorem for real closed fields (see Corollary 1.2.10 in [2]), it is clear that the number of roots of $\Phi(z)$ in R is $W_1 - W_2$, where W_1, W_2 are the numbers of sign variations in the lists $[(-1)^{\deg(\Phi_i;x)} v_i(\theta) \mid i = 0, \dots, m]$ and $[v_i(\theta) \mid i = 0, \dots, m]$ respectively. Since θ is positive and infinitely large over $\mathbb{R}$, we have $\operatorname{sign}_R(v_i(\theta)) = \operatorname{sign}(b_i)$ for $i = 0, \dots, m$. So we have $W_1 - W_2 = V_3 - V_4$.

Take arbitrarily an upper bound B_0 for every real root of $\operatorname{lcoeff}(\Phi(z), \theta)$. Obviously, there is a sufficiently large element $B \in \mathbb{R}$ such that $B_0 < B$, and $\operatorname{sign}(u_i(B)) = \operatorname{sign}(a_i)$ for $i = 0, \dots, m$. Observe that the leading coefficients of $\Phi_i(\theta, -B)$, $\Phi_i(\theta, B)$ as polynomials in θ are just $(-1)^{\deg(u_i;z)} u_i(-B)$, $u_i(B)$ respectively. In this case, the numbers of sign variations in the lists $[\Phi_i(\theta, -B) \mid i = 0, \dots, m]$, $[\Phi_i(\theta, B) \mid i = 0, \dots, m]$ are V_1, V_2 respectively. By Sturm's Theorem, the number of roots of $\Phi(z)$ in the open interval $]-B, B[$ is $V_1 - V_2$. According to Lemma 1.1, the number of bounded roots of $\Phi(z)$ in R is $V_1 - V_2$. This completes the proof. $\square$

Remark. By the proof of Proposition 1.2, it is easy to see that the numbers of negative and positive unbounded roots of $\Phi(z)$ in R are $V_3 - V_1$, $V_2 - V_4$ respectively.

By an argument similar to the proof of Lemma 1.1, we may further establish the following lemma.

Lemma 1.3. *Let the notations be as in Lemma* 1.1. *If* $]c_1, d_1[, \dots,]c_s, d_s[$ *are disjoint open intervals in* $\mathbb{R}$ *such that* $e \in \bigcup_{1 \leqslant k \leqslant s}]c_k, d_k[$ *for every real root* e *of* $\operatorname{lcoeff}(\Phi(z), \theta)$, *then every bounded root of* $\Phi(z)$ *in* R *belongs to* $\bigcup_{1 \leqslant k \leqslant s}]c_k, d_k[_R$.

Definition. Let $h(z)$ be an univariate polynomial in $\mathbb{R}[z]$. Open intervals $]c_1, d_1[, \dots,]c_t, d_t[$ in $\mathbb{R}$ is called a set of isolating intervals for $h(z)$, if the following conditions are satisfied:

(1) $-\infty < c_1 < d_1 \leqslant c_2 < d_2 \leqslant \cdots \leqslant c_t < d_t < \infty$.
(2) For every $k \in \{1, \dots, t\}$, there is exactly one root of $h(z)$ in $]c_k, d_k[$.
(3) Every root of $h(z)$ in $\mathbb{R}$ belongs to $\bigcup_{1 \leqslant k \leqslant t}]c_k, d_k[$.

In what follows, for an univariate polynomial $h(z) \in \mathbb{R}[z]$ and an open interval $]a, b[$ in $\mathbb{R}$ such that $h(z)$ has exactly one root in $]a, b[$, the only real root of $h(z)$ in $]a, b[$ is denoted by $(h(z); a, b)$.

By Lemma 1.3, we may prove the following

Theorem 1.4. *Let* $\Phi(z)$ *be as in Lemma* 1.1. *Then we may effectively compute an univariate polynomial* $h(z) \in \mathbb{R}[z]$ *and a finite number of open intervals* $]c_1, d_1[, \dots,]c_s, d_s[$ *in* $\mathbb{R}$ *such that the following statements are true*:

(1) $-\infty < c_1 < d_1 \leqslant c_2 < d_2 \leqslant \cdots \leqslant c_t < d_t < \infty$.
(2) *For every* $k \in \{1, \dots, s\}$, *there is exactly one root of* $h(z)$ *in* $]c_k, d_k[$.
(3) *For every* $k \in \{1, \dots, s\}$, *there is at least one root of* $\Phi(z)$ *in* $]c_k, d_k[_R$.
(4) *Every bounded root of* $\Phi(z)$ *in* R *belongs to* $\bigcup_{1 \leqslant k \leqslant s}]c_k, d_k[_R$.
(5) *If* α *is a root of* $\Phi(z)$ *in* $]c_\ell, d_\ell[_R$ *for some* $\ell \in \{1, \dots, s\}$, *then* $\pi(\alpha)$ *is the only root of* $h(z)$ *in* $]c_\ell, d_\ell[$.

Proof. According to Lemma 1.3, we implement the following effective computations:

Step 1. Take $h(z)$ as the leading coefficient $\operatorname{lcoeff}(\Phi(z), \theta)$ of $\Phi(z)$ as a polynomial in the variable θ. Note: $h(z) \in \mathbb{R}[z]$.

Step 2. By real root isolation for polynomials (see Algorithm 10.41 in [1]), find out a set of isolating intervals $]c_1, d_1[, \ldots,]c_t, d_t[$ for $h(z)$.

Step 3. For every $k \in \{1, \ldots, t\}$, by Sturm's Theorem, check whether $\Phi(z)$ has a root in the open interval $]c_k, d_k[_R$. Then collect all the indexes k such that $\Phi(z)$ has a root in $]c_k, d_k[_R$.

Then, we may assert that the polynomial $h(z)$ and the intervals $]c_1, d_1[, \ldots,]c_s, d_s[$ are as required in the theorem whenever $\{1, \ldots, s\}$ is the set of all the collected indexes.

Indeed, statements (1)–(3) in the theorem are obviously true. By Lemma 1.3, it is easy to see that statement (4) is true. Now assume that α is a root of $\Phi(z)$ in $]c_\ell, d_\ell[_R$ for some $\ell \in \{1, \ldots, s\}$. By the remark after Lemma 1.1, $\pi(\alpha)$ is a root of $h(z)$ in $\mathbb{R}$. From the inequalities $c_\ell < \alpha < d_\ell$, we get

$$c_\ell = \pi(c_\ell) \leqslant \pi(\alpha) \leqslant \pi(d_\ell) = d_\ell.$$

This implies that $\pi(\alpha)$ must be the only root of $h(z)$ in $]c_\ell, d_\ell[$, because the open interval $]c_\ell, d_\ell[$ contains exactly one real root of $h(z)$. Therefore, our assertion is verified. $\square$

In the following computations, we need a more general result, which involves the familiar theorem of Sylvester. Sylvester's Theorem may be found as Theorem 1.2.9 in [2] or Theorem 8.4.3 in [11].

Let f be a polynomial in $\mathbb{R}[x, y]$ such that $\frac{\partial f}{\partial y} \neq 0$, and put $f_1 := \frac{\partial f}{\partial y}$. Obviously, there exist $q_1, f_2 \in \mathbb{R}[x, y]$ such that

$$\operatorname{lcoeff}(f_1; y)^{2m_1} f = f_1 q_1 - f_2,$$

where m_1 is a non-negative integer, and $\deg(f_2; y) < \deg(f_1; y)$.

Whenever $f_2 \neq 0$, such a division algorithm may be continued for f_1 and f_2. In other words, we further have

$$\operatorname{lcoeff}(f_2; y)^{2m_2} f_1 = f_2 q_2 - f_3,$$

where m_2 is a non-negative integer, and $q_2, f_3 \in \mathbb{R}[x, y]$ such that $\deg(f_3; y) < \deg(f_2; y)$.

By repeating this division algorithm, we can obtain a final equality of the following form:

$$\operatorname{lcoeff}(f_s; y)^{2m_s} f_{s-1} = f_s q_s,$$

where m_s is a non-negative integer, and f_{s-1}, f_s, q_s are non-zero polynomials in $\mathbb{R}[x, y]$.

So, we obtain a sequence of non-zero polynomials $f, f_1, \ldots, f_s$ in $\mathbb{R}[x, y]$. Such a sequence $f, f_1, \ldots, f_s$ is called a Sturm sequence of f relative to y. Likewise, a Sturm sequence of f relative to x may be obtained.

Theorem 1.5. *Let $f(x, y)$ be a non-zero polynomial in $\mathbb{R}[x, y]$, let $g(z)$ be a non-zero polynomial in $\mathbb{R}[z]$ having the only root in an open interval $]c, d[$, and let $e, \delta \in \mathbb{R}$ with $e < \delta$. If a denotes $(g(z); c, d)$, and none of $c\eta$, $a\eta + e$, $a\eta + \delta$ and $d\eta$ is a root of $f(\eta, y)$, then $c\eta < a\eta + e <$*

$a\eta+\delta < d\eta$, and we may effectively compute the numbers of roots of $f(\eta, y)$ in $]c\eta, a\eta+e[_R$, $]a\eta+e, a\eta+\delta[_R$, $]a\eta+\delta, d\eta[_R$ respectively.

Proof. By the structure of the ordering of $\mathbb{R}(\eta)$, it is clear that $c\eta < a\eta+e < a\eta+\delta < d\eta$.

Let $f_0 := f, f_1, \dots, f_s$ be a Sturm sequence of $f(x, y)$ relative to y. Then, a Sturm sequence of $f(\eta, y)$ is as follows:

$$f_0(\eta, y), \quad f_1(\eta, y), \quad \dots, \quad f_s(\eta, y).$$

By Sturm's Theorem, it suffices to compute the number of sign variations in the list $[f_i(\eta, \alpha) \mid i = 0, \dots, s]$ for every $\alpha \in \{c\eta, a\eta+e, a\eta+\delta, d\eta\}$. According to the definition of the ordering of $\mathbb{R}(\eta)$, it is easy to compute the number of sign variations in the list $[f_i(\eta, \alpha) \mid i = 0, \dots, s]$ for every $\alpha \in \{c\eta, d\eta\}$.

In what follows, we proceed to compute the number of sign variations in the list $[f_i(\eta, \alpha) \mid i = 0, \dots, s]$ for $\alpha = a\eta+e$. For $\alpha = a\eta+\delta$, the number of sign variations in the list $[f_i(\eta, \alpha) \mid i = 0, \dots, s]$ may be similarly computed.

For every $i \in \{0, \dots, s\}$, the polynomial $f_i(\eta, \eta z+e) \in \mathbb{R}[\eta, z]$ may be expressed as follows:

$$f_i(\eta, \eta z+e) = u_{i0}(z)\eta^{n_i} + u_{i1}(z)\eta^{n_i-1} + \dots + u_{in_i}(z),$$

where $u_{ij}(z) \in \mathbb{R}[z]$, $j = 1, \dots, n_i$.

By Sylvester's Theorem, we can compute effectively the difference D_{ij} between the number of roots $g(z)$ in $]c, d[$ for which $u_{ij}(z)$ is positive and the number of roots $g(z)$ in $]c, d[$ for which $u_{ij}(z)$ is negative. Since $g(z)$ has the only root a in the open interval $]c, d[$, $D_{ij} \in \{0, 1, -1\}$, and $u_{ij}(a) = 0$, $u_{ij}(a) > 0$ or $u_{ij}(a) < 0$, according as $D_{ij} = 0$, 1 or -1. Then, by the definition of the ordering of $\mathbb{R}(\eta)$, the sign of $f_i(\eta, a\eta+e)$ may be determined. So we have computed the number of sign variations in the list $[f_i(\eta, \alpha) \mid i = 0, \dots, s]$ for $\alpha = a\eta+e$. This completes the proof. □

2. Infinite branches

In this section, we will present an algorithm for counting the infinitely elongating branches of a real plane algebraic curve. For the sake of simplicity, an infinitely elongating branch of a real plane algebraic curve will be called an infinite branch.

In this section, we let $f(x, y)$ be a non-constant polynomial in $\mathbb{R}[x, y]$, and $\mathcal{C}$ the curve in $\mathbb{R}^2$ defined by the equation $f(x, y) = 0$.

Proposition 2.1. *Let $\mathcal{C}$ be a curve in $\mathbb{R}^2$ defined by the equation $f(x, y) = 0$, where $f(x, y)$ is a non-constant polynomial in $\mathbb{R}[x, y]$. Then there exists a positive number $M \in \mathbb{R}$ such that the part of $\mathcal{C}$ in the region $\{(x, y) \in \mathbb{R}^2 \mid x > M\}$ is a finite number of disjoint infinite branches, and so is the part of $\mathcal{C}$ in the region $\{(x, y) \in \mathbb{R}^2 \mid x < -M\}$.*

Proof. Without loss of generality, we may assume that f is squarefree as a polynomial in $\mathbb{R}[x, y]$. Put $f_1 := \frac{\partial f}{\partial y}$, and write $g(x)$ for the resultant of f and f_1 relative to y. Then $g(x)$ is a non-zero polynomial in $\mathbb{R}[x]$.

Let $f_0 := f, f_1, \dots, f_s$ be a Sturm sequence of f relative to y. Write $u_i(x)$ for the leading coefficient $\operatorname{lcoeff}(f_i, y)$ of f_i as a polynomial in y over $\mathbb{R}[x]$, $i = 0, \dots, s$, and put $h(x) :=$

$g(x)\prod_{0\leqslant i\leqslant s} u_i(x)$. Then $h(x)\in\mathbb{R}[x]$, and there is a positive number $M\in\mathbb{R}$ such that $|\alpha|<M$ for all roots α of $h(x)$ in $\mathbb{R}$. By the intermediate value theorem for polynomials, the following statements are true:

(1) For all $a\in[M,+\infty[$, $g(a)\neq 0$, and $\operatorname{sign}(u_i(a))=\operatorname{sign}(u_i(M))$, $i=0,\dots,s$.
(2) For all $a\in\,]-\infty,-M]$, $g(a)\neq 0$, and $\operatorname{sign}(u_i(a))=\operatorname{sign}(u_i(-M))$, $i=0,\dots,s$.

Write d_i for the degree $\deg(f_i;y)$ of f_i relative to y, $i=0,\dots,s$. Obviously, for all $a\in\,]M,+\infty[$, the degree of $f(a,y)$ is also d_i, $i=0,\dots,s$. Denote by V_1, V_2 the numbers of sign variations in the lists $[(-1)^{d_i}u_i(M)\mid i=0,\dots,s]$ and $[u_i(M)\mid i=0,\dots,s]$ respectively. By statement (1), for every $a\in\,]M,+\infty[$, the numbers of sign variations in the lists $[(-1)^{d_i}u_i(a)\mid i=0,\dots,s]$ and $[u_i(a)\mid i=0,\dots,s]$ are V_1 and V_2 respectively. By Sturm's Theorem, the number of roots of $f(a,y)$ in $\mathbb{R}$ is V_1-V_2. Put $m:=V_1-V_2$, and write all the roots of $f(a,y)$ in $\mathbb{R}$ as follows:

$$\phi_1(a)<\phi_2(a)<\cdots<\phi_m(a),$$

where $\phi_i(x)$ is a function in the variable x defined on $[M,+\infty[$, $i=1,\dots,m$.

By theorem in [3] or Theorem 1.4 in [10], the function $\phi_i(x)$ is continuous, $i=1,\dots,m$. This implies that the part of $\mathcal{C}$ in the region $\{(x,y)\in\mathbb{R}^2\mid x>M\}$ consists of the infinite branches as follows: $y=\phi_1(x),\dots,y=\phi_m(x)$, where x varies over $]M,+\infty[$.

Moreover, for all $a\in\,]M,+\infty[$, $f(a,y)$ has no repeated root, as $g(a)\neq 0$. Thereby, the curve $\mathcal{C}$ has no node in the region $\{(x,y)\in\mathbb{R}^2\mid x>M\}$. Hence, the infinite branches of $\mathcal{C}$ in the region $\{(x,y)\in\mathbb{R}^2\mid x>M\}$ are mutually disjoint.

Likewise, the part of $\mathcal{C}$ in the region $\{(x,y)\in\mathbb{R}^2\mid x<-M\}$ is a finite number of disjoint infinite branches. This completes the proof. $\square$

According to Proposition 2.1, the infinite branches of $\mathcal{C}$ in the region $\{(x,y)\in\mathbb{R}^2\mid x>M\}$ are called the right-branches of $\mathcal{C}$, and the infinite branches of $\mathcal{C}$ in the region $\{(x,y)\in\mathbb{R}^2\mid x<-M\}$ are called the left-branches of $\mathcal{C}$. Caution: It is possible that the number of the right-branches (or left-branches) of $\mathcal{C}$ is 0.

As a result on the numbers of the right- and left-branches of $\mathcal{C}$, we can establish the following

Proposition 2.2. *Let $\mathcal{C}$ be as in Proposition* 2.1. *Then the number of the right- and left-branches of $\mathcal{C}$ is even.*

Proof. Denote by M such a positive number as obtained in the proof of Proposition 2.1, and write r for the number of the right- and left-branches of $\mathcal{C}$. By Proposition 2.1 and its proof, r is just the number of the roots of the polynomial $f(M,y)f(-M,y)$ in $\mathbb{R}$. Since the roots of $f(M,y)f(-M,y)$ in $\mathbb{R}(\sqrt{-1})\setminus\mathbb{R}$ appear as $\mathbb{R}$-conjugate pairs, r has the same parity as the degree of $f(M,y)f(-M,y)$. Observe that the degree of $f(M,y)f(-M,y)$ is even. Hence, r is even. This completes the proof. $\square$

Now, we proceed to seek the infinite branches of $\mathcal{C}$ other than the right- and left-branches for a curve $\mathcal{C}$ in $\mathbb{R}^2$ defined by the polynomial equation $f(x,y)=0$. According to Proposition 2.1, these infinite branches of $\mathcal{C}$ must lie in the strip region $\{(x,y)\in\mathbb{R}^2\mid -M<x<M\}$, where M is such a positive number as obtained in the proof of Proposition 2.1.

Proposition 2.3. *Let $\mathcal{C}$ be a curve in $\mathbb{R}^2$ defined by the polynomial equation $f(x, y) = 0$, and M is a positive number as in Proposition* 2.1. *Then there exists a positive number $N \in \mathbb{R}$ such that the part of $\mathcal{C}$ in the region $\{(x, y) \in \mathbb{R}^2 \mid -M < x < M,\ and\ y > N\}$ is a finite number of disjoint infinite branches, and so is the part of $\mathcal{C}$ in the region $\{(x, y) \in \mathbb{R}^2 \mid -M < x < M,\ and\ y < -N\}$.*

Proof. Without loss of generality, we may assume that f is squarefree as a polynomial in $\mathbb{R}[x, y]$. Write $g(y)$ for the resultant of f and $\frac{\partial f}{\partial y}$ relative to x. Then $g(y)$ is a non-zero polynomial in $\mathbb{R}[y]$.

Consider the polynomial $f(x, \eta)$ in $R[x]$, and all the roots of $f(x, \eta)$ in the open interval $]-M, M[_R$ are denoted as follows:

$$\alpha_1 < \cdots < \alpha_m,$$

where m is a non-negative integer.

Let θ be any element in R such that $\eta < \theta$. Then θ is obviously positive and infinitely large over $\mathbb{R}$. Thereby, when both $\mathbb{R}(\eta)$ and $\mathbb{R}(\theta)$ are regarded as two ordered subfields of R, there is an order-preserving $\mathbb{R}$-isomorphism σ such that $\sigma(\eta) = \theta$. Obviously, R is the real closure of both $\mathbb{R}(\eta)$ and $\mathbb{R}(\theta)$. By Lemma 3.8 in [12] and Zorn's Lemma, it may be proved that σ can be extended to an order-preserving automorphism of R, denoted still by σ. Thereby, all the roots of $f(x, \theta)$ in $]-M, M[_R$ are just as follows: $\sigma(\alpha_1) < \cdots < \sigma(\alpha_m)$. Clearly, $g(\theta) \neq 0$. Hence, the following sentence is valid in R:

$$\exists Y\Bigg(Y > 0 \wedge \forall y\Bigg(y > Y \longrightarrow \Bigg(g(y) \neq 0 \wedge \exists(x_1, \dots, x_m)\Bigg(\forall x\Bigg(f(x, y) = 0 \\ \longrightarrow \bigvee_{1\leqslant i\leqslant m} x = x_i\Bigg) \wedge \bigwedge_{1\leqslant i\leqslant m} f(x_i, y) = 0 \wedge \bigwedge_{1\leqslant i<j\leqslant m} x_i \neq x_j \\ \wedge \bigwedge_{1\leqslant i\leqslant m} -M < x_i < M\Bigg)\Bigg)\Bigg)\Bigg),$$

where $\to$ is used for implications.

Observe that all the constants in the above sentence belong to $\mathbb{R}$. By the familiar Transfer Principle for real closed fields (see Proposition 5.2.3 in [2]), the above sentence is also valid in $\mathbb{R}$. Hence, there exists a positive number N such that the following sentence is valid in $\mathbb{R}$:

$$\forall y\Bigg(y > N \longrightarrow \Bigg(g(y) \neq 0 \wedge \exists(x_1, \dots, x_m)\Bigg(\forall x\Bigg(f(x, y) = 0 \longrightarrow \bigvee_{1\leqslant i\leqslant m} x = x_i\Bigg) \\ \wedge \bigwedge_{1\leqslant i\leqslant m} f(x_i, y) = 0 \wedge \bigwedge_{1\leqslant i<j\leqslant m} x_i \neq x_j \wedge \bigwedge_{1\leqslant i\leqslant m} -M < x_i < M\Bigg)\Bigg)\Bigg),$$

where $\to$ is used for implications.

This sentence implies that $g(b) \neq 0$ and the polynomial $f(x, b)$ has exactly m roots in $]-M, M[$ for all $b \in]N, +\infty[$. Then, for every $b \in]N, +\infty[$, the roots of $f(x, b)$ in $]-M, M[$ may be denoted as follows:

$$\psi_1(b) < \psi_2(b) < \cdots < \psi_m(b),$$

where $\psi_i(y)$ is a function in the variable y defined on $]N, +\infty[$, $i = 1, \ldots, m$.

By theorem in [3] or Theorem 1.4 in [10], the function $\psi_i(y)$ is continuous, $i = 1, \ldots, m$. This implies that the part of $\mathcal{C}$ in the region $\{(x, y) \in \mathbb{R}^2 \mid -M < x < M,\ \text{and } y > N\}$ consists of the infinite branches as follows: $x = \psi_1(y), \ldots, x = \psi_m(y)$, where y varies over $]N, +\infty[$.

Moreover, for all $b \in]N, +\infty[$, $f(x, b)$ has no repeated root, as $g(b) \neq 0$. Thereby, the curve $\mathcal{C}$ has no node in the region $\{(x, y) \in \mathbb{R}^2 \mid -M < x < M,\ \text{and } y > N\}$. Hence, the infinite branches of $\mathcal{C}$ in the region $\{(x, y) \in \mathbb{R}^2 \mid -M < x < M,\ \text{and } y > N\}$ are mutually disjoint.

Likewise, the part of $\mathcal{C}$ in the region $\{(x, y) \in \mathbb{R}^2 \mid -M < x < M,\ \text{and } y < -N\}$ is a finite number of disjoint infinite branches. This completes the proof. □

According to Proposition 2.3, the infinite branches of $\mathcal{C}$ in the region $\{(x, y) \in \mathbb{R}^2 \mid -M < x < M,\ \text{and } y > N\}$ are called the up-branches of $\mathcal{C}$, and the infinite branches of $\mathcal{C}$ in the region $\{(x, y) \in \mathbb{R}^2 \mid -M < x < M,\ \text{and } y < -N\}$ are called the down-branches of $\mathcal{C}$.

In order to investigate the numbers of the up-branches and down-branches of $\mathcal{C}$, we need the following

Lemma 2.4. *Let $[a_i \mid i = 1, \ldots, s]$, $[b_i \mid i = 1, \ldots, s]$ be two lists of non-zero elements in R, and $e_i \in \{-1, 1\}$, $i = 1, \ldots, s$. If V_1, V_2, V_3 and V_4 are the numbers of sign variations in the lists $[a_i \mid i = 1, \ldots, s]$, $[b_i \mid i = 1, \ldots, s]$, $[e_i a_i \mid i = 1, \ldots, s]$ and $[e_i b_i \mid i = 1, \ldots, s]$ respectively, then $V_1 - V_2$ has the same parity as $V_3 - V_4$.*

Proof. By the definition of sign variations, we have

$$V_1 = \sum_{i=1}^{s-1} \frac{1}{2}\bigl(1 - \operatorname{sign}_R(a_i a_{i+1})\bigr),$$

$$V_2 = \sum_{i=1}^{s-1} \frac{1}{2}\bigl(1 - \operatorname{sign}_R(b_i b_{i+1})\bigr),$$

$$V_3 = \sum_{i=1}^{s-1} \frac{1}{2}\bigl(1 - \operatorname{sign}_R(e_i e_{i+1} a_i a_{i+1})\bigr),$$

$$V_4 = \sum_{i=1}^{s-1} \frac{1}{2}\bigl(1 - \operatorname{sign}_R(e_i e_{i+1} b_i b_{i+1})\bigr).$$

Then we have

$$\begin{aligned} V_3 - V_4 &= \sum_{i=1}^{s-1} \frac{1}{2}\bigl(\operatorname{sign}_R(e_i e_{i+1} b_i b_{i+1}) - \operatorname{sign}_R(e_i e_{i+1} a_i a_{i+1})\bigr), \\ &= \sum_{i=1}^{s-1} \frac{1}{2} e_i e_{i+1}\bigl(\operatorname{sign}_R(b_i b_{i+1}) - \operatorname{sign}_R(a_i a_{i+1})\bigr). \end{aligned}$$

Obviously, the following congruences hold:

$$e_i e_{i+1} \equiv 1 \pmod 2, \quad i = 1, \ldots, s-1.$$

Since

$$\frac{1}{2}\big(\operatorname{sign}_R(b_i b_{i+1}) - \operatorname{sign}_R(a_i a_{i+1})\big)$$

is an integer for $i = 1, \ldots, s-1$, we have

$$\frac{1}{2} e_i e_{i+1}\big(\operatorname{sign}_R(b_i b_{i+1}) - \operatorname{sign}_R(a_i a_{i+1})\big) \equiv \frac{1}{2}\big(\operatorname{sign}_R(b_i b_{i+1}) - \operatorname{sign}_R(a_i a_{i+1})\big) \pmod 2,$$

where $i = 1, \ldots, s-1$.

So we have

$$V_3 - V_4 \equiv \sum_{i=1}^{s-1} \frac{1}{2}\big(\operatorname{sign}_R(b_i b_{i+1}) - \operatorname{sign}_R(b_i b_{i+1})\big) \pmod 2,$$

i.e. $V_3 - V_4 \equiv V_1 - V_2 \pmod 2$. This completes the proof. □

Proposition 2.5. *Let $\mathcal{C}$ be as in Proposition* 2.1. *Then the number of the up- and down-branches of $\mathcal{C}$ is even.*

Proof. Denote by M such a positive number as obtained in the proof of Proposition 2.1, and write r for the number of the up- and down-branches of $\mathcal{C}$. By Proposition 2.3 and its proof, r is just the sum of the roots of $f(x, \eta)$ and $f(x, -\eta)$ in $]-M, M[_R$.

Let $f_0 := f,\ f_1 := \frac{\partial f}{\partial x},\ \ldots,\ f_s$ be a Sturm sequence of f relative to x. Then $f_0(x, \eta)$, $f_1(x, \eta), \ldots, f_s(x, \eta)$ is a Sturm sequence of $f(x, \eta)$ as an univariate polynomial over R. By Sturm's Theorem, the number of the roots of the polynomial $f(x, \eta)$ in $]-M, M[_R$ is $V_1 - V_2$, where V_1 and V_2 are the numbers of sign variations in the lists $[f_i(-M, \eta) \mid i = 0, \ldots, s]$ and $[f_i(M, \eta) \mid i = 0, \ldots, s]$ respectively. Likewise, the number of the roots of the polynomial $f(x, -\eta)$ in $]-M, M[_R$ is $V_3 - V_4$, where V_3 and V_4 are the numbers of sign variations in the lists $[f_i(-M, -\eta) \mid i = 0, \ldots, s]$ and $[f_i(M, -\eta) \mid i = 0, \ldots, s]$ respectively. By Proposition 2.3 and its proof, $r = (V_1 - V_2) + (V_3 - V_4)$.

Write d_i for the degree of f_i as a polynomial in y over $\mathbb{R}[x]$, $i = 0, \ldots, s$. Since neither M nor $-M$ is a root of $\operatorname{lcoeff}(f_i, y)$ for $i = 0, \ldots, s$, d_i is the degree of both $f_i(M, y)$ and $f_i(-M, y)$. Denote respectively by a_i and b_i the leading coefficients of $f_i(-M, \eta)$ and $f_i(M, \eta)$ as two polynomials in η, $i = 0, \ldots, s$. By the structure of the ordering of $\mathbb{R}[\eta]$, we have $\operatorname{sign}_R(f_i(-M, \eta)) = \operatorname{sign}(a_i)$ and $\operatorname{sign}_R(f_i(M, \eta)) = \operatorname{sign}(b_i)$, $i = 0, \ldots, s$. Hence, the numbers of sign variations in the lists $[a_i \mid i = 0, \ldots, s]$ and $[b_i \mid i = 0, \ldots, s]$ are V_1 and V_2 respectively. Observe that the leading coefficients of $f_i(-M, -\eta)$ and $f_i(M, -\eta)$ as two polynomials in η are $(-1)^{d_i} a_i$ and $(-1)^{d_i} b_i$ respectively, $i = 0, \ldots, s$. Thereby, the numbers of sign variations in the lists $[(-1)^{d_i} a_i \mid i = 0, \ldots, s]$ and $[(-1)^{d_i} b_i \mid i = 0, \ldots, s]$ are V_3 and V_4 respectively. By Lemma 2.4, r $(= (V_1 - V_2) + (V_3 - V_4))$ is an even number. The proof is complete. □

In Proposition 2.3 and its proof, the positive number M is involved for determining of the numbers of up-branches and down-branches. Actually, we may establish the result as follows:

Proposition 2.6. *Let $\mathcal{C}$ be a curve in $\mathbb{R}^2$ defined by the polynomial equation $f(x, y) = 0$, and M is such a positive number as required in Proposition 2.2. Then, for two natural numbers s and t, the following statements are equivalent*:

(1) *s and t are the numbers of up-branches and down-branches of $\mathcal{C}$ respectively.*
(2) *s and t are the numbers of roots of $f(x, \eta)$ and $f(x, -\eta)$ in $]-M, M[_R$ respectively.*
(3) *s and t are the numbers of bounded roots of $f(x, \eta)$ and $f(x, -\eta)$ in R respectively.*

Proof. The equivalence between statement (1) and statement (2) follows from Proposition 2.3 and its proof. Hence, it remains to prove that all bounded roots of $f(x, \eta)$ and $f(x, -\eta)$ lie in $]-M, M[_R$.

Suppose that $f(x, \eta)$ has some bounded root α such that $\alpha \notin]-M, M[_R$. Without loss of generality, we may assume $\alpha > M$. Put $a := \pi(\alpha)$. Then $a \geqslant M$, because π is order-preserving. Since $\alpha - a \in I$ is infinitesimal over $\mathbb{R}$, we have $\alpha - a < 1$. Hence $M < \alpha < a + 1$.

According to the proof of Proposition 2.1, we may assume that the right-branches of $\mathcal{C}$ are defined by the equations $y = \phi_i(x)$, where $\phi_i(x)$ is a continuous function on $[M, +\infty[$, $i = 1, \ldots, m$. By a familiar fact about continuous functions, there is a positive number $d \in \mathbb{R}$ such that

$$\phi_i(x) < d \quad \text{for all } i \in \{1, \ldots, m\} \text{ and for all } x \in [M, a + 1].$$

By Proposition 2.1, the following sentence is valid in R:

$$\forall(x, y)\big(\big(f(x, y) = 0 \wedge M \leqslant x \leqslant a + 1\big) \longrightarrow y < d\big).$$

By Transfer Principle for real closed fields, this sentence also is valid in R. Observe that $M < \alpha \leqslant a + 1$ and $f(\alpha, \eta) = 0$. Then we have $\eta < d$. This is impossible, because η is infinitely large over $\mathbb{R}$. Therefore, all bounded roots of $f(x, \eta)$ lie in $]-M, M[_R$. It may be similarly proved that all bounded roots of $f(x, -\eta)$ lie in $]-M, M[_R$. This completes the proof. □

3. Determination of asymptotes

In this section, we proceed to determine the asymptotes of a real plane algebraic curve $\mathcal{C}$. According to Propositions 2.1 and 2.3, the right-branches and the left-branches of $\mathcal{C}$ may be respectively numbered in the order from below to above, and the up-branches and the down-branches of $\mathcal{C}$ may be respectively numbered in the order from left to right.

First, we consider the up-branches and the down-branches.

Proposition 3.1. *Let $\mathcal{C}$ be a curve in $\mathbb{R}^2$ defined by the polynomial equation $f(x, y) = 0$. Then every up-branch and every down-branch of $\mathcal{C}$ has an asymptote, and the following statements are true*:

(1) *If the polynomial $f(x, \eta)$ has the bounded roots in R as follows*:

$$\alpha_1 < \cdots < \alpha_s,$$

then $\mathcal{C}$ has s up-branches with asymptotes $x = \pi(\alpha_i)$, $i = 1, \ldots, s$.

(2) *If the polynomial* $f(x, -\eta)$ *has the bounded roots in* R *as follows*:

$$\beta_1 < \cdots < \beta_t,$$

then $\mathcal{C}$ *has* t *down-branches with asymptotes* $x = \pi(\beta_i)$, $i = 1, \ldots, t$.

Proof. It suffices to prove statement (1). Statement (2) may be similarly proved.

For any element θ in R with $\eta < \theta$, θ is infinitely large over $\mathbb{R}$. As is indicated in the proof of Proposition 2.3, there is an order-preserving $\mathbb{R}$-automorphism σ of R such that $\sigma(\eta) = \theta$. Then, all the bounded roots of $f(x, \theta)$ in R are just as follows:

$$\sigma(\alpha_1) < \cdots < \sigma(\alpha_s).$$

Let $\epsilon \in \mathbb{R}$ be any positive number, and put $a_i = \pi(\alpha_i)$, $i = 1, \ldots, s$. Since $a_i - \alpha_i \in I$ is infinitesimal over $\mathbb{R}$ for $i = 1, \ldots, s$, we have $-\epsilon < a_i - \alpha_i < \epsilon$, $i = 1, \ldots, s$. Observing that σ is an order-preserving $\mathbb{R}$-automorphism of R, we have $-\epsilon < a_i - \sigma(\alpha_i) < \epsilon$, $i = 1, \ldots, s$. Thereby, the following sentence is valid in R:

$$\exists Y\Biggl(Y > 0 \wedge \forall(y, x_1, \ldots, x_s)\Biggl(\Biggl(Y < y \wedge x_1 < \cdots < x_s \wedge \bigwedge_{1 \leqslant i \leqslant s} f(x_i, y) = 0\Biggr) \longrightarrow \bigwedge_{1 \leqslant i \leqslant s} -\epsilon < a_i - x_i < \epsilon\Biggr)\Biggr),$$

where $\rightarrow$ is used for implications.

Observe that all the constants in the above sentence belong to $\mathbb{R}$. By the Transfer Principle, the above sentence also is valid in $\mathbb{R}$. Hence, there exists a positive number Δ such that the following sentence is valid in $\mathbb{R}$:

$$\forall(y, x_1, \ldots, x_s)\Biggl(\Biggl(\Delta < y \wedge x_1 < \cdots < x_s \wedge \bigwedge_{1 \leqslant i \leqslant s} f(x_i, y) = 0\Biggr) \longrightarrow \bigwedge_{1 \leqslant i \leqslant s} -\epsilon < a_i - x_i < \epsilon\Biggr)$$

where $\rightarrow$ is used for implications.

According to Proposition 2.3 and its proof, all the up-branches of $\mathcal{C}$ are defined by the equations $x = \psi_i(y)$, $i = 1, \ldots, s$, where $\psi_1(y), \ldots, \psi_s(y)$ are s continuous functions on $]N, +\infty[$ for some positive number N, and $\psi_1(y) < \cdots < \psi_s(y)$ for all $y \in]N, +\infty[$. By the final sentence, we have $|a_i - \psi_i(y)| < \epsilon$, $i = 1, \ldots, s$, whenever $y > \Delta$. This implies that the vertical line $x = a_i$ is the asymptote of the branch $x = \psi_i(y)$, $i = 1, \ldots, s$. The proof is complete. $\square$

Theorem 3.2. *Let* $\mathcal{C}$ *be as above. Then we can effectively compute the asymptotes of the up- and down-branches of* $\mathcal{C}$.

Proof. According to the preceding results, we implement the computations as follows:

(1) Extract the leading coefficient $h(x)$ (i.e. lcoeff$(f(x,\eta),\eta)$) of $f(x,\eta)$ as a polynomial in η. Moreover, by real root isolation for polynomials, compute a set of isolating intervals $]c_1,d_1[,\ldots,]c_s,d_s[$ for $h(x)$.

Note: the leading coefficient of $f(x,-\eta)$ as a polynomial in η is either $h(x)$ or $-h(x)$.

(2) Compute such a Sturm sequence of $f(x,y)$ relative to x as follows:

$$f_0 = f(x,y), \quad f_1 = \frac{\partial f}{\partial x}, \quad f_2, \quad \ldots, \quad f_m,$$

where $f_i \in \mathbb{R}[x,y]$, $i=0,\ldots,m$.

Thereby, we may obtain a Sturm sequence of $f(x,\eta)$ as follows:

$$f_0(x,\eta), \quad f_1(x,\eta), \quad f_2(x,\eta), \quad \ldots, \quad f_m(x,\eta),$$

and obtain a Sturm sequence of $f(x,-\eta)$ as follows:

$$f_0(x,-\eta), \quad f_1(x,-\eta), \quad f_2(x,-\eta), \quad \ldots, \quad f_m(x,-\eta).$$

(3) For $k=1,\ldots,s$, by Sturm's Theorem, count the numbers q_k, q'_k of roots of $f(x,\eta)$, $f(x,-\eta)$ in $]c_k,d_k[_R$ respectively.

Then, we have the following assertions:

(A1) The number of up-branches of $\mathcal{C}$ is $\sum_{k=1}^{s} q_k$, and in the order from left to right these up-branches have their asymptotes as follows:

$$\overbrace{x=a_1,\ldots,x=a_1}^{q_1}, \quad \overbrace{x=a_2,\ldots,x=a_2}^{q_2}, \quad \ldots, \quad \overbrace{x=a_s,\ldots,x=a_s}^{q_s},$$

where $a_k = (h(x); c_k, d_k)$, $k=1,\ldots,s$.

(A2) The number of down-branches of $\mathcal{C}$ is $\sum_{k=1}^{s} q'_k$, and in the order from left to right these down-branches have their asymptotes as follows:

$$\overbrace{x=a_1,\ldots,x=a_1}^{q'_1}, \quad \overbrace{x=a_2,\ldots,x=a_2}^{q'_2}, \quad \ldots, \quad \overbrace{x=a_s,\ldots,x=a_s}^{q'_s},$$

where $a_k = (h(x); c_k, d_k)$, $k=1,\ldots,s$.

Assertion (A1) may be verified as follows:

Let $\alpha_1 < \cdots < \alpha_m$ are all the bounded roots of $f(x,\eta)$ in R. By Proposition 3.1, in the order from left to right the up-branches of $\mathcal{C}$ have their asymptotes as follows:

$$x = \pi(\alpha_1), \quad \ldots, \quad x = \pi(\alpha_m).$$

By Theorem 1.4 and its proof, $\{\pi(\alpha_i) \mid i=1,\ldots,m\} = \{a_k \mid k=1,\ldots,s\}$, and for $i \in \{1,\ldots,m\}$ and $k \in \{1,\ldots,s\}$, $\pi(\alpha_i) = a_k$ if and only if $\alpha_i \in\,]c_k,d_k[$. Observe that $a_1 < \cdots < a_s$. Then assertion (A1) is verified.

Assertion (A2) may be similarly verified, and the proof is complete. □

Proposition 3.3. *Let the notations be as in Proposition* 3.1. *Then the number of up- and down-branches with the same asymptote is even.*

Proof. According to Theorem 3.2 and its proof, the number of up- and down-branches with asymptote $x = a$ is equal to the sum of bounded roots of $f(x,\eta)$ and $f(x,-\eta)$ in $]c,d[_R$, where $c,d \in \mathbb{R}$ such that $]c,d[$ is an isolating interval containing a.

Let $f_0 := f$, $f_1 := \frac{\partial f}{\partial x}, \ldots, f_s$ be a Sturm sequence of f relative to x. Then $f_0(x,\eta)$, $f_1(x,\eta), \ldots, f_s(x,\eta)$ is a Sturm sequence of $f(x,\eta)$ as an univariate polynomial over R. By Sturm's Theorem, the number of the roots of the polynomial $f(x,\eta)$ in $]c,d[_R$ is $V_1 - V_2$, where V_1 and V_2 are the numbers of sign variations in the lists $[f_i(c,\eta) \mid i = 0,\ldots,s]$ and $[f_i(d,\eta) \mid i = 0,\ldots,s]$ respectively. Likewise, the number of the roots of the polynomial $f(x,-\eta)$ in $]c,d[_R$ is $V_3 - V_4$, where V_3 and V_4 are the numbers of sign variations in the lists $[f_i(c,-\eta) \mid i = 0,\ldots,s]$ and $[f_i(d,-\eta) \mid i = 0,\ldots,s]$ respectively. Thereby, the number of up- and down-branches with asymptote $x - a$ is $(V_1 - V_2) + (V_3 - V_4)$.

Write d_i for the degree of f_i as a polynomial in y over $\mathbb{R}[x]$, $i = 0,\ldots,s$. Since both c and d may be arbitrarily chosen to approach a, we can assume that neither c nor d is a root of $\operatorname{lcoeff}(f_i, y)$ for $i = 0,\ldots,s$. In this case, d_i is the degree of both $f_i(c,y)$ and $f_i(d,y)$ for $i = 0,\ldots,s$. Denote respectively by a_i and b_i the leading coefficients of $f_i(c,\eta)$ and $f_i(d,\eta)$ as two polynomials in η, $i = 0,\ldots,s$. By the structure of the ordering of $\mathbb{R}(\eta)$, we have $\operatorname{sign}_R(f_i(c,\eta)) = \operatorname{sign}(a_i)$ and $\operatorname{sign}_R(f_i(d,\eta)) = \operatorname{sign}(b_i)$, $i = 0,\ldots,s$. Hence, the numbers of sign variations in the lists $[a_i \mid i = 0,\ldots,s]$ and $[b_i \mid i = 0,\ldots,s]$ are V_1 and V_2 respectively. Observe that the leading coefficients of $f_i(c,-\eta)$ and $f_i(d,-\eta)$ as two polynomials in η are $(-1)^{d_i}a_i$ and $(-1)^{d_i}b_i$ respectively, $i = 0,\ldots,s$. Thereby, the numbers of sign variations in the lists $[(-1)^{d_i}a_i \mid i = 0,\ldots,s]$ and $[(-1)^{d_i}b_i \mid i = 0,\ldots,s]$ are V_3 and V_4 respectively. By Lemma 2.4, $(V_1 - V_2) + (V_3 - V_4)$ is an even number. This completes the proof. □

As an application of Proposition 3.3, we proceed to treat the following example. In this example and its successors, our computations are implemented with the aid of the computer algebra system **Maple**. For the details of **Maple**, refer to [6].

Example 1. Let $\mathcal{C}$ be a curve in $\mathbb{R}^2$ defined by the following equation:

$$1 + 2x - y + x^3y + x^2y^3 - y^4x + y^2x - y^4 + 2x^2 + x^3 - y^3 = 0.$$

Compute all the asymptotes of the up- and down-branches of $\mathcal{C}$.

Process of Computing. Put $f := 2 + 2x - y + x^3y + x^2y^3 - y^4x + y^2x - y^4 + 2x^2 + x^3 - y^3$. According to Theorem 3.2 and its proof, we implement the computations as follows:

(1) Regarding $f(x,\eta)$ as a polynomial in η over $\mathbb{R}[x]$, we extract the leading coefficient of $f(x,\eta)$ as follows:

$$h(x) = -x - 1.$$

Moreover, an isolating interval $]-2,0[$ is computed for $h(x)$.

(2) Compute such a Sturm sequence of $f(x,y)$ relative to x as follows:

$$f_0 = 2 + 2x - y + x^3y + x^2y^3 - y^4x + y^2x - y^4 + 2x^2 + x^3 - y^3,$$
$$f_1 = 2 + 3x^2y + 2y^3x - y^4 + y^2 + 4x + 3x^2,$$

$$f_2 = \frac{2}{9}\big(y^6 + 3y^5 + 3y^4 + y^3 - 3y^2 - 6y - 2\big)(y+1)x \\ - \frac{1}{9}\big(y^7 - 10y^5 - 16y^4 - 11y^3 - 11y^2 + 9y + 14\big)(y+1),$$

$$f_3 = \frac{9}{4}\big(y^{15} + 9y^{14} + 32y^{13} + 52y^{12} + 14y^{11} - 108y^{10} - 260y^9 - 350y^8 \\ - 279y^7 - 37y^6 + 177y^5 + 221y^4 + 151y^3 + 13y^2 - 80y - 44\big).$$

Thereby, respective Sturm sequences of $f(x,\eta)$ and $f(x,-\eta)$ are as follows:

$$f_0(x,\eta), \quad f_1(x,\eta), \quad f_2(x,\eta), \quad f_3(x,\eta); \\ f_0(x,-\eta), \quad f_1(x,-\eta), \quad f_2(x,-\eta), \quad f_3(x,-\eta).$$

(3) By computation, we have

$$\begin{aligned} &\mathrm{lcoeff}\big(f_0(-2,\eta),\eta\big) = 1, && \mathrm{lcoeff}\big(f_1(-2,\eta),\eta\big) = -1, \\ &\mathrm{lcoeff}\big(f_2(-2,\eta),\eta\big) = -\frac{1}{9}, && \mathrm{lcoeff}\big(f_3(-2,\eta),\eta\big) = \frac{4}{9}; \\ &\mathrm{lcoeff}\big(f_0(0,\eta),\eta\big) = -1, && \mathrm{lcoeff}\big(f_1(0,\eta),\eta\big) = -1, \\ &\mathrm{lcoeff}\big(f_2(0,\eta),\eta\big) = -\frac{1}{9}, && \mathrm{lcoeff}\big(f_3(0,\eta),\eta\big) = \frac{4}{9}. \end{aligned}$$

Since the sign variations in the lists

$$\big[f_0(-2,\eta), f_1(-2,\eta), f_2(-2,\eta), f_3(-2,\eta)\big], \\ \big[f_0(0,\eta), f_1(0,\eta), f_2(0,\eta), f_3(,\eta)\big]$$

are the same as in $[1, -1, -\frac{1}{9}, \frac{4}{9}]$, $[-1, -1, -\frac{1}{9}, \frac{4}{9}]$ respectively, the numbers of sign variations in the lists

$$\big[f_0(-2,\eta), f_1(-2,\eta), f_2(-2,\eta), f_3(-2,\eta)\big], \\ \big[f_0(0,\eta), f_1(0,\eta), f_2(0,\eta), f_3(,\eta)\big]$$

are 2, 1 respectively. By Sturm's Theorem, $f(x,\eta)$ has the only root in $]-2,0[_R$.

Likewise, we have

$$\begin{aligned} &\mathrm{lcoeff}\big(f_0(-2,-\eta),\eta\big) = 1, && \mathrm{lcoeff}\big(f_1(-2,-\eta),\eta\big) = -1, \\ &\mathrm{lcoeff}\big(f_2(-2,-\eta),\eta\big) = -\frac{1}{9}, && \mathrm{lcoeff}\big(f_3(-2,-\eta),\eta\big) = -\frac{4}{9}; \\ &\mathrm{lcoeff}\big(f_0(0,-\eta),\eta\big) = -1, && \mathrm{lcoeff}\big(f_1(0,-\eta),\eta\big) = -1, \\ &\mathrm{lcoeff}\big(f_2(0,-\eta),\eta\big) = -\frac{1}{9}, && \mathrm{lcoeff}\big(f_3(0,-\eta),\eta\big) = -\frac{4}{9}. \end{aligned}$$

This implies that the numbers of sign variations in the lists

$$\big[f_0(-2,-\eta), f_1(-2,-\eta), f_2(-2,-\eta), f_3(-2,-\eta)\big],$$
$$\big[f_0(0,-\eta), f_1(0,-\eta), f_2(0,-\eta), f_3(0,-\eta)\big]$$

are 1, 0 respectively. By Sturm's Theorem, $f(x,-\eta)$ has the only root in $]-2,0[_R$.

Obviously, $(h(x); -2, 0) = -1$. According to Theorem 3.2 and its proof, the curve $\mathcal{C}$ has the only up-branch and the only down-branch, and the two branches have the same asymptote $x = -1$.

Now, we proceed to investigate the asymptotes of the right- and left-branches of $\mathcal{C}$.

Lemma 3.4. *Let $\mathcal{C}$ be as above. Then the asymptotes of the right- and left-branches of $\mathcal{C}$ do not encompass vertical lines in $\mathbb{R}^2$.*

Proof. Let $\mathcal{B}$ be any right-branch of $\mathcal{C}$. According to Proposition 2.1 and its proof, $\mathcal{B}$ may be defined by $y = \phi(x)$, where $\phi(x)$ is a continuous function on $[M, +\infty[$ for some positive number M. For any vertical line $x = a$ with $a \in \mathbb{R}$, it is easy to see that the distance between the line $x = a$ and the point $(x, \phi(x))$ of $\mathcal{B}$ is greater than 1 whenever $x > \max\{a+1, M\}$. Thereby, it is impossible that the line $x = a$ is the asymptote of $\mathcal{B}$. Hence, the asymptotes of the right-branches of $\mathcal{C}$ do not encompass vertical lines in $\mathbb{R}^2$. It may be similarly proved that the asymptotes of the left-branches of $\mathcal{C}$ do not encompass vertical lines in $\mathbb{R}^2$. This completes the proof. $\square$

Remark. According to Lemma 3.4, if some right- or left-branch of $\mathcal{C}$ has its asymptote, then the slope of this asymptote is a real number, i.e. the equation of this asymptote is of the form $y = ax + b$ where $a, b \in \mathbb{R}$.

By Lemma 3.4, Proposition 3.3 may be further improved as follows:

Proposition 3.5. *Let the notations be as in Proposition* 3.3. *Then the number of infinite branches of $\mathcal{C}$ with the same asymptote is even.*

Proof. Let ℓ be any asymptote of $\mathcal{C}$. By a suitable rotation of axes, ℓ may be converted into a vertical line ℓ'. By Lemma 3.4, all the branches of $\mathcal{C}$ with asymptote ℓ are exactly converted into all the up- and down-branches with asymptote ℓ'. According to Proposition 3.3, the number of these branches of $\mathcal{C}$ must be even. The proof is complete. $\square$

Lemma 3.6. *Let $\mathcal{C}$ be as above and z a new variable. If a is the slope of the asymptote of a right-branch* (*or left-branch*) *of $\mathcal{C}$, then there exists a bounded root α of $f(\eta, \eta z)$* (*or $f(-\eta, -\eta z)$*) *in R such that $\pi(\alpha) = a$.*

Proof. We consider only the case when a is the slope of the asymptote of a right-branch of $\mathcal{C}$. If a is the slope of the asymptote of a left-branch of $\mathcal{C}$, the conclusion may be deduced similarly.

By the hypothesis, we may assume that the equation of this asymptote is of the form $y = ax + b$ where $b \in \mathbb{R}$. Let ϵ be any positive number. By the definition of asymptotes, there is a positive number Δ such that following sentence is valid in $\mathbb{R}$:

$$\forall x\big(x > \Delta \longrightarrow \exists y\big(f(x,y) = 0 \wedge -\epsilon < y - ax - b < \epsilon\big)\big).$$

By the Transfer Principle, the above sentence also is valid in R. By the structure of the ordering of R, we have $\eta > \Delta$. According to this sentence, the polynomial $f(\eta, y)$ have a root ξ in R such that $-\epsilon < \xi - a\eta - b < \epsilon$. Then $-\frac{\epsilon}{\eta} < \frac{\xi}{\eta} - a - \frac{b}{\eta} < \frac{\epsilon}{\eta}$. So we have $\pi(-\frac{\epsilon}{\eta}) \leqslant \pi(\frac{\xi}{\eta} - a - \frac{b}{\eta}) \leqslant \pi(\frac{\epsilon}{\eta})$, and $0 \leqslant \pi(\frac{\xi}{\eta}) - a \leqslant 0$. Hence $\pi(\frac{\xi}{\eta}) = a$. Put $\alpha := \frac{\xi}{\eta}$. Then α is a bounded root of $f(\eta, \eta z)$ in R such that $\pi(\alpha) = a$. This completes the proof. □

Lemma 3.7. *Let $\mathcal{C}$ be as above, $a, b \in \mathbb{R}$, and w a new variable. Then the line $y = ax + b$ is the asymptote of a right-branch (or left-branch) of $\mathcal{C}$ if and only if $f(\eta, a\eta + w)$ (or $f(-\eta, -a\eta + w)$) has a bounded root β in R such that $\pi(\beta) = b$.*

Proof. *Necessity*: Now we only assume that the line $y = ax + b$ is the asymptote of a right-branch of $\mathcal{C}$. For the assumption that the line $y = ax + b$ is the asymptote of a left-branch of $\mathcal{C}$, the argument is similar.

According to the proof of Lemma 3.6, the polynomial $f(\eta, y)$ has a root ξ such that $-\epsilon < \xi - a\eta - b < \epsilon$ for any positive number ϵ. Put $\beta := \xi - a\eta$. Then β is a bounded root of $f(\eta, a\eta + w)$ in R. Moreover, for any positive number ϵ, we have $-\epsilon < \beta - b < \epsilon$, and $\beta - b \in I$. So we have $\pi(\beta) = b$. The "only if" part of Lemma 3.7 is proved.

Sufficiency: For simplicity, we only assume that $f(\eta, a\eta + w)$ has a bounded root β in R such that $\pi(\beta) = b$. For another assumption, the argument is similar.

Let $\beta_1 < \cdots < \beta_r$ be all roots of $f(\eta, a\eta + w)$ in R such that $\beta_k = \beta$ for some $k \in \{1, \ldots, r\}$. Put $\alpha_i := a\eta + \beta_i$, $i = 1, \ldots, r$. Then $\alpha_1 < \cdots < \alpha_r$ are all roots of $f(\eta, y)$ in R.

By Proposition 2.1 and its proof, for a positive number M, all the right-branches of $\mathcal{C}$ are defined by $y = \phi_1(x), \ldots, y = \phi_s(x)$ respectively, where $\phi_1(x), \ldots, \phi_s(x)$ are certain continuous functions on the interval $[M, +\infty[$ such that $\phi_1(x) < \cdots < \phi_s(x)$, and s is the number of the roots of $f(c, y)$ in $\mathbb{R}$ for every $c \in [M, +\infty[$. Observe that $\eta > M$. By the Transfer Principle, it is easy to see that s is the number of the roots of $f(\eta, y)$ in R. So we have $r = s$. In what follows, we shall prove that the line $y = ax + b$ is the asymptote of the right-branch defined by $y = \phi_k(x)$.

Let ϵ be any positive number. Since $\alpha_k - a\eta - b$ $(= \beta - b)$ is infinitesimal over $\mathbb{R}$, we have $-\epsilon < \alpha_k - a\eta - b < \epsilon$.

For any element θ in R with $\eta < \theta$, θ is infinitely large over $\mathbb{R}$. As is indicated in the proof of Proposition 2.3, there is an order-preserving $\mathbb{R}$-automorphism σ of R such that $\sigma(\eta) = \theta$. Then, all the roots of $f(\theta, y)$ in R are just as follows:

$$\sigma(\alpha_1) < \cdots < \sigma(\alpha_s).$$

Moreover, we have $-\epsilon < \sigma(\alpha_k - a\eta - b) < \epsilon$, and $-\epsilon < \sigma(\alpha_k) - a\theta - b < \epsilon$. Thereby, the following sentence is valid in R:

$$\exists X\Bigg(X > 0 \wedge \forall(x, y_1, \ldots, y_s)\Bigg(\Bigg(X < x \wedge y_1 < \cdots < y_s \wedge \bigwedge_{1 \leqslant i \leqslant s} f(x, y_i) = 0\Bigg)$$
$$\longrightarrow -\epsilon < y_k - ax - b < \epsilon\Bigg)\Bigg).$$

Observe that all the constants in the above sentence belong to $\mathbb{R}$. By the Transfer Principle, the above sentence also is valid in $\mathbb{R}$. Hence, there exists a positive number Δ such that the following sentence is valid in $\mathbb{R}$:

$$\forall(x, y_1, \ldots, y_s)\Bigg(\Bigg(\Delta < x \wedge y_1 < \cdots < y_s \wedge \bigwedge_{1 \leqslant i \leqslant s} f(x, y_i) = 0\Bigg)$$

$$\longrightarrow -\epsilon < y_k - ax - b < \epsilon\Bigg).$$

Observe that $\phi_1(x) < \cdots < \phi_s(x)$ are all the roots of $f(x, y)$ in $\mathbb{R}$ for all $x \in]M, +\infty[$. By the final sentence, we have $|\phi_k(x) - ax - b| < \epsilon$ whenever $x > \Delta$. This implies that the line $y = ax + b$ is the asymptote of the branch $y = \phi_k(x)$. This completes the proof. □

Actually, according to Lemma 3.7 and its proof, we may establish the following

Theorem 3.8. *Let $\mathcal{C}$ be as above, let $\alpha_1 < \cdots < \alpha_r$ be all the roots of $f(\eta, y)$ in R, and let $\beta_1 < \cdots < \beta_s$ be all the roots of $f(-\eta, y)$ in R. Then we have*

(1) *For $k \in \{1, \ldots, r\}$, in the order from below to above, the kth right-branch of $\mathcal{C}$ has its asymptote if and only if $\frac{\alpha_k}{\eta}$ is bounded and $\alpha_k - \pi(\frac{\alpha_k}{\eta})\eta$ is bounded. In this case, the line $y = ax + b$ is its asymptote, where $a = \pi(\frac{\alpha_k}{\eta})$, and $b = \pi(\alpha_k - a\eta)$.*
(2) *For $k \in \{1, \ldots, s\}$, in the order from below to above, the kth left-branch of $\mathcal{C}$ has its asymptote if and only if $\frac{\beta_k}{\eta}$ is bounded and $\beta_k - \pi(\frac{\beta_k}{\eta})\eta$ is bounded. In this case, the line $y = ax + b$ is its asymptote, where $a = -\pi(\frac{\beta_k}{\eta})$, and $b = \pi(\beta_k + a\eta)$.*

In what follows, we shall give an effective method to compute the asymptotes of the right- and left-branches of $\mathcal{C}$.

For a non-zero polynomial $f(x, y) \in \mathbb{R}[x, y]$ and a non-zero polynomial $g(z) \in \mathbb{R}[z]$, we may assert that $f(\eta, \eta z + w)$ and $g(z)$ have no non-constant common divisor as polynomials in $R[z, w]$, where z, w are two new variables. Indeed, if $g(z)$ and $f(\eta, \eta z + w)$ have a non-constant divisor $d(z)$ in $\mathbb{R}[z]$, $d(z)$ has at least one root b in $\mathbb{R}(\sqrt{-1})$ because $\mathbb{R}(\sqrt{-1})$ is algebraically closed. In this case, we have $f(\eta, \eta b + w) = 0$. Since η, $\eta b + w$ are algebraically independent over $\mathbb{R}(\sqrt{-1})$, $f(x, y)$ is a zero polynomial in $\mathbb{R}[x, y]$, a contradiction. Thereby, the resultant of $f(\eta, \eta z + w)$ and $g(z)$ relative to z is a non-zero polynomial in $R[w]$.

Theorem 3.9. *Let $\mathcal{C}$ be as above. Then we can effectively compute the asymptotes of the right- and left-branches of $\mathcal{C}$.*

Proof. First, we proceed to compute the asymptotes only for the right-branches of $\mathcal{C}$.

According to Lemma 3.6 and Theorem 3.8, we implement the computations as follows:

(1) By regarding $\phi(z)$ as $f(\eta, \eta z)$ in Theorem 1.4, we can compute an univariate polynomial $g(z) \in \mathbb{R}[z]$ and a finite number of open intervals $]c_1, d_1[, \ldots,]c_s, d_s[$ in $\mathbb{R}$ such that the statements (1)–(5) in Theorem 1.4 are true.
(2) Compute the resultant $\rho(w)$ of $g(z)$ and $f(\eta, \eta z + w)$ relative to z. Then $\rho(w) \in \mathbb{R}[\eta, w]$, and $\rho(w) \neq 0$.
(3) By respectively regarding z and $\Phi(z)$ as w and $\rho(w)$ in Theorem 1.4, we can compute an univariate polynomial $h(w) \in \mathbb{R}[w]$ and a finite number of open intervals $]e_1, \delta_1[, \ldots,]e_t, \delta_t[$ in $\mathbb{R}$ such that the statements in Theorem 1.4 are true.

(4) Write a_i for $(g(z); c_i, d_i)$, $i = 1, \dots, s$. For $i = 1, \dots, s-1$, by Theorem 1.5, count the number q_i of roots of $f(\eta, y)$ in $]a_i\eta + \delta_t, a_{i+1}\eta + e_1[_R$.

Moreover, count the numbers q_0 and q_s of roots of $f(\eta, y)$ in $]-\infty, a_1\eta + e_1[_R$ and $]a_s\eta + \delta_t, +\infty[_R$ respectively.

(5) For $i = 1, \dots, s$ and $j = 1, \dots, t$, by Theorem i.5, count the number q_{ij} of roots of $f(\eta, y)$ in $]a_i\eta + e_j, a_i\eta + \delta_j[_R$.

Then, we have the following assertions:

($\star$1) The number of right-branches of $\mathcal{C}$ is

$$\sum_{i=0}^{s} q_i + \sum_{i=1}^{s}\sum_{j=1}^{t} q_{ij}.$$

($\star$2) In the order from below to above, these right-branches have their respective asymptotes as follows:

$$\overbrace{\text{No}, \dots, \text{No}}^{q_0},$$
$$\overbrace{y = a_1x + b_1, \dots, y = a_1x + b_1}^{q_{11}}, \quad \dots, \quad \overbrace{y = a_1x + b_t, \dots, y = a_1x + b_t}^{q_{1t}},$$
$$\overbrace{\text{No}, \dots, \text{No}}^{q_1},$$
$$\overbrace{y = a_2x + b_1, \dots, y = a_2x + b_1}^{q_{21}}, \quad \dots, \quad \overbrace{y = a_2x + b_t, \dots, y = a_2x + b_t}^{q_{2t}},$$
$$\overbrace{\text{No}, \dots, \text{No}}^{q_2},$$
$$\vdots$$
$$\overbrace{y = a_sx + b_1, \dots, y = a_sx + b_1}^{q_{s1}}, \quad \dots, \quad \overbrace{y = a_sx + b_t, \dots, y = a_sx + b_t}^{q_{st}},$$
$$\overbrace{\text{No}, \dots, \text{No}}^{q_s},$$

where the word "No" means the corresponding branch has no asymptote, $a_i = (g(z); c_k, d_k)$, $i = 1, \dots, s$, and $b_j = (h(w); e_j, \delta_j)$, $j = 1, \dots, t$.

The above assertions may be verified as follows:
Let all the roots of $f(\eta, y)$ in R be as follows:

$$\alpha_1 < \alpha_2 < \cdots < \alpha_n.$$

By Proposition 1.2, Theorem 1.4 and their proofs, we have

$$n=\sum_{i=0}^{s} q_i+\sum_{i=1}^{s}\sum_{j=1}^{t} q_{ij}.$$

Denote by m the number of right-branches of $\mathcal{C}$. By Proposition 2.1 and its proof, there is a positive number M such that the number of roots of $f(a,y)$ in $\mathbb{R}$ is m whenever $a\in\mathbb{R}$ and $a>M$. Observe that $\eta>M$. By the Transfer Principle, it is easy to see that the number of roots of $f(\eta,y)$ in R is also m. Hence $m=n=\sum_{i=0}^{s} q_i+\sum_{i=1}^{s}\sum_{j=1}^{t} q_{ij}$.

Let α_k be an arbitrary root of $f(\eta,y)$ in R, $1\leqslant k\leqslant n$. By Theorem 3.8, it is easy to see that the kth right-branch of $\mathcal{C}$ has no asymptote if $\alpha_k\in\,]-\infty,a_1\eta+e_1[$ or $\alpha_k\in\,]a_s\eta+\delta_t,+\infty[$. This implies that both the first q_0 and the final q_s right-branches of $\mathcal{C}$ have no asymptotes in the order from bottom to top.

Now assume $\alpha_k\in\,]a_1\eta+e_1,a_s\eta+\delta_t[$. Then we have the following possible cases:

Case 1. $\alpha_k\in\,]a_\lambda\eta+e_\mu,a_\lambda\eta+\delta_\mu[$ for some $\lambda\in\{1,\dots,s\}$ and $\mu\in\{1,\dots,t\}$. By Theorem 3.8, it is easy to see that the line $y=a_\lambda x+b_\mu$ is the asymptote of the kth right-branch of $\mathcal{C}$.

Case 2. $\alpha_k\in\,]a_\lambda\eta+\delta_t,a_{\lambda+1}\eta+e_1[$ for some $\lambda\in\{1,\dots,s-1\}$. By Theorem 3.8, it is easy to see that the kth right-branch of $\mathcal{C}$ has no asymptote.

Moreover, by the structure of the ordering of $\mathbb{R}(\eta)$, we have

$$\begin{aligned}-\infty&<a_1\eta+e_1<a_1\eta+\delta_1<\cdots<a_1\eta+e_t<a_1\eta+\delta_t\\&<a_2\eta+e_1<a_2\eta+\delta_1<\cdots<a_2\eta+e_t<a_2\eta+\delta_t\\&\ \vdots\\&<a_s\eta+e_1<a_s\eta+\delta_1<\cdots<a_s\eta+e_t<a_s\eta+\delta_t<+\infty.\end{aligned}$$

According to the above arguments, assertions ($\star$1) and ($\star$2) have been verified.

As to the asymptotes of the left-branches of $\mathcal{C}$, we implement the computations as follows:

(1′) Implementing the computations as in (1), (2) and (3), we may obtain an univariate polynomial $g(z)\in\mathbb{R}[z]$, a finite number of open intervals $]c_1,d_1[,\dots,]c_s,d_s[$ in $\mathbb{R}$, an univariate polynomial $h(w)\in\mathbb{R}[w]$ and a finite number of open intervals $]e_1,\delta_1[,\dots,]e_t,\delta_t[$ in $\mathbb{R}$.

(2′) Write a'_i for $(g(z);c_{s-i+1},d_{s-i+1})$, $i=1,\dots,s$. Obviously, $-a'_1<-a'_2<\cdots<-a'_s$.

(3′) For $i=1,\dots,s-1$, by Theorem 1.5, count the number q'_i of roots of $f(-\eta,y)$ in $]-a'_i\eta+\delta_t,-a'_{i+1}\eta+e_1[_R$.

Moreover, count the numbers q'_0 and q'_s of roots of $f(-\eta,y)$ in $]-\infty,-a'_1\eta+e_1[_R$ and $]-a'_s\eta+\delta_t,+\infty[_R$ respectively.

(4′) For $i=1,\dots,s$ and $j=1,\dots,t$, by Theorem 1.5, count the number q'_{ij} of roots of $f(-\eta,y)$ in $]-a'_i\eta+e_j,-a'_i\eta+\delta_j[_R$.

Then, we have the following assertions:

($\star$3) The number of left-branches of $\mathcal{C}$ is

$$\sum_{i=0}^{s} q_i' + \sum_{i=1}^{s}\sum_{j=1}^{t} q_{ij}'.$$

($\star$4) In the order from below to above, these left-branches have their respective asymptotes as follows:

$$\overbrace{\text{No}, \ldots, \text{No}}^{q_0'},$$
$$\overbrace{y = a_1'x + b_1, \ldots, y = a_1'x + b_1}^{q_{11}'}, \quad \ldots, \quad \overbrace{y = a_1'x + b_t, \ldots, y = a_1'x + b_t}^{q_{1t}'},$$
$$\overbrace{\text{No}, \ldots, \text{No}}^{q_1'},$$
$$\overbrace{y = a_2'x + b_1, \ldots, y = a_2'x + b_1}^{q_{21}'}, \quad \ldots, \quad \overbrace{y = a_2'x + b_t, \ldots, y = a_2'x + b_t}^{q_{2t}'},$$
$$\overbrace{\text{No}, \ldots, \text{No}}^{q_2'},$$
$$\vdots$$
$$\overbrace{y = a_s'x + b_1, \ldots, y = a_s'x + b_1}^{q_{s1}'}, \quad \ldots, \quad \overbrace{y = a_s'x + b_t, \ldots, y = a_s'x + b_t}^{q_{st}'},$$
$$\overbrace{\text{No}, \ldots, \text{No}}^{q_s'},$$

where the word "No" means the corresponding branch has no asymptote, $a_i' = (g(z); c_{s-i+1}, d_{s-i+1})$, $i = 1, \ldots, s$, and $b_j = (h(w); e_j, \delta_j)$, $j = 1, \ldots, t$.

Assertions ($\star$3) and ($\star$4) may be verified similarly. We leave these verifications to the reader. The proof is complete. $\square$

As an application of Theorem 3.9, we proceed to treat the following example, which is the complement of Example 1. For the sake of convenience, for a non-zero polynomial $\Phi(x)$ in $R[x]$, write $\Phi(+\infty)$, $\Phi(-\infty)$ for the leading coefficients of $\Phi(x)$, $\Phi(-x)$ respectively.

Example 2. Let $\mathcal{C}$ be as in Example 1. Compute all asymptotes of the right- and left-branches of $\mathcal{C}$.

Process of Computing. Put $f := 2 + 2x - y + x^3y + x^2y^3 - y^4x + y^2x - y^4 + 2x^2 + x^3 - y^3$. According to Theorem 3.9 and its proof, we implement the computations as follows:

(1) As a polynomial over $\mathbb{R}[z]$ in the variable η, the leading coefficient of $f(\eta, \eta z)$ is $g(z) = z^3 - z^4$. By real root isolation for $g(z)$, find out a set of isolating intervals as follows:

$$\left]-1, \frac{1}{2}\right[, \quad \left]\frac{1}{2}, 2\right[.$$

(2) As a polynomial over $\mathbb{R}[w]$ in the variable η, the leading coefficient of the resultant of $f(\eta, \eta z + w)$ and $g(z)$ relative to z is $h(w) = -(w+1)^3 w$. By real root isolation for $h(w)$, find out a set of isolating intervals as follows:

$$\left]-2, -\frac{1}{2}\right[, \quad \left]-\frac{1}{2}, 1\right[.$$

(3) A Sturm sequence of f relative to y is computed as follows:

$$\begin{aligned}
f_0 &= 2 + 2x - y + x^3 y + x^2 y^3 - y^4 x + y^2 x - y^4 + 2x^2 + x^3 - y^3, \\
f_1 &= \frac{\partial f}{\partial y} = -1 + x^3 + 3x^2 y^2 - 4y^3 x + 2yx - 4y^3 - 3y^2, \\
f_2 &= -\left(3x^3 - 3x^2 + 5x + 3\right) y^2 - \left(12x^3 + 2x^2 - 2x - 12\right) y - x^4 - 15x^3 \\
&\quad - 2x^2 - 31x - 33, \\
f_3 &= \left(6x^8 + 18x^7 + 28x^6 - 28x^5 - 158x^4 - 192x^3 - 118x^2 - 30x + 18\right) y \\
&\quad + 12x^8 + 56x^7 + 156x^6 + 229x^5 + 167x^4 - 34x^3 - 214x^2 - 255x - 117, \\
f_4 &= 36x^{20} + 324x^{19} + 1884x^{18} + 7608x^{17} + 24820x^{16} + 67256x^{15} \\
&\quad + 158296x^{14} + 330815x^{13} + 618887x^{12} + 1057430x^{11} + 1623870x^{10} \\
&\quad + 2266105x^9 + 2879565x^8 + 3283388x^7 + 3348636x^6 + 3001889x^5 \\
&\quad + 2280669x^4 + 1423890x^3 + 682290x^2 + 204687x + 26487.
\end{aligned}$$

Then, respective Sturm sequences of $f(\eta, y)$ and $f(-\eta, y)$ are as follows:

$$\begin{gathered}
f_0(\eta, y), \quad f_1(\eta, y), \quad f_2(\eta, y), \quad f_3(\eta, y), \quad f_4(\eta, y); \\
f_0(-\eta, y), \quad f_1(-\eta, y), \quad f_2(-\eta, y), \quad f_3(-\eta, y), \quad f_4(-\eta, y).
\end{gathered}$$

(4) Write a_1, a_2 for $(g(z); -1, \frac{1}{2})$, $(g(z); \frac{1}{2}, 2)$ respectively. By Theorem 1.5, for $\alpha = -\infty$, $a_1\eta - 2$, $a_1\eta - \frac{1}{2}$, $a_1\eta + 1$, $a_2\eta - 2$, $a_2\eta - \frac{1}{2}$, $a_2\eta + 1$, $+\infty$, count the number of sign variations in the list $[f_i(\eta, \alpha) \mid i = 0, \ldots, 4]$ as follows:

$$3, \ 3, \ 2, \ 2, \ 2, \ 2, \ 1, \ 1.$$

Moreover, for $\alpha = -\infty$, $-a_2\eta - 2$, $-a_2\eta - \frac{1}{2}$, $-a_2\eta + 1$, $-a_1\eta - 2$, $-a_1\eta - \frac{1}{2}$, $-a_1\eta + 1$, $+\infty$, count the number of sign variations in the list $[f_i(-\eta, \alpha) \mid i = 0, \ldots, 4]$ as follows:

$$4, \ 4, \ 4, \ 3, \ 2, \ 1, \ 1, \ 0.$$

Denoting $b_1 := (h(w); -2, -\frac{1}{2})$ and $b_2 := (h(w); -\frac{1}{2}, 1)$, we have the conclusions:

($\star$1) The number of right-branches of $\mathcal{C}$ is $3 - 1 = 2$.
($\star$2) By Sturm's Theorem, the numbers of roots of $f(\eta, y)$ in $]a_1\eta - 2, a_1\eta - \frac{1}{2}[_R$, $]a_2\eta - \frac{1}{2}, a_2\eta + 1[_R$ are 1, 1 respectively. Then, in the order from below to above, the right-branches of $\mathcal{C}$ have their respective asymptotes as follows: $y = a_1x + b_1$, $y = a_2x + b_2$, i.e. $y = -1$, $y = x$.
($\star$3) The number of left-branches of $\mathcal{C}$ is $4 - 0 = 4$.
($\star$4) By Sturm's Theorem, the numbers of roots of $f(-\eta, y)$ in $]-a_2\eta - \frac{1}{2}, -a_2\eta + 1[_R$, $]-a_2\eta + 1, -a_1\eta - 2[_R$, $]-a_1\eta - 2, -a_1\eta - \frac{1}{2}[_R$, $]-a_1\eta + 1, +\infty[_R$ are 1, 1, 1, 1 respectively. Then, in the order from below to above, the left-branches of $\mathcal{C}$ have their respective asymptotes as follows: $y = a_2x + b_2$, No, $y = a_1x + b_1$, No, i.e. $y = x$, No, $y = -1$, No.

With the aid of the computer algebra system **Maple**, the algorithms above have been made into a general program to count the branches and compute the asymptotes for a real plane algebraic curve defined by a polynomial equation with rational coefficients. Applying this program to Examples 1 and 2, all the computations cost CPU time 0.64s. The following example were done on a Pentium IV computer with 128 MB RAM.

Example 3. Let $\mathcal{C}$ be a curve in $\mathbb{R}^2$ defined by the following equation:

$$\begin{aligned} &y^3x^3 - x^4y^2 - 3x^4y + 3x^5 + 2x^2y^3 - 2x^3y^2 + 3x^3y - 3x^4 - xy^5 + y^4x^2 \\ &\quad + y^5 - y^4x - 6xy^3 + 6x^2y^2 + 9x^2y - 9x^3 + 3y^3 - 3y^2x - 9xy + 9x^2 + 1 = 0. \end{aligned}$$

Compute all the asymptotes of $\mathcal{C}$.

At the cost of CPU time 1.65 s, the following results appeared on the screen:

The asymptotes of up-branches are as follows:

$$\big[1, x = (z - 1, 0, 2)\big].$$

The asymptotes of down-branches are as follows:

$$\big[1, x = (z - 1, 0, 2)\big].$$

The asymptotes of right-branches are as follows:

$$\left[1, y = \left(-z^3 + z, -2, -\frac{1}{2}\right)x + (w, -1, 1)\right], \quad [1, \text{No}], \quad [1, \text{No}],$$
$$\left[2, y = \left(-z^3 + z, \frac{1}{2}, 2\right)x + (w, -1, 1)\right].$$

The asymptotes of left-branches are as follows:

$$\left[2, y = \left(-z^3 + z, \frac{1}{2}, 2\right)x + (w, -1, 1)\right], \quad \left[1, y = \left(-z^3 + z, -2, -\frac{1}{2}\right)x + (w, -1, 1)\right].$$

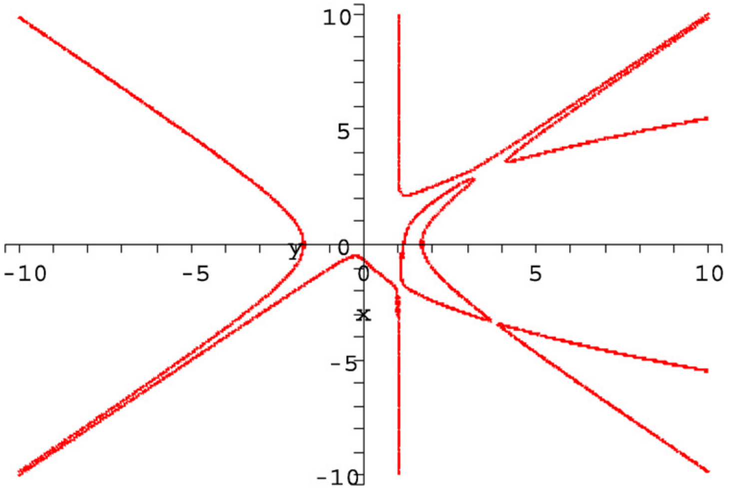

Fig. 1.

This indicates the facts as follows:

(1) $\mathcal{C}$ has the only up-branch with asymptote $x = (z - 1, 0, 2)$ (i.e. $x = 1$).
(2) $\mathcal{C}$ has the only down-branch with asymptote $x = (z - 1, 0, 2)$ (i.e. $x = 1$).
(3) $\mathcal{C}$ has 5 right-branches, and the respective asymptotes are distributed, in the order from below to above, as follows:

$$y = \left(z^3 - z, -2, -\frac{1}{2}\right)x + (w, -1, 1) \quad \text{(i.e. } y = -x\text{)}, \quad \text{No}, \quad \text{No},$$

$$y = \left(z^3 - z, \frac{1}{2}, 2\right)x + (w, -1, 1), \qquad y = \left(z^3 - z, \frac{1}{2}, 2\right)x + (w, -1, 1) \quad \text{(i.e. } y = x\text{)}.$$

(4) $\mathcal{C}$ has 3 left-branches, and the respective asymptotes are distributed, in the order from below to above, as follows:

$$y = \left(z^3 - z, \frac{1}{2}, 2\right)x + (w, -1, 1), \qquad y = \left(z^3 - z, \frac{1}{2}, 2\right)x + (w, -1, 1) \quad \text{(i.e. } y = x\text{)},$$

$$y = \left(z^3 - z, -2, -\frac{1}{2}\right)x + (w, -1, 1) \qquad \text{(i.e. } y = -x\text{)}.$$

The diagram of Example 3 is shown in Fig. 1.

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