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Hesitant fuzzy information aggregation in decision making ☆

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ABSTRACT

As a generalization of fuzzy set, hesitant fuzzy set is a very useful tool in situations where there are some difficulties in determining the membership of an element to a set caused by a doubt between a few different values. The aim of this paper is to develop a series of aggregation operators for hesitant fuzzy information. We first discuss the relationship between intutionistic fuzzy set and hesitant fuzzy set, based on which we develop some operations and aggregation operators for hesitant fuzzy elements. The correlations among the aggregation operators are further discussed. Finally, we give their application in solving decision making problems.

1. Introduction

Since fuzzy set [1] was introduced, several extensions have been developed, such as intutionistic fuzzy set [2], type-2 fuzzy set [3,4], type-*n* fuzzy set [3], fuzzy multiset [5,6] and hesitant fuzzy set [7,8]. Intuitionistic fuzzy set has three main parts: membership function, non-membership function and hesitancy function. Type-2 fuzzy set allows the membership of a given element as a fuzzy set. Type-*n* fuzzy set generalizes type-2 fuzzy set permitting membership to be type-$n-1$ fuzzy set. In fuzzy multiset, the elements can be repeated more than once. Hesitant fuzzy set permits the membership having a set of possible values. A lot of work has been done about the first four types of fuzzy sets, however, little has been done about the hesitant fuzzy set. Torra [7,8] discussed the relationship between hesitant fuzzy set and other three kinds of fuzzy set, and showed that the envelope of hesitant fuzzy set is an intuitionistic fuzzy set. He also proved that the operations he proposed are consistent with the ones of intitionistic fuzzy set when applied to the envelope of hesitant fuzzy set.

Hesitant fuzzy set can be applied in many decision making problems. To get the optimal alternative in a decision making problem with multiple attributes and multiple persons, there are usually two ways: (1) aggregate the decision makers' opinions under each attribute for alternatives, then aggregate the collective values of attributes for each alternative; (2) aggregate the attribute values given by the decision makers for each alternative, and then aggregate the decision makers' opinions for each alternative. For example, for a decision making problem with four attributes $G_j(j=1,2,3,4)$, five decision makers $d_k(k=1,2,\ldots,5)$ are required to give the attribute values of three alternatives $Y_i(i=1,2,3)$. If we have known that d_1 is familiar with c_1, d_2 with c_2, d_3 with c_3, d_4 and d_5 with c_4, then it is better to let the decision maker evaluate the attribute he/she is familiar to, so as to make the decision information more reasonable. However, in some practical problems, anonymity is required in order to protect the decision makers' privacy or avoid influencing each other, for example, the presidential election

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or the blind peer review of thesis, in which we do not know which attributes that the decision makers are respectively familiar with, and thus, leading us to consider all the situations in order to get more reasonable decision results. But the existing methods only consider the minor situations that each decision maker is good at evaluating all the attributes, which hardly happen. Hesitant fuzzy set is very useful in avoiding such issues in which each attribute can be described as a hesitant fuzzy set defined in terms of the opinions of decision makers [8]. Then the aggregation techniques should be given to aggregate the values for each alternative under the attributes, which is just the focus of this paper. In order to do that, we organize the remainder of the paper as follows. In Section 2, we discuss the relationship between the hesitant fuzzy set and intutionsitic fuzzy set. Section 3 develops some operators for aggregating hesitant fuzzy information. Based on the developed operators, Section 4 gives a method for decision making with hesitant fuzzy information. Section 5 gives some concluding remarks.

2. Intuitionistic fuzzy set and hesitant fuzzy set

Intuitionistic fuzzy set (IFS), as a generalization form of fuzzy set (FS) [1], was introduced by Atanassov [2]. Since it assigns to each element a membership degree, a non-membership degree and a hesitancy degree, IFS is more powerful in dealing with vagueness and uncertainty than FS. Since its appearance, IFS has attracted more and more attention from researchers [9–11].

Definition 1 [2]. Let X be fixed, an intuitionistic fuzzy set (IFS) A on X is defined as follows:

$$A = \{< x, \mu_A(x), v_A(x) > | x \in X\}, \tag{1}$$

where the functions $\mu_A(x)$ and $v_A(x)$ denote the degrees of membership and non-membership of the element $x \in X$ to the set A, respectively, with the condition:

$$0 \leqslant \mu_A(x) \leqslant 1, \quad 0 \leqslant v_A(x) \leqslant 1, \quad 0 \leqslant \mu_A(x) + v_A(x) \leqslant 1 \tag{2}$$

and $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$ is usually called the degree of indeterminacy of x to A. Xu [12] named $\alpha = (\mu_\alpha, v_\alpha)$ an intuitionistic fuzzy value (IFV), and let V be the set of all IFVs.

For α, α_1, $\alpha_2 \in V$, Xu and Yager [12,13] gave some operations on them, shown as:

(1) $\alpha^c = (v_\alpha, \mu_\alpha)$;
(2) $\alpha_1 \cup \alpha_2 = \left(\max\left(\mu_{\alpha_1}, \mu_{\alpha_2}\right), \min(v_{\alpha_1}, v_{\alpha_2})\right)$;
(3) $\alpha_1 \cap \alpha_2 = \left(\min\left(\mu_{\alpha_1}, \mu_{\alpha_2}\right), \max(v_{\alpha_1}, v_{\alpha_2})\right)$;
(4) $\alpha_1 \oplus \alpha_2 = \left(\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2}, v_{\alpha_1}v_{\alpha_2}\right)$;
(5) $\alpha_1 \otimes \alpha_2 = \left(\mu_{\alpha_1}\mu_{\alpha_2}, v_{\alpha_1} + v_{\alpha_2} - v_{\alpha_1}v_{\alpha_2}\right)$;
(6) $\lambda\alpha = \left(1 - (1 - \mu_\alpha)^\lambda, v_\alpha^\lambda\right), \lambda > 0$;
(7) $\alpha^\lambda = \left(\mu_\alpha^\lambda, 1 - (1 - v_\alpha)^\lambda\right), \lambda > 0$.

However, when giving the membership degree of an element, the difficulty of establishing the membership degree is not because we have a margin of error, or some possibility distribution on the possibility values, but because we have several possible values. For such cases, Torra [7,8] proposed another generation of FS.

Definition 2 (*[7,8]*). Let X be a fixed set, a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of [0,1].

To be easily understood, we express the HFS by a mathematical symbol:

$$E = \{< x, h_E(x) > | x \in X\}, \tag{3}$$

where $h_E(x)$ is a set of some values in [0,1], denoting the possible membership degrees of the element $x \in X$ to the set E. For convenience, we call $h = h_E(x)$ a hesitant fuzzy element (HFE) and H the set of all HFEs.

Given three HFEs represented by h, h_1 and h_2, Torra [7,8] defined some operations on them, which can be described as:

(1) $h^c = \cup_{\gamma \in h}\{1 - \gamma\}$;
(2) $h_1 \cup h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$;
(3) $h_1 \cap h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}$.

Torra [7,8] showed that the envelop of a HFE is an IFV, expressed in the following definition:

Definition 3 (*[7,8]*). Given a HFE h, we define the IFV $A_{env}(h)$ as the envelope of h, where $A_{env}(h)$ can be represented as $(h^-, 1 - h^+)$, with $h^- = \min\{\gamma | \gamma \in h\}$ and $h^+ = \max\{\gamma | \gamma \in h\}$.

Then, he gave the further study of the relationship between HFEs and IFVs:

(1) $A_{env}(h^c) = (A_{env}(h))^c$;
(2) $A_{env}(h_1 \cup h_2) = A_{env}(h_1) \cup A_{env}(h_2)$;
(3) $A_{env}(h_1 \cap h_2) = A_{env}(h_1) \cap A_{env}(h_2)$.

To compare the HFEs, we define the following comparison laws:

Definition 4. For a HFE $h, s(h) = \frac{1}{\#h}\sum_{\gamma \in h}\gamma$ is called the score function of h, where $\#h$ is the number of the elements in h. For two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

Based on the relationship between the HFEs and IFVs, we define some new operations on the HFEs h, h_1 and h_2:

(1) $h^\lambda = \cup_{\gamma \in h}\{\gamma^\lambda\}$;
(2) $\lambda h = \cup_{\gamma \in h}\{1-(1-\gamma)^\lambda\}$;
(3) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;
(4) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1\gamma_2\}$.

In fact, all the above operations on HFEs can be suitable for HFS. Some relationships can be further established for these operations on HFEs.

Theorem 1. *For three HFEs h, h_1 and h_2, the followings are valid:*
(1) $h_1^c \cup h_2^c = (h_1 \cap h_2)^c$;
(2) $h_1^c \cap h_2^c = (h_1 \cup h_2)^c$;
(3) $(h^c)^\lambda = (\lambda h)^c$;
(4) $\lambda(h^c) = (h^\lambda)^c$;
(5) $h_1^c \oplus h_2^c = (h_1 \otimes h_2)^c$;
(6) $h_1^c \otimes h_2^c = (h_1 \oplus h_2)^c$.

Proof. For three HFEs h, h_1 and h_2, we have

(1) $h_1^c \cup h_2^c = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{1-\gamma_1, 1-\gamma_2\} = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{1-\min\{\gamma_1, \gamma_2\}\} = (h_1 \cap h_2)^c$;
(2) $h_1^c \cap h_2^c = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{1-\gamma_1, 1-\gamma_2\} = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{1-\max\{\gamma_1, \gamma_2\}\} = (h_1 \cup h_2)^c$;
(3) $(h^c)^\lambda = \cup_{\gamma \in h}\{(1-\gamma)^\lambda\} = (\cup_{\gamma \in h}\{1-(1-\gamma)^\lambda\})^c = (\lambda h)^c$;
(4) $\lambda h^c = \cup_{\gamma \in h}\{1-(1-(1-\gamma))^\lambda\} = \cup_{\gamma \in h}\{1-\gamma^\lambda\} = (h^\lambda)^c$;
(5) $h_1^c \oplus h_2^c = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{(1-\gamma_1)+(1-\gamma_2)-(1-\gamma_1)(1-\gamma_2)\} = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{1-\gamma_1\gamma_2\} = (h_1 \otimes h_2)^c$;
(6) $h_1^c \otimes h_2^c = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{(1-\gamma_1)(1-\gamma_2)\} = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{1-(\gamma_1+\gamma_2-\gamma_1\gamma_2)\} = (h_1 \oplus h_2)^c$,

which complete the proof of the theorem. □

The relationship between the IFVs and HFEs can be further discussed:

Theorem 2. *Let h, h_1 and h_2 be three HFEs, then*

(1) $A_{env}(h^\lambda) = (A_{env}(h))^\lambda$;
(2) $A_{env}(\lambda h) = \lambda(A_{env}(h))$;
(3) $A_{env}(h_1 \oplus h_2) = A_{env}(h_1) \oplus A_{env}(h_2)$;
(4) $A_{env}(h_1 \otimes h_2) = A_{env}(h_1) \otimes A_{env}(h_2)$.

Proof. For any three HFEs h, h_1, h_2, we have

(1) $A_{env}(h^\lambda) = A_{env}(\{\gamma^\lambda | \gamma \in h\}) = ((h^-)^\lambda, 1-(h^+)^\lambda)$, $(A_{env}(h))^\lambda = (h^-, 1-h^+)^\lambda = ((h^-)^\lambda, 1-(1-(1-h^+))^\lambda) = ((h^-)^\lambda, 1-(h^+)^\lambda)$;
(2) $A_{env}(\lambda h) = A_{env}(\{1-(1-\gamma)^\lambda | \gamma \in h\}) = (1-(1-h^-)^\lambda, 1-(1-(1-h^+)^\lambda)) = (1-(1-h^-)^\lambda, \quad (1-h^+)^\lambda)$, $\quad \lambda(A_{env}(h)) = \lambda(h^-, 1-h^+) = \left(1-(1-h^-)^\lambda, (1-h^+)^\lambda\right)$;
(3) $A_{env}(h_1 \oplus h_2) = A_{env}(\{\gamma_1+\gamma_2-\gamma_1\gamma_2 | \gamma_1 \in h_1, \gamma_2 \in h_2\}) = \left(h_1^- + h_2^- - h_1^-h_2^-, 1-\left(h_1^+ + h_2^+ - h_1^+h_2^+\right)\right) = \left(h_1^- + h_2^- - h_1^-h_2^-, (1-h_1^+)(1-h_2^+)\right)$, $A_{env}(h_1) \oplus A_{env}(h_2) = \left(h_1^-, 1-h_1^+\right) \oplus \left(h_2^-, 1-h_2^+\right) = \left(h_1^- + h_2^- - h_1^-h_2^-, (1-h_1^+)(1-h_2^+)\right)$;
(4) $A_{env}(h_1 \otimes h_2) = A_{env}(\{\gamma_1\gamma_2 | \gamma_1 \in h_1, \gamma_2 \in h_2\}) = \left(h_1^-h_2^-, 1-h_1^+h_2^+\right)$,
$A_{env}(h_1) \otimes A_{env}(h_2) = \left(h_1^-, 1-h_1^+\right) \oplus \left(h_2^-, 1-h_2^+\right) = \left(h_1^-h_2^-, (1-h_1^+)+(1-h_2^+)-(1-h_1^+)(1-h_2^+)\right) = \left(h_1^-h_2^-, 1-h_1^+h_2^+\right)$.

Thus the proof is completed. □

3. Aggregation operators for hesitant fuzzy information

Since its appearance, the ordered weighted averaging (OWA) operator, introduced by Yager [14], has received more and more attention [15–21]. Xu and Yager [12,13] gave some intuitionistic fuzzy aggregation operators as listed below:

For a collection of IFVs $\alpha_i (i = 1,2,\ldots,n)$, then

(1) The intuitionistic fuzzy weighted averaging (IFWA) operator [12]:

$$\mathrm{IFWA}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \mathop{\oplus}_{j=1}^{n}(w_j\alpha_j) = \left(1-\prod_{j=1}^{n}(1-\mu_{\alpha_j})^{w_j}, \prod_{j=1}^{n}(\nu_{\alpha_j})^{w_j}\right) \tag{4}$$

where $w = (w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ is the weight vector of $(\alpha_1,\alpha_2,\ldots,\alpha_n)$ with $w_j \in [0,1]$, $j = 1, 2,\ldots,n$, $\sum_{j=1}^{n} w_j = 1$.

(2) The intuitionistic fuzzy weighted geometric (IFWG) operator [13]:

$$\mathrm{IFWG}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \mathop{\oplus}_{j=1}^{n}\alpha_j^{w_j} = \left(\prod_{j=1}^{n}(\mu_{\alpha_j})^{w_j}, 1-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{w_j}\right), \tag{5}$$

where $w = (w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ is the weight vector of $(\alpha_1,\alpha_2,\ldots,\alpha_n)$ with $w_j \in [0,1]$, $j = 1, 2,\ldots,n$, $\sum_{j=1}^{n} w_j = 1$.

(3) The intuitionistic fuzzy ordered weighted averaging (IFOWA) operator [12]:

$$\mathrm{IFOWA}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \mathop{\oplus}_{j=1}^{n}\left(\omega_j\alpha_{\sigma(j)}\right) = \left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_{\sigma(j)}}\right)^{\omega_j}, \prod_{j=1}^{n}(\nu_{\alpha_{\sigma(j)}})^{\omega_j}\right), \tag{6}$$

where $\alpha_{\sigma(j)}$ is the jth largest of α_i $(i = 1,2,\ldots,n)$, and $\omega = (\omega_1,\omega_2,\ldots,\omega_n)^{\mathrm{T}}$ is the aggregation-associated vector such that $\omega_j \in [0,1]$, $j = 1,2,\ldots,n$, $\sum_{j=1}^{n}\omega_j = 1$.

(4) The intuitionistic fuzzy ordered weighted geometric (IFOWG) operator [13]:

$$\mathrm{IFOWG}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \mathop{\oplus}_{j=1}^{n}\alpha_{\sigma(j)}^{\omega_j} = \left(\prod_{j=1}^{n}\left(\mu_{\alpha_{\sigma(j)}}\right)^{\omega_j}, 1-\prod_{j=1}^{n}\left(1-\nu_{\alpha_{\sigma(j)}}\right)^{\omega_j}\right), \tag{7}$$

where $\alpha_{\sigma(j)}$ is the jth largest of α_i $(i = 1,2,\ldots,n)$, and $\omega = (\omega_1,\omega_2,\ldots,\omega_n)^{\mathrm{T}}$ is the aggregation-associated vector such that $\omega_j \in [0,1]$, $j = 1, 2,\ldots,n$, $\sum_{j=1}^{n}\omega_j = 1$.

(5) The intuitionistic fuzzy hybrid averaging (IFHA) operator [12]:

$$\mathrm{IFHA}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \mathop{\oplus}_{j=1}^{n}\left(\omega_j\dot{\alpha}_{\sigma(j)}\right) = \left(1-\prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\sigma(j)}}\right)^{\omega_j}, \prod_{j=1}^{n}\left(\nu_{\dot{\alpha}_{\sigma(j)}}\right)^{\omega_j}\right), \tag{8}$$

where $\dot{\alpha}_{\sigma(j)}$ is the jth largest of $\dot{\alpha}_i = nw_i\alpha_i (i = 1,2,\ldots,n)$, $w = (w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ is the weight vector of $(\alpha_1,\alpha_2,\ldots,\alpha_n)$ with $w_j \in [0,1]$, $j = 1, 2,\ldots,n$, $\sum_{j=1}^{n} w_j = 1$ and $\omega = (\omega_1,\omega_2,\ldots,\omega_n)^{\mathrm{T}}$ is the aggregation-associated vector such that $\omega_j \in [0,1]$, $j = 1, 2,\ldots,n$, and $\sum_{j=1}^{n}\omega_j = 1$.

(6) The intuitionistic fuzzy hybrid geometric (IFHG) operator [13]:

$$\mathrm{IFHG}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \mathop{\oplus}_{j=1}^{n}\ddot{\alpha}_{\sigma(j)}^{\omega_j} = \left(\prod_{j=1}^{n}\left(\mu_{\ddot{\alpha}_{\sigma(j)}}\right)^{\omega_j}, 1-\prod_{j=1}^{n}\left(1-\nu_{\ddot{\alpha}_{\sigma(j)}}\right)^{\omega_j}\right), \tag{9}$$

where $\ddot{\alpha}_{\sigma(j)}$ is the jth largest of $\ddot{\alpha}_i = \alpha_i^{nw_i} (i = 1,2,\ldots,n)$, $w = (w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ is the weight vector of $(\alpha_1,\alpha_2,\ldots,\alpha_n)$ with $w_j \in [0,1]$, $j = 1, 2,\ldots,n$, $\sum_{j=1}^{n} w_j = 1$ and $\omega = (\omega_1,\omega_2,\ldots,\omega_n)^{\mathrm{T}}$ is the aggregation-associated vector such that $\omega_j \in [0,1]$, $j = 1, 2,\ldots,n$, and $\sum_{j=1}^{n}\omega_j = 1$.

Yager [22] defined a generalized ordered weighted averaging (GOWA) operator, Zhao et al. [23] extended it to accommodate situations where the input arguments are IFVs.

Definition 5 [23]. A generalized intuitionistic fuzzy ordered weighted averaging (GIFOWA) operator of dimension n is a mapping GIFOWA: $V^n \to V$, which has the following form:

$$\mathrm{GIFOWA}_{\lambda}(\alpha_1,\alpha_2,\ldots,\alpha_m) = \left(\mathop{\oplus}_{j=1}^{n}\left(w_j\alpha_{\sigma(j)}^{\lambda}\right)\right)^{1/\lambda} = \left(\left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_{\sigma(j)}}^{\lambda}\right)^{w_j}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^{n}(1-(1-\nu_{\alpha_{\sigma(j)}})^{\lambda})^{w_j}\right)^{1/\lambda}\right), \tag{10}$$

where $\lambda > 0$, $w = (w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ is the weight vector of $(\alpha_1,\alpha_2,\ldots,\alpha_n)$ with $w_j \in [0,1]$, $j = 1, 2,\ldots,n$ and $\sum_{j=1}^{n} w_j = 1$, and $\alpha_{\sigma(j)}$ is the jth largest of α_i $(i = 1, 2,\ldots,n)$.

Furthermore, Torra and Narukawa [8] proposed an aggregation principle for HFEs:

Definition 6 [8]. Let $E = \{h_1, h_2, \ldots, h_n\}$ be a set of n HFEs, Θ be a function on E, Θ: $[0,1]^N \to [0,1]$, then

$$\Theta_E = \cup_{\gamma \in \{h_1 \times h_2 \times \cdots \times h_n\}} \{\Theta(\gamma)\}. \tag{11}$$

Based on Definition 6 and the defined operations for HFEs, we will give a series of new specific aggregation operators for HFEs, and investigate their desirable properties:

Definition 7. Let $h_j (j = 1,2,\ldots,n)$ be a collection of HFEs. A hesitant fuzzy weighted averaging (HFWA) operator is a mapping $H^n \to H$ such that

$$\mathrm{HFWA}(h_1, h_2, \ldots, h_n) = \mathop{\oplus}_{j=1}^{n} (w_j h_j) = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left\{ 1 - \prod_{j=1}^{n} (1 - \gamma_j)^{w_j} \right\}, \tag{12}$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight vector of $h_j (j = 1,2,\ldots,n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. Especially, if $w = (1/n, 1/n, \ldots, 1/n)^{\mathrm{T}}$, then the HFWA operator reduces to the hesitant fuzzy averaging (HFA) operator:

$$\mathrm{HFA}(h_1, h_2, \ldots, h_n) = \mathop{\oplus}_{j=1}^{n} \left(\frac{1}{n} h_j \right) = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left\{ 1 - \prod_{j=1}^{n} (1 - \gamma_j)^{1/n} \right\}. \tag{13}$$

Definition 8. Let $h_j (j = 1,2,\ldots,n)$ be a collection of HFEs and let HFWG: $H^n \to H$, if

$$\mathrm{HFWG}(h_1, h_2, \ldots, h_n) = \mathop{\otimes}_{j=1}^{n} h_j^{w_j} = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left\{ \prod_{j=1}^{n} \gamma_j^{w_j} \right\}, \tag{14}$$

then HFWG is called a hesitant fuzzy weighted geometric (HFWG) operator, where $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight vector of $h_j (j = 1,2,\ldots,n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. In the case where $w = (1/n, 1/n, \ldots, 1/n)^{\mathrm{T}}$, the HFWA operator reduces to the hesitant fuzzy geometric (HFG) operator:

$$\mathrm{HFG}(h_1, h_2, \ldots, h_n) = \mathop{\otimes}_{j=1}^{n} h_j^{1/n} = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left\{ \prod_{j=1}^{n} \gamma_j^{1/n} \right\}. \tag{15}$$

Lemma 1 ([24,25]). *Let $x_j > 0$, $\lambda_j > 0$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} \lambda_j = 1$, then*

$$\prod_{j=1}^{n} x_j^{\lambda_j} \leqslant \sum_{j=1}^{n} \lambda_j x_j \tag{16}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 3. *Assume that $h_j (j = 1,2,\ldots,n)$ is a collection of HFEs, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of them, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, then*

$$\mathrm{HFWG}(h_1, h_2, \ldots, h_n) \leqslant \mathrm{HFWA}(h_1, h_2, \ldots, h_n). \tag{17}$$

Proof. For any $\gamma_1 \in h_1$, $\gamma_2 \in h_2, \ldots, \gamma_n \in h_n$, based on Lemma 1, we have

$$\prod_{j=1}^{n} \gamma_j^{w_j} \leqslant \sum_{j=1}^{n} w_j \gamma_j = 1 - \sum_{j=1}^{n} w_j (1 - \gamma_j) \leqslant 1 - \prod_{j=1}^{n} (1 - \gamma_j)^{w_j}, \tag{18}$$

which implies that $\otimes_{j=1}^{n} \left(h_j^{w_j} \right) \leqslant \oplus_{j=1}^{n} (w_j h_j)$, and completes the proof of Theorem 3. □

Theorem 3 shows that the values obtained by the HFWG operator are not bigger than the ones obtained by the HFWA operator.

Definition 9. For a collection of the HFEs h_j $(j = 1,2,\ldots,n)$, a generalized hesitant fuzzy weighted averaging (GHFWA) operator is a mapping GHFWA: $H^n \to H$ such that

$$\mathrm{GHFWA}_\lambda(h_1, h_2, \ldots, h_n) = \left(\mathop{\oplus}_{i=1}^{n} \left(w_j h_j^{\lambda} \right) \right)^{1/\lambda} = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left\{ \left(1 - \prod_{j=1}^{n} \left(1 - \gamma_j^{\lambda} \right)^{w_j} \right)^{1/\lambda} \right\}, \tag{19}$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight vector of $h_j (j = 1,2,\ldots,n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. Especially, if $\lambda = 1$, then the GHFWA operator reduces to the HFWA operator.

Theorem 4. *Let $h_j (j = 1,2,\ldots,n)$ be a collection of HFEs having the weight vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1, \lambda > 0$, then*

$$\text{HFWG}(h_1, h_2, \ldots, h_n) \leqslant \text{GHFWA}_\lambda(h_1, h_2, \ldots, h_n). \tag{20}$$

Proof. For any $\gamma_1 \in h_1$, $\gamma_2 \in h_2, \ldots, \gamma_n \in h_n$, based on Lemma 1, we have

$$\prod_{j=1}^n \gamma_j^{w_j} = \left(\prod_{j=1}^n \left(\gamma_j^\lambda\right)^{w_j}\right)^{1/\lambda} \leqslant \left(\sum_{j=1}^n w_j \gamma_j^\lambda\right)^{1/\lambda} = \left(1 - \sum_{j=1}^n w_j \left(1 - \gamma_j^\lambda\right)\right)^{1/\lambda} \leqslant \left(1 - \prod_{j=1}^n \left(1 - \gamma_j^\lambda\right)^{w_j}\right)^{1/\lambda}, \tag{21}$$

which implies that $\otimes_{j=1}^n \left(h_j^{w_j}\right) \leqslant \left(\oplus_{i=1}^n \left(w_j h_j^\lambda\right)\right)^{1/\lambda}$, and completes the proof of the theorem. □

From Theorem 4, we can conclude that the values obtained by the HFWG operator are not bigger than the ones obtained by the GHFWA operator for any $\lambda > 0$.

Definition 10. Let $h_j (j = 1,2,\ldots,n)$ be a collection of HFEs, $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of them, such that $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$. A generalized hesitant fuzzy weighted geometric (GHFWG) operator is a mapping $H^n \to H$, and

$$\text{GHFWG}_\lambda(h_1, h_2, \ldots, h_n) = \frac{1}{\lambda}\left(\overset{n}{\underset{j=1}{\otimes}} (\lambda h_j)^{w_j}\right) = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left\{1 - \left(1 - \prod_{j=1}^n (1 - (1 - \gamma_j)^\lambda)^{w_j}\right)^{1/\lambda}\right\}. \tag{22}$$

If $\lambda = 1$, then the GHFWG operator becomes the HFWG operator.

Theorem 5. *For a collection of HFEs $h_j (j = 1,2,\ldots,n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector such that $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1, \lambda > 0$, then*

$$\text{GHFWG}_\lambda(h_1, h_2, \ldots, h_n) \leqslant \text{HFWA}(h_1, h_2, \ldots, h_n). \tag{23}$$

Proof. Let $\gamma_1 \in h_1$, $\gamma_2 \in h_2, \ldots, \gamma_n \in h_n$, based on Lemma 1, we can obtain

$$\begin{aligned} 1 - \left(1 - \prod_{j=1}^n (1 - (1 - \gamma_j)^\lambda)^{w_j}\right)^{1/\lambda} &\leqslant 1 - \left(1 - \sum_{j=1}^n w_j (1 - (1 - \gamma_j)^\lambda)\right)^{1/\lambda} = 1 - \left(\sum_{j=1}^n w_j (1 - \gamma_j)^\lambda\right)^{1/\lambda} \\ &\leqslant 1 - \left(\prod_{j=1}^n (1 - \gamma_j)^{w_j \lambda}\right)^{1/\lambda} = 1 - \prod_{j=1}^n (1 - \gamma_j)^{w_j}, \end{aligned} \tag{24}$$

which implies that $\frac{1}{\lambda}\left(\otimes_{j=1}^n (\lambda h_j)^{w_j}\right) \leqslant \oplus_{j=1}^n (w_j h_j)$, and completes the proof of the theorem. □

Theorem 5 gives us the result that the values obtained by the GHFWG operator are not bigger than the ones obtained by the HFWA operator, no matter how the parameter $\lambda (\lambda > 0)$ changes.

Example 1. Let $h_1 = (0.2, 0.3, 0.5)$, $h_2 = (0.4, 0.6)$ be two HFEs, $w = (0.7, 0.3)^T$ be the weight vector of them, then by Definitions 7–10, we have

$$\begin{aligned} \text{GHFWA}_1(h_1, h_2) = \text{HFWA}(h_1, h_2) &= \overset{2}{\underset{j=1}{\oplus}} (w_j h_j) \\ &= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{1 - \prod_{j=1}^2 (1 - \gamma_j)^{w_j}\right\} = \left\{1 - (1 - 0.2)^{0.7} \times (1 - 0.4)^{0.3}, 1 - (1 - 0.2)^{0.7} \times (1 - 0.6)^{0.3}, \right. \\ &\quad 1 - (1 - 0.3)^{0.7} \times (1 - 0.4)^{0.3}, 1 - (1 - 0.3)^{0.7} \times (1 - 0.6)^{0.3}, 1 - (1 - 0.5)^{0.7} \times (1 - 0.4)^{0.3}, \\ &\quad \left. 1 - (1 - 0.5)^{0.7} \times (1 - 0.6)^{0.3}\right\} = \{0.2661, 0.3316, 0.3502, 0.4082, 0.4719, 0.5324\}. \end{aligned}$$

$$\begin{aligned} \text{GHFWA}_6(h_1, h_2) = \left(\overset{2}{\underset{j=1}{\oplus}} (w_j h_j)\right)^{1/6} &= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{\left(1 - \prod_{j=1}^2 (1 - \gamma_j^6)^{w_j}\right)^{1/6}\right\} \\ &= \{(1 - (1 - 0.2^6)^{0.7} \times (1 - 0.4^6)^{0.3})^{1/6}, (1 - (1 - 0.2^6)^{0.7} \times (1 - 0.6^6)^{0.3})^{1/6}, \\ &\quad (1 - (1 - 0.3^6)^{0.7} \times (1 - 0.4^6)^{0.3})^{1/6} (1 - (1 - 0.3^6)^{0.7} \times (1 - 0.6^6)^{0.3})^{1/6}, (1 - (1 - 0.5^6)^{0.7} \\ &\quad \times (1 - 0.4^6)^{0.3})^{1/6}, (1 - (1 - 0.5^6)^{0.7} \times (1 - 0.6^6)^{0.3})^{1/6}\} \\ &= \{0.3293, 0.3468, 0.4707, 0.4925, 0.4951, 0.5409\}. \end{aligned}$$

$$\mathrm{GHFWG}_1(h_1,h_2)=\mathrm{HFWG}(h_1,h_2)=\overset{2}{\underset{j=1}{\oplus}}\left(h_j^{w_j}\right)=\cup_{\gamma_1\in h_1,\gamma_2\in h_2}\left\{\prod_{j=1}^{2}\gamma_j^{w_j}\right\}$$
$$=\left\{0.2^{0.7}\times 0.4^{0.3},0.2^{0.7}\times 0.6^{0.3},0.3^{0.7}\times 0.4^{0.3},0.3^{0.7}\times 0.6^{0.3},0.5^{0.7}\times 0.4^{0.3},0.5^{0.7}\times 0.6^{0.3}\right\}$$
$$=\{0.2462,0.2781,0.3270,0.3693,0.4676,0.5281\}.$$

$$\mathrm{GHFWG}_6(h_1,h_2)=\frac{1}{6}\left(\overset{2}{\underset{j=1}{\otimes}}(6h_j)^{w_j}\right)=\cup_{\gamma_1\in h_1,\gamma_2\in h_2}\left\{1-\left(1-\prod_{j=1}^{2}(1-(1-\gamma_j)^6)^{w_j}\right)^{1/6}\right\}$$
$$=\Big\{1-(1-(1-(1-0.2)^6)^{0.7}\times(1-(1-0.4)^6)^{0.3})^{1/6},$$
$$1-(1-(1-(1-0.2)^6)^{0.7}\times(1-(1-0.6)^6)^{0.3})^{1/6},1-(1-(1-(1-0.3)^6)^{0.7}\times(1-(1-0.4)^6)^{0.3})^{1/6},$$
$$1-(1-(1-(1-0.3)^6)^{0.7}\times(1-(1-0.6)^6)^{0.3})^{1/6},1-(1-(1-(1-0.5)^6)^{0.7}\times(1-(1-0.4)^6)^{0.3})^{1/6},$$
$$1-(1-(1-(1-0.5)^6)^{0.7}\times(1-(1-0.6)^6)^{0.3})^{1/6}\Big\}$$
$$=\{0.2333,0.2400,0.3222,0.3369,0.4591,0.5203\}.$$

In the following, we discuss the relationships among the developed aggregation operators:

Theorem 6. *Let $h_j(j = 1,2,\ldots,n)$ be a collection of HFEs with the weight vector $w = (w_1,w_2,\ldots,w_n)^T$ such that $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1, \lambda > 0$, then*

(1) $\oplus_{j=1}^n w_j h_j^c = \left(\otimes_{j=1}^n h_j^{w_j}\right)^c$;
(2) $\otimes_{j=1}^n \left(h_j^c\right)^{w_j} = \left(\oplus_{j=1}^n w_j h_j\right)^c$;
(3) $\left(\oplus_{j=1}^n w_j \left(h_j^c\right)^{\lambda}\right)^{1/\lambda} = \left(\frac{1}{\lambda}\left(\otimes_{j=1}^n (\lambda h_j)^{w_j}\right)\right)^c$;
(4) $\frac{1}{\lambda}\left(\otimes_{j=1}^n \left(\lambda h_j^c\right)^{w_j}\right) = \left(\left(\oplus_{j=1}^n \left(w_j h_j^{\lambda}\right)\right)^{1/\lambda}\right)^c$.

Proof.

(1) $\oplus_{j=1}^n\left(w_j h_j^c\right) = \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{1-\prod_{j=1}^n(\gamma_j)^{w_j}\right\} = \left(\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{\prod_{j=1}^n(\gamma_j)^{w_j}\right\}\right)^c = \left(\otimes_{j=1}^n h_j^{w_j}\right)^c$;
(2) $\otimes_{j=1}^n\left(h_j^c\right)^{w_j} = \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{\prod_{j=1}^n(1-\gamma_j)^{w_j}\right\} = \left(\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{1-\prod_{j=1}^n(1-\gamma_j)^{w_j}\right\}\right)^c = \left(\oplus_{j=1}^n w_j h_j\right)^c$;
(3) $\left(\oplus_{i=1}^n\left(w_j\left(h_j^c\right)^{\lambda}\right)\right)^{1/\lambda} = \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{\left(1-\prod_{j=1}^n(1-(1-\gamma_j)^{\lambda})^{w_j}\right)^{1/\lambda}\right\}$
$= \left(\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{1-\left(1-\prod_{j=1}^n(1-(1-\gamma_j)^{\lambda})^{w_j}\right)^{1/\lambda}\right\}\right)^c = \left(\frac{1}{\lambda}\left(\otimes_{j=1}^n(\lambda h_j)^{w_j}\right)\right)^c$;
(4) $\frac{1}{\lambda}\left(\otimes_{j=1}^n\left(\lambda h_j^c\right)^{w_j}\right) = \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{1-\left(1-\prod_{j=1}^n\left(1-\gamma_j^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right\} = \left(\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{\left(1-\prod_{j=1}^n\left(1-\gamma_j^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right\}\right)^c$
$= \left(\oplus_{j=1}^n w_j h_j^{\lambda}\right)^{1/\lambda}$. □

Theorem 7. *Let $h_j(j = 1,2,\ldots,n)$ be a collection of HFEs associated with the weight vector $w = (w_1,w_2,\ldots,w_n)^T$ such that $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1, \lambda > 0$, then*

(1) $A_{env}\left(\oplus_{j=1}^n w_j h_j\right) = \oplus_{j=1}^n (w_j A_{env}(h_j))$;
(2) $A_{env}\left(\otimes_{j=1}^n w_j h_j\right) = \otimes_{j=1}^n (w_j A_{env}(h_j))$;
(3) $A_{env}\left(\left(\oplus_{j=1}^n w_j (h_j)^{\lambda}\right)^{1/\lambda}\right) = \left(\oplus_{j=1}^n w_j (A_{env}(h_j))^{\lambda}\right)^{1/\lambda}$;
(4) $A_{env}\left(\frac{1}{\lambda}\left(\otimes_{j=1}^n (\lambda h_j)^{w_j}\right)\right) = \frac{1}{\lambda}\left(\otimes_{j=1}^n (\lambda A_{env}(h_j))^{w_j}\right)$.

Proof. Based on Definition 3, we can get

(1) $A_{env}\left(\oplus_{j=1}^n (w_j h_j)\right) = A_{env}\left(\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{1-\prod_{j=1}^n(1-\gamma_j)^{w_j}\right\}\right) = \left(1-\prod_{j=1}^n\left(1-h_j^-\right)^{w_j}, 1-\left(1-\prod_{j=1}^n\left(1-h_j^+\right)^{w_j}\right)\right)$
$= \left(1-\prod_{j=1}^n\left(1-h_j^-\right)^{w_j}, \prod_{j=1}^n\left(1-h_j^+\right)^{w_j}\right) = \oplus_{j=1}^n\left(w_j\left(h_j^-, 1-h_j^+\right)\right) = \oplus_{j=1}^n (w_j A_{env}(h_j))$;

(2) $A_{env}\left(\otimes_{j=1}^{n} h_j^{w_j}\right) = A_{env}\left(\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{\prod_{j=1}^{n}\gamma_j^{w_j}\right\}\right) = \left(\prod_{j=1}^{n}\left(h_j^-\right)^{w_j}, 1-\prod_{j=1}^{n}\left(h_j^+\right)^{w_j}\right) = \left(\prod_{j=1}^{n}\left(h_j^-\right)^{w_j},\right.$
$\left. 1-\prod_{j=1}^{n}\left(1-\left(1-h_j^+\right)\right)^{w_j}\right) = \otimes_{i=1}^{n}\left(h_j^-, 1-h_j^+\right)^{w_j} = \otimes_{i=1}^{n}(A_{env}(h_j))^{w_j}$;

(3) $A_{env}\left(\left(\oplus_{j=1}^{n} w_j(h_j)^{\lambda}\right)^{1/\lambda}\right) = A_{env}\left(\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{\left(1-\prod_{j=1}^{n}(1-\gamma_j^{\lambda})^{w_j}\right)^{1/\lambda}\right\}\right)$
$= \left(\left(1-\prod_{j=1}^{n}(1-(h_j^-)^{\lambda})^{w_j}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^{n}\left(1-\left(h_j^+\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right) = \left(\left(1-\prod_{j=1}^{n}(1-(h_j^-)^{\lambda})^{w_j}\right)^{1/\lambda},\right.$
$\left. 1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-h_j^+\right)\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right) = \left(\oplus_{j=1}^{n} w_j\left(h_j^-, 1-h_j^+\right)^{\lambda}\right)^{1/\lambda} = \left(\oplus_{j=1}^{n} w_j(A_{env}(h_j))^{\lambda}\right)^{1/\lambda}$;

(4) $A_{env}\left(\frac{1}{\lambda}\left(\otimes_{j=1}^{n}(\lambda h_j)^{w_j}\right)\right) = A_{env}\left(\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{1-\left(1-\prod_{j=1}^{n}\left(1-(1-\gamma_j)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right\}\right) = \left(1-\left(1-\prod_{j=1}^{n}\right.\right.$
$\left.\left.\left(1-(1-h_j^-)^{\lambda}\right)^{w_j}\right)^{1/\lambda}, \left(1-\prod_{j=1}^{n}\left(1-\left(1-h_j^+\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right) = \frac{1}{\lambda}\left(\otimes_{j=1}^{n}\left(\lambda\left(h_j^-, 1-h_j^+\right)\right)^{w_j}\right) = \frac{1}{\lambda}\left(\otimes_{j=1}^{n}(\lambda A_{env}(h_j))^{w_j}\right)$. □

Definition 11. Let $h_j (j = 1,2,\ldots,n)$ be a collection of HFEs, $h_{\sigma(j)}$ be the jth largest of them, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector such that $\omega_j \in [0,1]$ and $\sum_{j=1}^{n}\omega_j = 1$, then

(1) A hesitant fuzzy ordered weighted averaging (HFOWA) operator is a mapping HFOWA: $H^n \to H$, where

$$\mathrm{HFOWA}(h_1,h_2,\ldots,h_n) = \overset{n}{\underset{j=1}{\oplus}}(\omega_j h_{\sigma(j)}) = \cup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\ldots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{1-\prod_{j=1}^{n}(1-\gamma_{\sigma(j)})^{\omega_j}\right\}. \quad (25)$$

(2) A hesitant fuzzy ordered weighted geometric (HFOWG) operator is a mapping HFOWG: $H^n \to H$, where

$$\mathrm{HFOWG}(h_1,h_2,\ldots,h_n) = \overset{n}{\underset{j=1}{\otimes}} h_{\sigma(j)}^{\omega_j} = \cup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\ldots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{\prod_{j=1}^{n}\gamma_{\sigma(j)}^{\omega_j}\right\}. \quad (26)$$

(3) A generalized hesitant fuzzy ordered weighted averaging (GHFOWA) operator is a mapping GHFOWA: $H^n \to H$, where

$$\mathrm{GHFOWA}_{\lambda}(h_1,h_2,\ldots,h_n) = \left(\overset{n}{\underset{j=1}{\oplus}}\left(\omega_j h_{\sigma(j)}^{\lambda}\right)\right)^{1/\lambda} = \cup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\ldots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{\left(1-\prod_{j=1}^{n}\left(1-\gamma_{\sigma(j)}^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}\right\} \quad (27)$$

with $\lambda > 0$.

(4) A generalized hesitant fuzzy ordered weighted geometric (GHFOWG) operator is a mapping $H^n \to H$, where

$$\mathrm{GHFOWG}_{\lambda}(h_1,h_2,\ldots,h_n) = \frac{1}{\lambda}\left(\overset{n}{\underset{j=1}{\otimes}}(\lambda h_{\sigma(j)})^{\omega_j}\right) = \cup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\ldots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{1-\left(1-\prod_{j=1}^{n}\left(1-(1-\gamma_{\sigma(j)})^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}\right\} \quad (28)$$

with $\lambda > 0$.

In the case where $\omega = (1/n, 1/n, \ldots, 1/n)^T$, the HFOWA operator reduces to the HFA operator, and the HFOWG operator becomes the HFG operator; in the case where $\lambda = 1$, the GHFOWA operator reduces to the HFOWA operator and the GHFOWG operator reduces to the HFOWG operator.

The HFOWA, HFOWG, GHFOWA, and HFOWG operators are developed based on the idea of the OWA operator [14]. The main characterization of the OWA operator is its reordering step. Several methods have been developed to obtain the OWA weights. Yager [14] used linguistic quantifiers to compute the OWA weights. O'Hagan [26] generated the OWA weights with a predefined degree of orness by maximizing the entropy of the OWA weights. Filev and Yager [27] obtained the OWA weights based on the exponential smoothing. Yager and Filev [28] got the OWA weights from a collection of samples with the relevant aggregated data. Xu and Da [29] obtained the OWA weights under partial weight information by establishing a linear objective-programming model. Especially, based on the normal distribution (Gaussian distribution), Xu [18] developed a method to obtain the OWA weights, whose prominent characteristic is that it can relieve the influence of unfair arguments on the decision result by assigning low weights to those "false" or "biased" ones.

Example 2. Let $h_1 = (0.1, 0.4)$, $h_2 = (0.3, 0.5)$ and $h_3 = (0.2, 0.5, 0.8)$ be three HFEs, and suppose that the aggregation-associated vector is $\omega = (0.25, 0.4, 0.35)^T$.

By Definition 4, we calculate the score values of h_1, h_2 and h_3:

$$s(h_1) = \frac{0.1+0.4}{2} = 0.25, \quad s(h_2) = \frac{0.3+0.5}{2} = 0.4, \quad s(h_3) = \frac{0.2+0.5+0.8}{3} = 0.5.$$

Since

$$s(h_3) > s(h_2) > s(h_1)$$

then

$$h_{\sigma(1)} = h_3 = (0.2, 0.5, 0.8), \quad h_{\sigma(2)} = h_2 = (0.3, 0.5), \quad h_{\sigma(3)} = h_1 = (0.1, 0.4).$$

By Definition 11, we have

$$\begin{aligned}\text{GHFOWA}_1(h_1,h_2,h_3) = \text{HFOWA}(h_1,h_2,h_3) = \overset{3}{\underset{j=1}{\oplus}}(\omega_j h_{\sigma(j)}) = \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\gamma_3\in h_3}\left\{1-(1-\gamma_3)^{0.25}(1-\gamma_2)^{0.4}(1-\gamma_1)^{0.35}\right\}\\ = \{0.2097, 0.2973, 0.3092, 0.3143, 0.3858, 0.3903, 0.4006, 0.4412, 0.4671, 0.5115, 0.5151, 0.5762\}.\end{aligned}$$

$$\begin{aligned}\text{GHFOWA}_2(h_1,h_2,h_3) = \left(\overset{3}{\underset{j=1}{\oplus}}(\omega_j h^2_{\sigma(j)})\right)^{1/2} = \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\gamma_3\in h_3}\left\{\left(1-(1-\gamma_3^2)^{0.25}(1-\gamma_2^2)^{0.4}(1-\gamma_1^2)^{0.35}\right)^{1/2}\right\}\\ = \{0.2239, 0.3213, 0.3271, 0.3476, 0.3961, 0.4123, 0.4165, 0.4687, 0.5067, 0.5461, 0.5586, 0.5920\}.\end{aligned}$$

$$\begin{aligned}\text{GHFOWG}_1(h_1,h_2,h_3) = \text{HFOWG}(h_1,h_2,h_3) = \overset{3}{\underset{j=1}{\oplus}}\left(h^{\omega_j}_{\sigma(j)}\right) = \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\gamma_3\in h_3}\left\{\gamma_3^{0.25}\gamma_2^{0.4}\gamma_1^{0.35}\right\}\\ = \{0.1845, 0.2264, 0.2321, 0.2610, 0.2847, 0.2998, 0.3202, 0.3678, 0.3770, 0.4240, 0.4624, 0.5201\}.\end{aligned}$$

$$\begin{aligned}\text{GHFWG}_2(h_1,h_2,h_3) &= \frac{1}{2}\left(\overset{3}{\underset{j=1}{\otimes}}(2h_{\sigma(j)})^{\omega_j}\right)\\ &= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\gamma_3\in h_3}\left\{1-\left(1-\left(1-(1-\gamma_3)^2\right)^{0.25}\left(1-(1-\gamma_2)^2\right)^{0.4}\left(1-(1-\gamma_1)^2\right)^{0.35}\right)^{1/2}\right\}\\ &= \{0.1820, 0.2165, 0.2238, 0.2403, 0.2678, 0.2882, 0.2972, 0.3601, 0.3740, 0.4057, 0.4610, 0.5047\}.\end{aligned}$$

From Definitions 7–11, it is noted that the HFWA, HFWG, GHFWA and GHFWG operators only weight the hesitant fuzzy argument itself, but ignores the importance of the ordered position of the argument, while the HFOWA, HFOWG, GHFOWA and GHFOWG operators only weight the ordered position of each given argument, but ignore the importance of the argument. To solve this drawback, it is necessary to introduce some hybrid aggregation operators for hesitant fuzzy arguments, which weight all the given arguments and their ordered positions.

Definition 12. For a collection of HFEs $h_j (j = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of them with $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$, n is the balancing coefficient which plays a role of balance, then we define the following aggregation operators, which are all based on the mapping $H^n \to H$ with an aggregation-associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j \in [0,1]$ and $\sum_{j=1}^n \omega_j = 1$:

(1) The hesitant fuzzy hybrid averaging (HFHA) operator:

$$\text{HFHA}(h_1, h_2, \ldots, h_n) = \overset{n}{\underset{j=1}{\oplus}}(\omega_j \dot{h}_{\sigma(j)}) = \cup_{\dot{\gamma}_{\sigma(1)}\in\dot{h}_{\sigma(1)},\dot{\gamma}_{\sigma(2)}\in\dot{h}_{\sigma(2)},\ldots,\dot{\gamma}_{\sigma(n)}\in\dot{h}_{\sigma(n)}}\left\{1-\prod_{j=1}^n(1-\dot{\gamma}_{\sigma(j)})^{\omega_j}\right\}, \tag{29}$$

where $\dot{h}_{\sigma(j)}$ is the jth largest of $\dot{h} = nw_k h_k (k = 1, 2, \ldots, n)$.

(2) The hesitant fuzzy hybrid geometric (HFHG) operator:

$$\text{HFHG}(h_1, h_2, \ldots, h_n) = \overset{n}{\underset{j=1}{\otimes}}\ddot{h}^{\omega_j}_{\sigma(j)} = \cup_{\ddot{\gamma}_{\sigma(1)}\in\ddot{h}_{\sigma(1)},\ddot{\gamma}_{\sigma(2)}\in\ddot{h}_{\sigma(2)},\ldots,\ddot{\gamma}_{\sigma(n)}\in\ddot{h}_{\sigma(n)}}\left\{\prod_{j=1}^n\ddot{\gamma}^{\omega_j}_{\sigma(j)}\right\}, \tag{30}$$

where $\ddot{h}_{\sigma(j)}$ is the jth largest of $\ddot{h}_k = h_k^{nw_k} (k = 1, 2, \ldots, n)$.

(3) The generalized hesitant fuzzy hybrid averaging (HFHA) operator:

$$\text{GHFHA}(h_1, h_2, \ldots, h_n) = \left(\overset{n}{\underset{j=1}{\oplus}}\left(\omega_j \dot{h}^{\lambda}_{\sigma(j)}\right)\right)^{1/\lambda} = \cup_{\dot{\gamma}_{\sigma(1)}\in\dot{h}_{\sigma(1)},\dot{\gamma}_{\sigma(2)}\in\dot{h}_{\sigma(2)},\ldots,\dot{\gamma}_{\sigma(n)}\in\dot{h}_{\sigma(n)}}\left\{\left(1-\prod_{j=1}^n\left(1-\dot{\gamma}^{\lambda}_{\sigma(j)}\right)^{\omega_j}\right)^{1/\lambda}\right\}, \tag{31}$$

where $\lambda > 0$, $\dot{h}_{\sigma(j)}$ is the jth largest of $\dot{h} = nw_k h_k (k = 1, 2, \ldots, n)$.

(4) The generalized hesitant fuzzy hybrid geometric (GHFHG) operator:

$$\mathrm{GHFHG}(h_1,h_2,\ldots,h_n)=\frac{1}{\lambda}\left(\overset{n}{\underset{j=1}{\otimes}}\left(\lambda\ddot{h}_{\sigma(j)}\right)^{\omega_j}\right)=\cup_{\ddot{\gamma}_{\sigma(1)}\in\ddot{h}_{\sigma(1)},\ddot{\gamma}_{\sigma(2)}\in\ddot{h}_{\sigma(2)},\ldots,\ddot{\gamma}_{\sigma(n)}\in\ddot{h}_{\sigma(n)}}\left\{1-\left(1-\prod_{j=1}^{n}\left(1-(1-\ddot{\gamma}_{\sigma(j)})^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}\right\},\quad(32)$$

where $\lambda>0,\ddot{h}_{\sigma(j)}$ is the jth largest of $\ddot{h}_k=h_k^{nw_k}(k=1,2,\ldots,n)$.

Especially, if $w=(1/n,1/n,\ldots,1/n)^{\mathrm{T}}$, then the HFHA operator reduces to the HFOWA operator, the HFHG operator reduces to the HFOWG operator, the GHFHA operator reduces to the GHFOWA operator, and the GHFHG operator becomes the GHFOWG operator; if $\lambda=1$, then the GHFHA operator reduces to the HFHA operator, and the GHFHG operator becomes the HFHG operator.

Example 3. Let $h_1=(0.2,0.4,0.5)$, $h_2=(0.2,0.6)$ and $h_3=(0.1,0.3,0.4)$ be three HFEs, whose weight vector is $w=(0.15,0.3,0.55)^{\mathrm{T}}$, and the aggregation-associated vector is $\omega=(0.3,0.4,0.3)^{\mathrm{T}}$. Then we can obtain

$$\dot{h}_1=(1-(1-0.2)^{3\times0.15},1-(1-0.4)^{3\times0.15},1-(1-0.5)^{3\times0.15})=(0.0955,0.2054,0.2680),$$

$$\dot{h}_2=(1-(1-0.2)^{3\times0.3},1-(1-0.6)^{3\times0.3})=(0.1819,0.5616),$$

$$\dot{h}_3=(1-(1-0.1)^{3\times0.55},1-(1-0.3)^{3\times0.55},1-(1-0.4)^{3\times0.55})=(0.1596,0.4448,0.5695)$$

and

$$s(\dot{h}_1)=0.1896,\quad s(\dot{h}_2)=0.3718,\quad s(\dot{h}_3)=0.3913.$$

Since

$$s(\dot{h}_3)>s(\dot{h}_2)>s(\dot{h}_1)$$

then

$$\dot{h}_{\sigma(1)}=\dot{h}_3=(0.1596,0.4448,0.5695),\quad \dot{h}_{\sigma(2)}=\dot{h}_2=(0.1819,0.5616),$$

$$\dot{h}_{\sigma(3)}=\dot{h}_1=(0.0955,0.2054,0.2680).$$

By Definition 12, we have

$$\begin{aligned}\mathrm{GHFHA}_1(h_1,h_2,h_3)&=\mathrm{HFHA}(h_1,h_2,h_3)=\overset{3}{\underset{j=1}{\oplus}}(\omega_j\dot{h}_{\sigma(j)})=\cup_{\dot{\gamma}_1\in\dot{h}_1,\dot{\gamma}_2\in\dot{h}_2,\dot{\gamma}_3\in\dot{h}_3}\{1-(1-\dot{\gamma}_3)^{0.3}(1-\dot{\gamma}_2)^{0.4}(1-\dot{\gamma}_1)^{0.3}\}\\&=\{0.1501,0.1825,0.2023,0.2494,0.2781,0.2956,0.3046,0.3311,0.3378,0.3474,0.3630,\\&\quad 0.3785,0.4152,0.4375,0.4512,0.4582,0.4788,0.4915\}.\\\mathrm{GHFHA}_3(h_1,h_2,h_3)&=\left(\overset{3}{\underset{j=1}{\oplus}}(\omega_j\dot{h}^3_{\sigma(j)})\right)^{1/3}=\cup_{\dot{\gamma}_1\in\dot{h}_1,\dot{\gamma}_2\in\dot{h}_2,\dot{\gamma}_3\in\dot{h}_3}\left\{\left(1-(1-\dot{\gamma}_3^3)^{0.3}(1-\dot{\gamma}_2^3)^{0.4}(1-\dot{\gamma}_1^3)^{0.3}\right)^{1/3}\right\}\\&=\{0.1573,0.1840,0.2112,0.3102,0.3179,0.3279,0.3957,0.4243,0.4283,0.4336,0.4649,\\&\quad 0.4681,0.4725,0.4003,0.4065,0.5069,0.5095,0.5130\}.\end{aligned}$$

If we use the GHFHG operator to aggregate the HFEs h_1, h_2 and h_3, then

$$\ddot{h}_{\sigma(1)}=\ddot{h}_1=(0.2^{3\times0.15},0.4^{3\times0.15},0.5^{3\times0.15})=(0.4847,0.6621,0.7320),$$

$$\ddot{h}_{\sigma(2)}=\ddot{h}_2=(0.2^{3\times0.3},0.6^{3\times0.3})=(0.2349,0.6314),$$

$$\ddot{h}_{\sigma(3)}=\ddot{h}_3=(0.1^{3\times0.55},0.3^{3\times0.55},0.4^{3\times0.55})=(0.0224,0.1372,0.2205).$$

$$\begin{aligned}\mathrm{GHFHG}_1(h_1,h_2,h_3)&=\mathrm{HFHG}(h_1,h_2,h_3)=\overset{3}{\underset{j=1}{\oplus}}\left(\ddot{h}^{\omega_j}_{\sigma(j)}\right)=\cup_{\ddot{\gamma}_1\in\ddot{h}_1,\ddot{\gamma}_2\in\ddot{h}_2,\ddot{\gamma}_3\in\ddot{h}_3}\{\gamma_1^{0.3}\gamma_2^{0.4}\gamma_3^{0.3}\}\\&=\{0.1442,0.1584,0.1632,0.2142,0.2352,0.2424,0.2484,0.2728,0.2811,\\&\quad 0.2864,0.3145,0.3241,0.3690,0.4051,0.4175,0.4254,0.4671,0.4814\}.\end{aligned}$$

$$\begin{aligned}\mathrm{GHFHG}_3(h_1,h_2,h_3)&=\frac{1}{3}\left(\overset{3}{\underset{j=1}{\otimes}}(3\ddot{h}_{\sigma(j)})^{\omega_j}\right)=\cup_{\ddot{\gamma}_1\in\ddot{h}_1,\ddot{\gamma}_2\in\ddot{h}_2,\ddot{\gamma}_3\in\ddot{h}_3}\left\{1-(1-(1-(1-\ddot{\gamma}_1)^2)^{0.25}(1-(1-\ddot{\gamma}_2)^2)^{0.4}(1-(1-\ddot{\gamma}_3)^2)^{0.35})^{1/3}\right\}\\&=\{0.1264,0.1312,0.1322,0.1633,0.1698,0.1710,0.2361,0.2467,0.2487,0.2772,0.2905,\\&\quad 0.2930,0.3222,0.3390,0.3423,0.3902,0.4138,0.4185\}.\end{aligned}$$

4. Decision making based on hesitant fuzzy information

In some practical problems, for example, the presidential election or the blind peer review of thesis, anonymity is required in order to protect the decision makers' privacy or avoid influencing each other. In this section, we apply the hesitant fuzzy aggregation operators to multi-attribute decision making with anonymity. Suppose that there are m alternatives $Y_i (i = 1,2,\dots,m)$ and n attributes G_j $(j = 1,2,\dots,n)$ with the attribute weight vector $w = (w_1,w_2,\dots,w_n)^T$ such that $w_j \in [0,1]$, $j = 1,2,\dots,n$. If the decision makers provide several values for the alternative Y_i under the attribute G_j with anonymity, these values can be considered as a hesitant fuzzy element h_{ij}. In the case where two decision makers provide the same value, then the value emerges only once in h_{ij}.

Based on the above analysis, we give the following decision making method:

Step 1. The decision makers provide their evaluations about the alternative Y_i under the attribute G_j, denoted by the hesitant fuzzy elements $h_{ij} (i = 1,2,\dots,m; j = 1,2,\dots,n)$.

Step 2. Utilize the developed aggregation operators to obtain the hesitant fuzzy elements h_i $(i = 1,2,\dots,m)$ for the alternatives $Y_i (i = 1,2,\dots,m)$, i.e.,

$$h_i = \text{GHFWA}_\lambda(h_{i1},h_{i2},\dots,h_{i2}) = \left(\overset{n}{\underset{j=1}{\oplus}}\left(w_j h_{ij}^\lambda\right)\right)^{1/\lambda} = \cup_{\gamma_{i1}\in h_{i1},\gamma_{i2}\in h_{i2},\dots,\gamma_{in}\in h_{in}}\left\{\left(1-\prod_{j=1}^{n}(1-\gamma_{ij}^\lambda))^{w_j}\right)^{1/\lambda}\right\}, \quad i = 1,2,\dots,m. \tag{33}$$

Step 3. Compute the score values $s(h_i) (i = 1,2,\dots,m)$ of $h_i (i = 1,2,\dots,m)$ by Definition 4.

Step 4. Get the priority of the alternatives Y_i $(i = 1,2,\dots,m)$ by ranking $s(h_i) (i = 1,2,\dots,m)$.

Example 4 [30]. The enterprise's board of directors, which includes five members, is to plan the development of large projects (strategy initiatives) for the following five years. Suppose there are four possible projects Y_i $(i = 1,2,3,4)$ to be evaluated. It is necessary to compare these projects to select the most important of them as well as order them from the point of view of their importance, taking into account four attributes suggested by the Balanced Scorecard methodology [31] (it should be noted that all of them are of the maximization type): G_1: financial perspective, G_2: the customer satisfaction, G_3: internal business process perspective, and G_4: learning and growth perspective. And suppose that the weight vector of the attributes is $w = (0.2,0.3,0.15,0.35)^T$.

In the following, we use the developed method to get the optimal project.

Step 1. In order to avoid influencing each other, the decision makers are required to provide their preferences in anonymity and the decision matrix $H = (h_{ij})_{n\times n}$ is presented in Table 1, where $h_{ij} (i,j = 1,2,3,4)$ are in the form of HFEs.

Step 2. Utilize the GHFWA operator to obtain the hesitant fuzzy elements $h_i (i = 1,2,3,4)$ for the projects $Y_i (i = 1,2,\dots,m)$. Take project Y_4 for an example, and let $\lambda = 1$, we have

$$h_4 = \text{GHFWA}_1(h_{41},h_{42},h_{43},h_{44}) = \text{HFWA}((0.3,0.5,0.6),(0.2,0.4),(0.5,0.6,0.7),(0.8,0.9)) = \overset{4}{\underset{j=1}{\oplus}}(w_j h_{4j})$$
$$= \cup_{\gamma_{41}\in d_{41},\gamma_{42}\in d_{42},\gamma_{43}\in d_{43},\gamma_{44}\in d_{44}}\left\{1-\prod_{j=1}^{4}(1-\gamma_{4j})^{w_j}\right\}$$
$$= \{0.5532, 0.5679, 0.5822, 0.5861, 0.5901, 0.5960, 0.6005, 0.6036, 0.6131, 0.6136, 0.6168, 0.6203, 0.6294, 0.6299, 0.6335, 0.6450, 0.6456, 0.6494, 0.6605, 0.6610, 0.6753, 0.6722, 0.6784, 0.6830, 0.6865, 0.6890, 0.6964, 0.6969, 0.6993, 0.7021, 0.7092, 0.7097, 0.7125, 0.7215, 0.7219, 0.7337\}.$$

As the parameter λ changes we can get different results for each alternative, here we will not list them for vast amounts of data.

Step 3. Compute the score values $s(h_i) (i = 1,2,3,4)$ of $h_i (i = 1,2,3,4)$ by Definition 4. The score values for the alternatives are shown in Table 2.

Table 1
Hesitant fuzzy decision matrix.

	G_1	G_2	G_3	G_4
Y_1	(0.2,0.4,0.7)	(0.2,0.6,0.8)	(0.2,0.3,0.6,0.7,0.9)	(0.3,0.4,0.5,0.7,0.8)
Y_2	(0.2,0.4,0.7,0.9)	(0.1,0.2,0.4,0.5)	(0.3,0.4,0.6,0.9)	(0.5,0.6,0.8,0.9)
Y_3	(0.3,0.5,0.6,0.7)	(0.2,0.4,0.5,0.6)	(0.3,0.5,0.7,0.8)	(0.2,0.5,0.6,0.7)
Y_4	(0.3,0.5,0.6)	(0.2,0.4)	(0.5,0.6,0.7)	(0.8,0.9)

Table 2
Score values obtained by the GHFWA operator and the rankings of alternatives.

	Y_1	Y_2	Y_3	Y_4	Ranking
$GHFWA_1$	0.5634	0.6009	0.5178	0.6524	$Y_4 > Y_2 > Y_1 > Y_3$
$GHFWA_2$	0.5847	0.6278	0.5337	0.6781	$Y_4 > Y_2 > Y_1 > Y_3$
$GHFWA_5$	0.6324	0.6807	0.5723	0.7314	$Y_4 > Y_2 > Y_1 > Y_3$
$GHFWA_{10}$	0.6730	0.7235	0.6087	0.7745	$Y_4 > Y_2 > Y_1 > Y_3$
$GHFWA_{20}$	0.7058	0.7576	0.6410	0.8077	$Y_4 > Y_2 > Y_1 > Y_3$

Table 3
Score values obtained by the GHFWG operator and the rankings of alternatives.

	Y_1	Y_2	Y_3	Y_4	Ranking
$GHFWG_1$	0.4783	0.4625	0.4661	0.5130	$Y_4 > Y_1 > Y_3 > Y_2$
$GHFWG_2$	0.4546	0.4295	0.4526	0.4755	$Y_4 > Y_1 > Y_3 > Y_2$
$GHFWG_5$	0.4011	0.3706	0. 4170	0.4082	$Y_3 > Y_4 > Y_1 > Y_2$
$GHFWG_{10}$	0.3564	0.3264	0.3809	0.3609	$Y_3 > Y_4 > Y_1 > Y_2$
$GHFWG_{20}$	0.3221	0.2919	0.3507	0.3266	$Y_3 > Y_4 > Y_1 > Y_2$

Step 4. By ranking $s(h_i)(i = 1,2,3,4)$, we can get the priorities of the alternatives $Y_i(i = 1,2,3,4)$ as the parameter λ changes, which are listed in Table 2.
From Table 2, we can find that the score values obtained by the GHFWA operator become bigger as the parameter λ increases for the same aggregation arguments, and the decision makers can choose the values of λ according to their preferences.

In Step 2, if we use the GHFWG operator instead of the GHFWA operator to aggregation the values of the alternatives, the score values and the rankings of the alternatives are listed in Table 3.

It is pointed out that the ranking of the alternatives may change when the parameter λ in the GHFWG operator changes. By analyzing Tables 2 and 3, we can find that the score values obtained by the GHFWG operator become smaller as the parameter λ increases for the same aggregation arguments, but the values obtained by the GHFWA operator are always greater than the ones obtained by the GHFWG operator for the same value of the parameter λ and the same aggregation values.

5. Concluding remarks

In this paper, we have given an intensive study on hesitant fuzzy information aggregation techniques and their application in decision making. Some hesitant fuzzy operational rules have been developed based on the interconnection between the hesitant fuzzy set and the intuitionsitic fuzzy set. To aggregate the hesitant fuzzy information, a series of operators have been developed under various situations, the relationships among them have been discussed. Moreover, we have applied the developed aggregation operators to solve the decision making problems with anonymity. By the illustrative example, we have roughly shown the change trends of the results derived by the developed aggregation operators with the increase of the parameter λ.

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