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Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method

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ABSTRACT

In this paper, we introduce an iterative method based on the extragradient method for finding a common element of the set of a general system of variational inequalities and the set of fixed points of a strictly pseudocontractive mapping in a real Hilbert space. Furthermore, we prove that the studied iterative method strongly converges to a common element of the set of a general system of variational inequalities and the set of fixed points of a strictly pseudocontractive mapping under some mild conditions imposed on algorithm parameters.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. For a given nonlinear operator $A : C \to H$, we consider the following variational inequality problem of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

The set of solutions of the variational inequality (1.1) is denoted by $VI(A, C)$. Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. The variational inequality problem has been extensively studied in the literature. See, e.g. [1–7] and the references therein. For finding an element of $\mathrm{Fix}(S) \cap VI(A, C)$ under the assumption that a set $C \subset H$ is closed and convex, a mapping S of C into itself is nonexpansive and a mapping A of C into H is α-inverse strongly monotone, Takahashi and Toyoda [8] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 0,$$

where P_C is the metric projection of H onto C, $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $\mathrm{Fix}(S) \cap VI(A, C)$ is nonempty, then the sequence $\{x_n\}$ converges weakly to some $z \in \mathrm{Fix}(S) \cap VI(A, C)$. Recently, Nadezhkina and Takahashi [9] and Zeng and Yao [10] proposed some so-called extragradient method motivated by the idea of Korpelevich [11] for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem. Further, these iterative methods are extended in [12] to develop a new iterative method for finding elements in $\mathrm{Fix}(S) \cap VI(A, C)$.

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Let $A, B : C \to H$ be two mappings. Now we concern the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.2}$$

which is called a general system of variational inequalities where $\lambda > 0$ and $\mu > 0$ are two constants. The set of solutions of (1.2) is denoted by GSVI(A, B, C). In particular, if $A = B$, then problem (1.2) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.3}$$

which is defined by Verma [13] (see also [14]) and is called the new system of variational inequalities. Further, if we add up the requirement that $x^* = y^*$, then problem (1.3) reduces to the classical variational inequality problem (1.1). For solving problem (1.2), recently, Ceng et al. [15] introduced and studied a relaxed extragradient method. It is clear that their results unified and extended many results in the literature.

Motivated and inspired by the above works, in this paper, we introduce an iterative method based on the extragradient method for finding a common element of the set of a general system of variational inequalities and the set of fixed points of a strictly pseudocontractive mapping in a real Hilbert space. Furthermore, we prove that the studied iterative method strongly converges to a common element of the set of a general system of variational inequalities and the set of fixed points of a strictly pseudocontractive mapping under some mild conditions imposed on algorithm parameters.

2. Preliminaries

In this section, we collect some notations and lemmas. Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that a mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists a real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

A mapping $S : C \to C$ is said to be a *strictly pseudo-contractive* if there exists a constant $0 \leq k < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.1}$$

For such case, we also say that S is a *k-strict pseudo-contraction.* It is clear that, in a real Hilbert space H, (2.1) is equivalent to the following

$$\langle Sx - Sy, x - y \rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.2}$$

From [16], we know that if S is a k-strictly pseudocontractive mapping, then S satisfies Lipschitz condition $\|Sx - Sy\| \leq \frac{1+k}{1-k}\|x - y\|$ for all $x, y \in C$. We use Fix(S) to denote the set of fixed points of S. It is well-known that the class of strict pseudo-contractions strictly includes the class of nonexpansive mappings which are mappings $S : C \to C$ such that $\|Sx - Sy\| \leq \|x - y\|$, $\forall x, y \in C$. A mapping $Q : C \to C$ is called *contraction* if there exists a constant $\rho \in [0, 1)$ such that $\|Qx - Qy\| \leq \rho\|x - y\|$ for all $x, y \in C$.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

The mapping P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

It is known that $P_C x$ is characterized by the following property:

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \quad \forall x \in H, \ y \in C. \tag{2.3}$$

In order to prove our main results in the next section, we need the following lemmas.

Lemma 2.1 (*[15]*)**.** *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.2) if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)], \quad \forall x \in C,$$

where $y^* = P_C(x^* - \mu Bx^*)$.

In particular, if the mappings $A, B : C \to H$ are α-inverse strongly monotone and β-inverse strongly monotone, respectively, then the mapping G is nonexpansive provided $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$.

Throughout this paper, the set of fixed points of the mapping G is denoted by Γ.

Lemma 2.2 (*[17]*)**.** *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.3 ([17]). *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a k-strictly pseudo-contractive mapping. Then, the mapping $I - T$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \to x^*$ weakly and $(I - T)x_n \to y$ strongly, then $(I - T)x^* = y$.*

Lemma 2.4 ([18]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \le (1 - \gamma_n)a_n + \delta_n$ where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that*

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Now we state and prove our main results.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a k-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers. Assume $(\gamma + \delta)k \le \gamma$. Then*

$$\|\gamma(x - y) + \delta(Sx - Sy)\| \le (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C.$$

Proof. From (2.1) and (2.2), we have

$$\begin{aligned}\|\gamma(x - y) + \delta(Sx - Sy)\|^2 &= \gamma^2\|x - y\|^2 + \delta^2\|Sx - Sy\|^2 + 2\gamma\delta\langle Sx - Sy, x - y\rangle \\ &\le \gamma^2\|x - y\|^2 + \delta^2[\|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2] \\ &\quad + 2\gamma\delta\left[\|x - y\|^2 - \frac{1-k}{2}\|(I - S)x - (I - S)y\|^2\right] \\ &= (\gamma + \delta)^2\|x - y\|^2 + \delta[(\gamma + \delta)k - \gamma]\|(I - S)x - (I - S)y\|^2 \\ &\le (\gamma + \delta)^2\|x - y\|^2,\end{aligned}$$

which implies that

$$\|\gamma(x - y) + \delta(Sx - Sy)\| \le (\gamma + \delta)\|x - y\|. \quad \square \tag{3.1}$$

Theorem 3.2. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let the mappings $A, B : C \to H$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\Omega := \mathrm{Fix}(S) \cap \Gamma \ne \emptyset$. Let $Q : C \to C$ be a ρ-contraction with $\rho \in \left[0, \frac{1}{2}\right)$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{cases} z_n = P_C(x_n - \mu Bx_n), \\ y_n = \alpha_n Qx_n + (1 - \alpha_n)P_C(z_n - \lambda Az_n), \\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda Az_n) + \delta_n Sy_n, \quad \forall n \ge 0, \end{cases} \tag{3.2}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four sequences in $[0, 1]$ such that:

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \le \gamma_n < (1 - 2\rho)\delta_n$ *for all* $n \ge 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\lim_{n\to\infty}\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) = 0$.

Then the sequence $\{x_n\}$ generated by (3.2) converges strongly to $x^* = P_\Omega \cdot Qx^*$ and (x^*, y^*) is a solution of the general system of variational inequalities (1.2), where $y^* = P_C(x^* - \mu Bx^*)$.

Proof. We divide the proof into several steps.

Step 1. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

First, we can write (3.2) as $x_{n+1} = \beta_n x_n + (1 - \beta_n)u_n$, $n \ge 0$, where $u_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$. It follows that

$$\begin{aligned}u_{n+1} - u_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\gamma_{n+1}P_C(z_{n+1} - \lambda Az_{n+1}) + \delta_{n+1}Sy_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n P_C(z_n - \lambda Az_n) + \delta_n Sy_n}{1 - \beta_n} \\ &= \frac{\gamma_{n+1}[P_C(z_{n+1} - \lambda Az_{n+1}) - P_C(z_n - \lambda Az_n)] + \delta_{n+1}(Sy_{n+1} - Sy_n)}{1 - \beta_{n+1}} \\ &\quad + \left(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}\right)P_C(z_n - \lambda Az_n) + \left(\frac{\delta_{n+1}}{1 - \beta_{n+1}} - \frac{\delta_n}{1 - \beta_n}\right)Sy_n.\end{aligned} \tag{3.3}$$

From Lemma 3.1 and (3.2), we get

$$\begin{aligned}
&\|\gamma_{n+1}[P_C(z_{n+1}-\lambda Az_{n+1})-P_C(z_n-\lambda Az_n)]+\delta_{n+1}(Sy_{n+1}-Sy_n)\| \\
&\quad\le \|\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Sy_{n+1}-Sy_n)\|+\gamma_{n+1}\|[P_C(z_{n+1}-\lambda Az_{n+1})-y_{n+1}]+[y_n-P_C(z_n-\lambda Az_n)]\| \\
&\quad\le (\gamma_{n+1}+\delta_{n+1})\|y_{n+1}-y_n\|+\gamma_{n+1}\alpha_{n+1}\|Qx_{n+1}-P_C(z_{n+1}-\lambda Az_{n+1})\| \\
&\qquad+\gamma_{n+1}\alpha_n\|Qx_n-P_C(z_n-\lambda Az_n)\|. \qquad (3.4)
\end{aligned}$$

Since A, B are α-inverse strongly monotone mapping and β-inverse strongly monotone mapping, respectively, then we have

$$\begin{aligned}
\|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|x-y\|^2-2\lambda\langle Ax-Ay, x-y\rangle+\lambda^2\|Ax-Ay\|^2 \\
&\le \|x-y\|^2+\lambda(\lambda-2\alpha)\|Ax-Ay\|^2, \qquad (3.5)
\end{aligned}$$

and

$$\|(I-\mu B)x-(I-\mu B)y\|^2 \le \|x-y\|^2+\mu(\mu-2\beta)\|Bx-By\|^2. \qquad (3.6)$$

It is clear that if $0\le\lambda\le 2\alpha$ and $0\le\mu\le 2\beta$, then $(I-\lambda A)$ and $(I-\mu B)$ are nonexpansive. It follows that

$$\begin{aligned}
\|P_C(z_{n+1}-\lambda Az_{n+1})-P_C(z_n-\lambda Az_n)\| &\le \|z_{n+1}-\lambda Az_{n+1}-(z_n-\lambda Az_n)\| \\
&\le \|z_{n+1}-z_n\| \\
&= \|P_C(x_{n+1}-\mu Bx_{n+1})-P_C(x_n-\mu Bx_n)\| \\
&\le \|(x_{n+1}-\mu Bx_{n+1})-(x_n-\mu Bx_n)\| \\
&\le \|x_{n+1}-x_n\|.
\end{aligned}$$

Then,

$$\begin{aligned}
\|y_{n+1}-y_n\| &\le \|P_C(z_{n+1}-\lambda Az_{n+1})-P_C(z_n-\lambda Az_n)\|+\alpha_{n+1}\|Qx_{n+1}-P_C(z_{n+1}-\lambda Az_{n+1})\| \\
&\quad+\alpha_n\|Qx_n-P_C(z_n-\lambda Az_n)\| \\
&\le \|x_{n+1}-x_n\|+\alpha_n\|Qx_n-P_C(z_n-\lambda Az_n)\|+\alpha_{n+1}\|Qx_{n+1}-P_C(z_{n+1}-\lambda Az_{n+1})\|. \qquad (3.7)
\end{aligned}$$

Therefore, from (3.3), (3.4) and (3.7), we have

$$\begin{aligned}
\|u_{n+1}-u_n\| &\le \|x_{n+1}-x_n\|+\left(1+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\right)\alpha_n\|Qx_n-P_C(z_n-\lambda Az_n)\| \\
&\quad+\left(1+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\right)\alpha_{n+1}\|Qx_{n+1}-P_C(z_{n+1}-\lambda Az_{n+1})\| \\
&\quad+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|(\|P_C(z_n-\lambda Az_n)\|+\|Sy_n\|).
\end{aligned}$$

This implies that

$$\limsup_{n\to\infty}(\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Hence by Lemma 2.2 we get $\lim_{n\to\infty}\|u_n-x_n\|=0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|u_n-x_n\|=0. \qquad (3.8)$$

Step 2. $\lim_{n\to\infty}\|Az_n-Ay^*\|=0$ and $\lim_{n\to\infty}\|Bx_n-Bx^*\|=0$.

Let $x^*\in\Omega$. From Lemma 2.1, we have $x^*=Sx^*$ and

$$x^*=P_C[P_C(x^*-\mu Bx^*)-\lambda AP_C(x^*-\mu Bx^*)].$$

Put $y^*=P_C(x^*-\mu Bx^*)$. Then $x^*=P_C(y^*-\lambda Ay^*)$. From (3.5) and (3.6), we have

$$\begin{aligned}
\|P_C(z_n-\lambda Az_n)-P_C(y^*-\lambda Ay^*)\|^2 &\le \|(z_n-\lambda Az_n)-(y^*-\lambda Ay^*)\|^2 \\
&\le \|z_n-y^*\|^2+\lambda(\lambda-2\alpha)\|Az_n-Ay^*\|^2, \qquad (3.9)
\end{aligned}$$

and

$$\begin{aligned}
\|z_n-y^*\|^2 &= \|P_C(x_n-\mu Bx_n)-P_C(x^*-\mu Bx^*)\|^2 \\
&\le \|(x_n-\mu Bx_n)-(x^*-\mu Bx^*)\|^2 \\
&\le \|x_n-x^*\|^2+\mu(\mu-2\beta)\|Bx_n-Bx^*\|^2. \qquad (3.10)
\end{aligned}$$

It follows from (3.2), (3.9) and (3.10) that

$$\begin{aligned}\|y_n - x^*\|^2 &\le (1-\alpha_n)\|P_C(z_n - \lambda Az_n) - P_C(y^* - \lambda Ay^*)\|^2 + \alpha_n\|Qx_n - x^*\|^2 \\ &\le \alpha_n\|Qx_n - x^*\|^2 + \|z_n - y^*\|^2 + \lambda(\lambda - 2\alpha)\|Az_n - Ay^*\|^2 \\ &\le \alpha_n\|Qx_n - x^*\|^2 + \|x_n - x^*\|^2 + \mu(\mu - 2\beta)\|Bx_n - Bx^*\|^2 + \lambda(\lambda - 2\alpha)\|Az_n - Ay^*\|^2. \end{aligned} \tag{3.11}$$

Using the convexity of $\|\cdot\|$, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\beta_n(x_n - x^*) + (1-\beta_n)\frac{1}{1-\beta_n}[\gamma_n(P_C(z_n - \lambda Az_n) - x^*) + \delta_n(Sy_n - x^*)]\|^2 \\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left\|\frac{\gamma_n}{1-\beta_n}(P_C(z_n - \lambda Az_n) - x^*) + \frac{\delta_n}{1-\beta_n}(Sy_n - x^*)\right\|^2 \\ &= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|\frac{\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)}{1-\beta_n} + \frac{\alpha_n}{1-\beta_n}(P_C(z_n - \lambda Az_n) - Qx_n)\|^2 \\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left\|\frac{\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)}{1-\beta_n}\right\|^2 + M\alpha_n \\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + M\alpha_n, \end{aligned} \tag{3.12}$$

where $M > 0$ is some appropriate constant. So, by (3.11) and (3.12), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 + \mu(\mu - 2\beta)(1-\beta_n)\|Bx_n - Bx^*\|^2 \\ &\quad + \lambda(\lambda - 2\alpha)(1-\beta_n)\|Az_n - Ay^*\|^2 + (M + \|Qx_n - x^*\|^2)\alpha_n. \end{aligned}$$

Therefore,

$$\begin{aligned}&\lambda(2\alpha - \lambda)(1-\beta_n)\|Az_n - Ay^*\|^2 + \mu(2\beta - \mu)(1-\beta_n)\|Bx_n - Bx^*\|^2 \\ &\quad \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + (M + \|Qx_n - x^*\|^2)\alpha_n \\ &\quad \le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| + (M + \|Qx_n - x^*\|^2)\alpha_n. \end{aligned}$$

Since $\liminf_{n\to\infty}\lambda(2\alpha - \lambda)(1-\beta_n) > 0$, $\liminf_{n\to\infty}\mu(2\beta - \mu)(1-\beta_n) > 0$, $\|x_n - x_{n+1}\| \to 0$ and $\alpha_n \to 0$, we have

$$\lim_{n\to\infty}\|Az_n - Ay^*\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|Bx_n - Bx^*\| = 0.$$

Step 3. $\lim_{n\to\infty}\|Sy_n - y_n\| = 0$.

Set $v_n = P_C(z_n - \lambda Az_n)$. Noting that P_C is firmly nonexpansive, then we have

$$\begin{aligned}\|z_n - y^*\|^2 &= \|P_C(x_n - \mu Bx_n) - P_C(x^* - \mu Bx^*)\|^2 \\ &\le \langle (x_n - \mu Bx_n) - (x^* - \mu Bx^*), z_n - y^*\rangle \\ &= \frac{1}{2}(\|x_n - x^* - \mu(Bx_n - Bx^*)\|^2 + \|z_n - y^*\|^2 - \|(x_n - x^*) - \mu(Bx_n - Bx^*) - (z_n - y^*)\|^2) \\ &\le \frac{1}{2}(\|x_n - x^*\|^2 + \|z_n - y^*\|^2 - \|(x_n - z_n) - \mu(Bx_n - Bx^*) - (x^* - y^*)\|^2) \\ &= \frac{1}{2}(\|x_n - x^*\|^2 + \|z_n - y^*\|^2 - \|x_n - z_n - (x^* - y^*)\|^2 \\ &\quad + 2\mu\langle x_n - z_n - (x^* - y^*), Bx_n - Bx^*\rangle - \mu^2\|Bx_n - Bx^*\|^2), \end{aligned}$$

and

$$\begin{aligned}\|v_n - x^*\| &= \|P_C(z_n - \lambda Az_n) - P_C(y^* - \lambda Ay^*)\|^2 \\ &\le \langle z_n - \lambda Az_n - (y^* - \lambda Ay^*), v_n - x^*\rangle \\ &= \frac{1}{2}\left(\|z_n - \lambda Az_n - (y^* - \lambda Ay^*)\|^2 + \|v_n - x^*\|^2 - \|z_n - \lambda Az_n - (y^* - \lambda Ay^*) - (v_n - x^*)\|^2\right) \\ &\le \frac{1}{2}(\|z_n - y^*\|^2 + \|v_n - x^*\|^2 - \|z_n - v_n + (x^* - y^*)\|^2 \\ &\quad + 2\lambda\langle Az_n - Ay^*, z_n - v_n + (x^* - y^*)\rangle - \lambda^2\|Az_n - Ay^*\|^2) \\ &\le \frac{1}{2}(\|x_n - x^*\|^2 + \|v_n - x^*\|^2 - \|z_n - v_n + (x^* - y^*)\|^2 + 2\lambda\langle Az_n - Ay^*, z_n - v_n + (x^* - y^*)\rangle). \end{aligned}$$

Thus, we have

$$\|z_n - y^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - z_n - (x^* - y^*)\|^2 + 2\mu\langle x_n - z_n - (x^* - y^*), Bx_n - Bx^*\rangle - \mu^2\|Bx_n - Bx^*\|^2, \tag{3.13}$$

and

$$\|v_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|z_n - v_n + (x^* - y^*)\|^2 + 2\lambda\|Az_n - Ay^*\|\,\|z_n - v_n + (x^* - y^*)\|.$$

It follows that

$$\begin{aligned}\|y_n - x^*\|^2 &\le \alpha_n\|Qx_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2\\ &\le \alpha_n\|Qx_n - x^*\|^2 + \|v_n - x^*\|^2\\ &\le \alpha_n\|Qx_n - x^*\|^2 + \|x_n - x^*\|^2 - \|z_n - v_n + (x^* - y^*)\|^2 + 2\lambda\|Az_n - Ay^*\|\,\|z_n - v_n + (x^* - y^*)\|.\end{aligned} \tag{3.14}$$

By (3.11), (3.12) and (3.13), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|Qx_n - x^*\|^2 + (1-\beta_n)\|z_n - y^*\|^2 + M\alpha_n\\ &\le \|x_n - x^*\|^2 - (1-\beta_n)\|x_n - z_n - (x^* - y^*)\|^2\\ &\quad + 2(1-\beta_n)\mu\|x_n - z_n - (x^* - y^*)\|\,\|Bx_n - Bx^*\| + (M + (1-\beta_n)\alpha_n)\alpha_n.\end{aligned}$$

It follows that

$$\begin{aligned}(1-\beta_n)\|x_n - z_n - (x^* - y^*)\|^2 &\le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| + (M + (1-\beta_n)\alpha_n)\alpha_n\\ &\quad + 2(1-\beta_n)\mu\|x_n - z_n - (x^* - y^*)\|\,\|Bx_n - Bx^*\|.\end{aligned}$$

Note that $\|x_{n+1} - x_n\| \to 0$, $\alpha_n \to 0$ and $\|Bx_n - Bx^*\| \to 0$. Then we immediately deduce

$$\lim_{n\to\infty}\|x_n - z_n + (x^* - y^*)\| = 0. \tag{3.15}$$

By (3.12) and (3.14), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 - (1-\beta_n)\|z_n - v_n + (x^* - y^*)\|^2\\ &\quad + 2\lambda(1-\beta_n)\|Az_n - Ay^*\|\,\|z_n - v_n + (x^* - y^*)\| + (M + \|Qx_n - x^*\|^2)\alpha_n.\end{aligned}$$

So, we obtain

$$\begin{aligned}(1-\beta_n)\|z_n - v_n + (x^* - y^*)\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\lambda(1-\beta_n)\|Az_n - Ay^*\|\\ &\quad \times \|z_n - v_n + (x^* - y^*)\| + (M + \|Qx_n - x^*\|^2)\alpha_n.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\|z_n - v_n - (x^* - y^*)\| = 0.$$

This together with $\|y_n - v_n\| \le \alpha_n\|Qx_n - v_n\| \to 0$ imply that

$$\lim_{n\to\infty}\|z_n - y_n - (x^* - y^*)\| = 0. \tag{3.16}$$

Thus, from (3.15) and (3.16), we deduce that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0.$$

Since

$$\begin{aligned}\|\delta_n(Sy_n - x_n)\| &\le \|x_{n+1} - x_n\| + \gamma_n\|P_C(z_n - \lambda Az_n) - x_n\|\\ &\le \|x_{n+1} - x_n\| + \gamma_n\|y_n - x_n\| + \gamma_n\alpha_n\|Qx_n - x_n\|.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\|Sy_n - x_n\| = 0 \quad\text{and}\quad \lim_{n\to\infty}\|Sy_n - y_n\| = 0.$$

Step 4. $\limsup_{n\to\infty}\langle Qx^* - x^*, x_n - x^*\rangle \le 0$ where $x^* = P_\Omega \cdot Qx^*$.

As $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \to v$ weakly. First, it is clear from Lemma 2.3 that $v \in \mathrm{Fix}(S)$. Next, we prove that $v \in \Gamma$. We note that

$$\begin{aligned}\|y_n - G(y_n)\| &\le \alpha_n\|Qx_n - G(y_n)\| + (1-\alpha_n)\|P_C[P_C(x_n - \mu Bx_n) - \lambda AP_C(x_n - \mu Bx_n)] - G(y_n)\|\\ &= \alpha_n\|Qx_n - G(y_n)\| + (1-\alpha_n)\|G(x_n) - G(y_n)\|\\ &\le \alpha_n\|Qx_n - G(y_n)\| + (1-\alpha_n)\|x_n - y_n\|\\ &\to 0.\end{aligned}$$

According to Lemma 2.3 we obtain $v \in \Gamma$. Therefore, $v \in \Omega$. Hence, it follows from (2.3) that

$$\begin{aligned}\limsup_{n\to\infty}\langle Qx^* - x^*, x_n - x^*\rangle &= \lim_{i\to\infty}\langle Qx^* - x^*, x_{n_i} - x^*\rangle \\ &= \langle Qx^* - x^*, v - x^*\rangle \\ &\le 0.\end{aligned}$$

Step 5. $\lim_{n\to\infty} x_n = x^*$.

From (3.2) and the convexity of $\|\cdot\|$, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*) + \gamma_n\alpha_n(P_C(z_n - \lambda Az_n) - Qx_n)\|^2 \\ &\le \|\beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)\|^2 + 2\gamma_n\alpha_n\langle P_C(z_n - \lambda Az_n) - Qx_n, x_{n+1} - x^*\rangle \\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left\|\frac{1}{1-\beta_n}[\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)]\right\|^2 \\ &\quad + 2\gamma_n\alpha_n\langle P_C(z_n - \lambda Az_n) - x^*, x_{n+1} - x^*\rangle + 2\gamma_n\alpha_n\langle x^* - Qx_n, x_{n+1} - x^*\rangle. \end{aligned} \tag{3.17}$$

By Lemma 3.1 and (3.17), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + 2\gamma_n\alpha_n\|P_C(z_n - \lambda Az_n) - x^*\|\,\|x_{n+1} - x^*\| \\ &\quad + 2\gamma_n\alpha_n\langle x^* - Qx_n, x_{n+1} - x^*\rangle \\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[(1-\alpha_n)\|z_n - y^*\|^2 + 2\alpha_n\langle Qx_n - x^*, y_n - x^*\rangle] \\ &\quad + 2\gamma_n\alpha_n\|z_n - y^*\|\,\|x_{n+1} - x^*\| + 2\gamma_n\alpha_n\langle x^* - Qx_n, x_{n+1} - x^*\rangle.\end{aligned}$$

From (3.10), we note that $\|z_n - y^*\| \le \|x_n - x^*\|$. Hence, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n(1-\beta_n)\langle Qx_n - x^*, y_n - x^*\rangle \\ &\quad + 2\gamma_n\alpha_n\|x_n - x^*\|\,\|x_{n+1} - x^*\| + 2\gamma_n\alpha_n\langle x^* - Qx_n, x_{n+1} - x^*\rangle \\ &\le [1 - (1-\beta_n)\alpha_n]\|x_n - x^*\|^2 + 2\alpha_n\gamma_n\langle Qx_n - x^*, y_n - x_{n+1}\rangle \\ &\quad + 2\alpha_n\delta_n\langle Qx_n - x^*, y_n - x^*\rangle + 2\alpha_n\gamma_n\|x_n - x^*\|\,\|x_{n+1} - x^*\| \\ &\le [1 - (1-\beta_n)\alpha_n]\|x_n - x^*\|^2 + 2\alpha_n\gamma_n\|Qx_n - x^*\|\,\|y_n - x_{n+1}\| \\ &\quad + 2\alpha_n\delta_n\langle Qx_n - x^*, x_n - x^*\rangle + 2\alpha_n\delta_n\langle Qx_n - x^*, y_n - x_n\rangle + 2\alpha_n\gamma_n\|x_n - x^*\|\,\|x_{n+1} - x^*\| \\ &\le [1 - (1-\beta_n)\alpha_n]\|x_n - x^*\|^2 + 2\alpha_n\gamma_n\|Qx_n - x^*\|\,\|y_n - x_{n+1}\| \\ &\quad + 2\alpha_n\delta_n\rho\|x_n - x^*\|^2 + 2\alpha_n\delta_n\langle Qx^* - x^*, x_n - x^*\rangle \\ &\quad + 2\alpha_n\delta_n\|Qx_n - x^*\|\,\|y_n - x_n\| + 2\alpha_n\gamma_n\|x_n - x^*\|\,\|x_{n+1} - x^*\| \\ &\le [1 - (1-\beta_n)\alpha_n]\|x_n - x^*\|^2 + 2\alpha_n\gamma_n\|Qx_n - x^*\|\,\|y_n - x_{n+1}\| \\ &\quad + 2\alpha_n\delta_n\rho\|x_n - x^*\|^2 + 2\alpha_n\delta_n\langle Qx^* - x^*, x_n - x^*\rangle \\ &\quad + 2\alpha_n\delta_n\|Qx_n - x^*\|\,\|y_n - x_n\| + \alpha_n\gamma_n(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2),\end{aligned}$$

that is,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \left[1 - \frac{(1-2\rho)\delta_n - \gamma_n}{1-\alpha_n\gamma_n}\alpha_n\right]\|x_n - x^*\|^2 + \frac{[(1-2\rho)\delta_n - \gamma_n]\alpha_n}{1-\alpha_n\gamma_n} \\ &\quad \times \left\{\frac{2\gamma_n}{(1-2\rho)\delta_n - \gamma_n}\|Qx_n - x^*\|\,\|y_n - x_{n+1}\|\right. \\ &\quad \left. + \frac{2\delta_n}{(1-2\rho)\delta_n - \gamma_n}\|Qx_n - x^*\|\,\|y_n - x_n\| + \frac{2\delta_n}{(1-2\rho)\delta_n - \gamma_n}\langle Qx^* - x^*, x_n - x^*\rangle\right\}.\end{aligned}$$

Note that $\liminf_{n\to\infty}\frac{(1-2\rho)\delta_n - \gamma_n}{1-\alpha_n\gamma_n} > 0$. It follows that $\sum_{n=0}^{\infty}\frac{(1-2\rho)\delta_n-\gamma_n}{1-\alpha_n\gamma_n}\alpha_n = \infty$. It is clear that

$$\begin{aligned}&\limsup_{n\to\infty}\left\{\frac{2\gamma_n}{(1-2\rho)\delta_n - \gamma_n}\|Qx_n - x^*\|\,\|y_n - x_{n+1}\| + \frac{2\delta_n}{(1-2\rho)\delta_n - \gamma_n}\|Qx_n - x^*\|\,\|y_n - x_n\|\right. \\ &\quad \left. + \frac{2\delta_n}{(1-2\rho)\delta_n - \gamma_n}\langle Qx^* - x^*, x_n - x^*\rangle\right\} \le 0.\end{aligned}$$

Therefore, all conditions of Lemma 2.4 are satisfied. Therefore, we immediately deduce that $x_n \to x^*$. This completes the proof. □

Corollary 3.3. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let the mappings $A, B : C \to H$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\Omega := \mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. For fixed $u \in C$ and given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ y_n = \alpha_n u + (1-\alpha_n) P_C(z_n - \lambda A z_n), \\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda A z_n) + \delta_n S y_n, \quad \forall n \geq 0, \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four sequences in $[0, 1]$ such that:

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n < \delta_n$ *for all* $n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;
(iv) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right) = 0$.

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega u$ and (x^*, y^*) is a solution of the general system of variational inequalities (1.2), where $y^* = P_C(x^* - \mu B x^*)$.

Corollary 3.4. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let the mappings $A, B : C \to H$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $S : C \to C$ be a nonexpansive mapping such that $\Omega := \mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. Let $Q : C \to C$ be a ρ-contraction with $\rho \in \left[0, \frac{1}{2}\right)$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ y_n = \alpha_n Q x_n + (1-\alpha_n) P_C(z_n - \lambda A z_n), \\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda A z_n) + \delta_n S y_n, \quad \forall n \geq 0, \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four sequences in $[0, 1]$ such that:

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $\gamma_n < (1-2\rho)\delta_n$ *for all* $n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \gamma_n > 0$;
(iv) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right) = 0$.

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega \cdot Q x^*$ and (x^*, y^*) is a solution of the general system of variational inequalities (1.2), where $y^* = P_C(x^* - \mu B x^*)$.

Corollary 3.5. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let the mappings $A, B : C \to H$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $S : C \to C$ be a nonexpansive mapping such that $\Omega := \mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. For fixed $u \in C$ and given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ y_n = \alpha_n u + (1-\alpha_n) P_C(z_n - \lambda A z_n), \\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda A z_n) + \delta_n S y_n, \quad \forall n \geq 0, \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four sequences in $[0, 1]$ such that:

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $\gamma_n < (1-2\rho)\delta_n$ *for all* $n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \gamma_n > 0$;
(iv) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right) = 0$.

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega \cdot Q x^*$ and (x^*, y^*) is a solution of the general system of variational inequalities (1.2), where $y^* = P_C(x^* - \mu B x^*)$.

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