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The Kirchhoff indexes of some composite networks

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ABSTRACT

In electric circuit theory, it is of great interest to compute the effective resistance between any pair of vertices of a network, as well as the Kirchhoff index. During the past decade these parameters have been applied in organic chemistry as natural structural indexes different from the usual ones in order to achieve an improvement in the discrimination between different molecules that have similar structural behaviours. This new application has started an important and fruitful line of research which has carried the computation of the Kirchhoff index into some symmetrical networks such as distance-regular grahs or circulant graphs. Moreover, a wide range of generalized Kirchhoff indexes for some networks have been introduced. The objective of the present work is to obtain the Kirchhoff index for composite networks such as corona or cluster networks.

1. Introduction

The Kirchhoff index, also known as the Total Resistance of a network, was introduced in chemistry as a better alternative to the other parameters used for discriminating among different molecules with similar shapes and structures; see [11].

Chemically, the Kirchhoff index of a molecular graph is the sum of the squared atomic displacements produced by molecular vibrations of atoms from their equilibrium positions (see [9]). Small values of the Kirchhoff index indicate that the atoms are very rigid in the molecule. For instance, in 2,2,3-trimethylbutane (shown on Fig. 1) the smallest displacement is obtained for the carbon atom connected to three methyl groups, that is, the carbon atom connected only to other carbon atoms. In view of the applications of the Kirchhoff index, a new line of research with a considerable amount of progress has been developed and the Kirchhoff index has been computed for some classes of graphs with symmetries; see for instance [1,4,10,14] and the references therein. Moreover, it is of great interest to calculate this parameter for composite networks and find possible relations between the Kirchhoff indexes of the original networks and those of their composite networks; see for instance [15].

In [5,6] a generalization of the Kirchhoff Index of a finite network, which is based on defining the generalization of the effective resistance between any pair of vertices with respect to a value $\lambda \geq 0$ and a weight ω on the vertex set, was introduced. Here we show that this generalization is essential to obtain the expression for the Kirchhoff index of a composite network in terms of the Kirchhoff indexes of the factors.

In this work the role of Green's function is crucial in order to evaluate the effective resistances of the network. So, after the introduction of the main definitions of the involved operators and their properties, we obtain the expression of Green's function for some composite networks in terms of the Green's functions of the factors. Therefore, as a by-product, we obtain the expression of the effective resistances and hence of the Kirchhoff index, all of them in terms of the corresponding parameters of each factor network.

Although the techniques and notations used in this work belong to a discrete framework, they have their origin on the continuum: they are similar to the ones used in partial differential equations research. The adaptation of these techniques to

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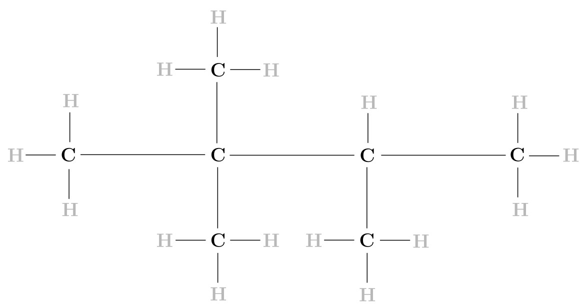

Fig. 1. 2,2,3-trimethylbutane.

the discrete context is shown to be suitable and for that reason we can work with functions instead of vectors or operators instead of matrices.

Given a finite set V, the set of real valued functions on V is denoted by $\mathcal{C}(V)$. The standard inner product on $\mathcal{C}(V)$ is denoted by $\langle\cdot,\cdot\rangle$ and hence, if $u, v \in \mathcal{C}(V)$, then $\langle u, v\rangle = \sum_{x\in V} u(x)\, v(x)$. For any $x \in V$, $\varepsilon_x \in \mathcal{C}(V)$ stands for the Dirac function at x and 1 is the function defined by $1(x) = 1$, for any $x \in V$. On the other hand, $\omega \in \mathcal{C}(V)$ is called a *weight* if it satisfies that $\omega(x) > 0$ for any $x \in V$ and moreover $\langle\omega,\omega\rangle = 1$. The set of weights on V is denoted by $\Omega(V)$.

The triple $\Gamma = (V, E, c)$ denotes a finite network; that is, a finite connected graph without loops or multiple edges, with vertex set V, whose cardinality equals n, and edge set E, in which each edge $\{x, y\}$ has been assigned a *conductance* $c(x, y) > 0$. So, the conductance can be considered as a symmetric function $c\colon V \times V \longrightarrow [0, +\infty)$ such that $c(x, x) = 0$ for any $x \in V$ and moreover, vertex x is adjacent to vertex y iff $c(x, y) > 0$.

The *combinatorial Laplacian*, or simply the *Laplacian* of the network Γ is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \sum_{y\in V} c(x, y)\left(u(x) - u(y)\right), \quad x \in V.$$

Given $q \in \mathcal{C}(V)$, the *Schrödinger operator* on Γ with *potential* q is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$, where $qu \in \mathcal{C}(V)$ is defined as $(qu)(x) = q(x)u(x)$; see for instance [2,7]. It is well-known that any Schrödinger operator is self-adjoint, and we are interested in those Schrödinger operators that are positive semi-definite. The characterization of this type of operator was obtained in [2] by considering, for any $\omega \in \Omega(V)$, the *potential determined by* ω defined as the function $q_\omega = -\omega^{-1}\mathcal{L}(\omega)$. The Schrödinger operator given by this kind of potential, $\mathcal{L}_{q_\omega}$, contains as a particular case the *normalized Laplacian* introduced by Chung and Langlands in [8].

Proposition 1.1 (*[2, Proposition 3.3]*)**.** *The Schrödinger operator $\mathcal{L}_q$ is positive semi-definite iff there exist $\omega \in \Omega(V)$ and $\lambda \geq 0$ such that $q = q_\omega + \lambda$. Moreover, ω and λ are uniquely determined. In addition, $\mathcal{L}_q$ is singular iff $\lambda = 0$, in which case $\langle\mathcal{L}_{q_\omega}(v), v\rangle = 0$ iff $v = a\omega$, $a \in \mathbb{R}$. In any case, λ is the lowest eigenvalue of $\mathcal{L}_q$ and its associated eigenfunctions are multiples of ω.*

Theorem 1.2 (*Fredholm Alternative*)**.** *Given $f \in \mathcal{C}(V)$, $\mathcal{L}_{q_\omega}(u) = f$ is solvable if and only if $\langle f, \omega\rangle = 0$. Moreover, there exists only one solution $u \in \mathcal{C}(V)$ with $\langle u, \omega\rangle = 0$.*

If $\mathcal{L}_q$ is positive definite, then it is invertible and its inverse is called the *Green operator*. On the other hand, when $\mathcal{L}_q$ is positive semi-definite and singular the operator that assigns to each function $f \in \mathcal{C}(V)$ the unique $u \in \mathcal{C}(V)$ such that $\mathcal{L}_q(u) = f - \langle f, \omega\rangle\omega$ and $\langle u, \omega\rangle = 0$ is called the *Green operator*. In any case, the Green operator is denoted by $\mathcal{G}_q$, see [5]. Moreover, the function $G_q : V \times V \longrightarrow \mathbb{R}$, defined as $G_q(x, y) = \mathcal{G}_q(\varepsilon_y)(x)$ for any $x, y \in V$, is called *Green function*. Observe that $\mathcal{G}_q(\omega) = \lambda^\dagger\omega$, where $\lambda^\dagger = \lambda^{-1}$ when $\lambda > 0$ and $\lambda^\dagger = 0$ when $\lambda = 0$. Moreover, $\mathcal{G}_q$ is self-adjoint as a consequence of the Fredholm Alternative and $\mathcal{G}_q$ is a symmetric function.

Observe that $u = G_q(\cdot, y)$ is the only solution of the problem $\mathcal{L}_q(u) = \varepsilon_y - \omega(y)\omega$, $\langle u, \omega\rangle = 0$ when $\lambda = 0$ and $\mathcal{L}_q(u) = \varepsilon_y$ when $\lambda > 0$. In addition, if $\lambda > 0$ then $\mathcal{G}_q(\mathcal{L}_q(u)) = u$ for all $u \in \mathcal{C}(V)$ and if $\lambda = 0$ then $\mathcal{G}_{q_\omega}(\mathcal{L}_{q_\omega}(u)) = \mathcal{L}_{q_\omega}(\mathcal{G}_{q_\omega}(u)) = u - \langle u, \omega\rangle\omega$. In both situations $\mathcal{G}_q(\omega) = \lambda^\dagger\omega$.

In [5,6], a generalization of the concept of Kirchhoff index was introduced by defining the effective resistance and the Kirchhoff index with respect to a value $\lambda \geq 0$ and a weight $\omega \in \Omega(V)$. Specifically, if we consider the functional on $\mathcal{C}(V)$ defined as

$$\mathfrak{J}_{x,y}(u) = 2\left[\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right] - \langle\mathcal{L}_q(u), u\rangle,$$

then we can give the following definitions.

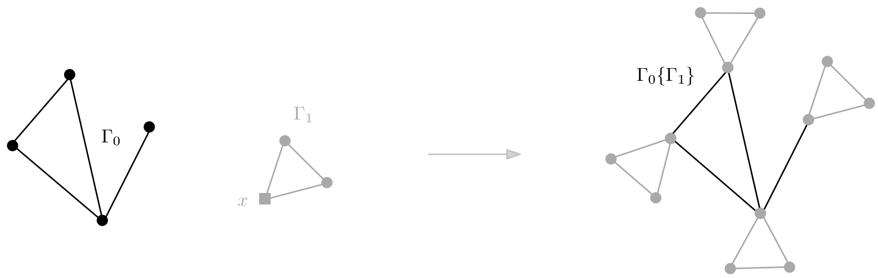

Fig. 2. Cluster.

Definition 1.3. Given $x, y \in V$, the *effective resistance between x and y with respect to* λ *and* ω, is the value

$$R_{\lambda,\omega}(x, y) = \max_{u\in\mathcal{C}(V)} \{\mathfrak{J}_{x,y}(u)\}.$$

Moreover, the *Kirchhoff index of* Γ *with respect to* λ *and* ω, is the value

$$\mathsf{k}(\lambda, \omega) = \frac{1}{2} \sum_{x,y\in V} R_{\lambda,\omega}(x, y)\omega^2(x)\,\omega^2(y).$$

In addition, if we consider the functional

$$\mathfrak{J}_x(u) = 2\left[\frac{u(x)}{\omega(x)} - \langle u, \omega\rangle\right] - \langle \mathcal{L}_q(u), u\rangle$$

the *total resistance at* $x \in V$ *with respect to* λ *and* ω is defined as

$$r_{\lambda,\omega}(x) = \max_{u\in\mathcal{C}(V)} \{\mathfrak{J}_x(u)\}.$$

In the following we drop the expression with respect to λ and ω when it does not lead to confusion. When $\lambda = 0$ we usually omit the subindex λ in the above expressions and when ω is constant we also omit the subindex ω. Therefore, nR is nothing else than the standard effective resistance of the network, whereas $\frac{\mathsf{k}}{n}$ is the Kirchhoff index introduced in the context of organic chemistry, see for instance [13].

The following formulae, which express the different parameters in terms of Green's functions, will be crucial for obtaining the main results of the present paper, see [5] for the proofs.

Proposition 1.4 ([5, Proposition 4.3]). *For any* $x, y \in V$, *it is satisfied that*

$$r_{\lambda,\omega}(x) = \frac{G_q(x, x)}{\omega^2(x)} - \lambda^\dagger \quad \textit{and} \quad R_{\lambda,\omega}(x, y) = \frac{G_q(x, x)}{\omega^2(x)} + \frac{G_q(y, y)}{\omega^2(y)} - \frac{2G_q(x, y)}{\omega(x)\omega(y)}.$$

Therefore,

$$\mathsf{k}(\lambda, \omega) = \sum_{x\in V} r_{\lambda,\omega}(x)\omega^2(x) = \sum_{x\in V} G_q(x, x) - \lambda^\dagger.$$

From now on we will work on two different composite networks: cluster and corona networks. In fact, we will follow the techniques used in [3] in order to obtain the effective resistances and the Kirchhoff indexes of join networks in terms of the same parameters on the factors.

2. Cluster networks

The standard cluster $\Gamma = \Gamma_0\{\Gamma_1\}$ of two graphs Γ_0 and Γ_1 with vertex sets V_0 and V_1 (with a distinguished vertex $x \in V_1$) and edge sets E_0 and E_1 consists of $m = |V_0|$ copies of the graph Γ_1, each one attached to a vertex of Γ_0 identifying the vertex x with this one, as shown in Fig. 2. The edges are maintained as in the original graphs.

For this composite graph the expression of the Kirchhoff index was obtained in [15].

In this section we consider the generalization of the cluster graph to the case of $m + 1$ different networks and we obtain the expression for Green's function, the effective resistances and the Kirchhoff index with respect to a weight in terms of the corresponding parameters of the factors.

Let $\Gamma_0 = (V_0, E_0, c_0)$ be a connected network with vertex set given by $V_0 = \{x_1, \dots, x_m\}$ and let $\Gamma_i = (V_i, E_i, c_i)$, $i = 1, \dots, m$, be connected networks such that $x_i \in V_i$. Let $V = \coprod_{i=1}^m V_i$ be the disjoint union of all vertex sets and consider $\omega \in \Omega(V)$.

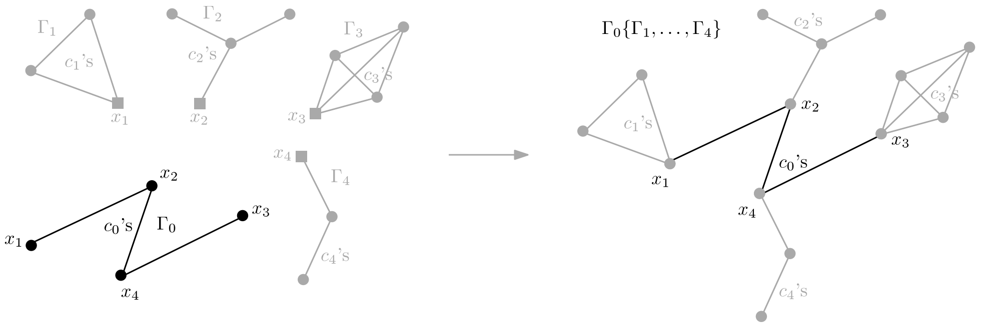

Fig. 3. Cluster network $\Gamma_0\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$.

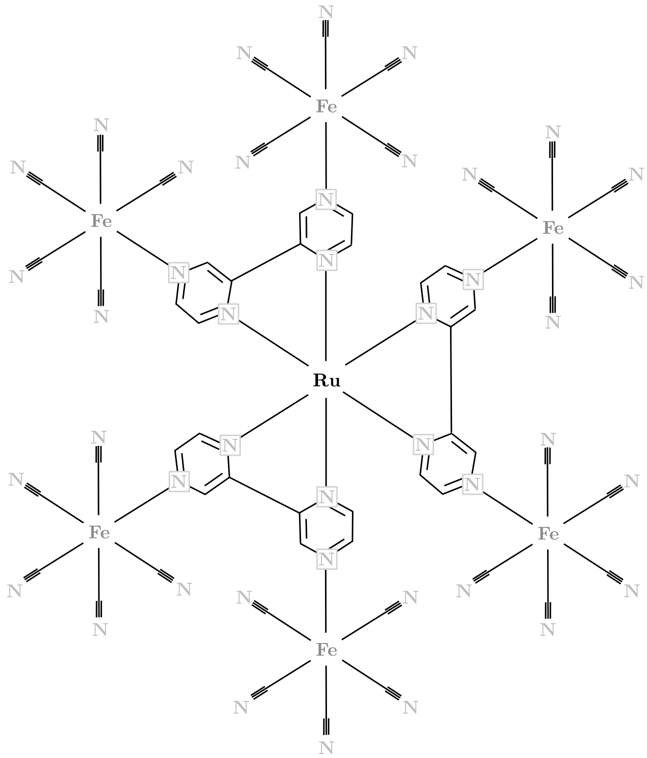

Fig. 4. $[\mathrm{Ru(bpz)_3\,\{Fe(CN)_5\}_6}]^{16}$ metal complex.

We call *cluster network with base Γ_0, satellites $\{\Gamma_i\}_{i=1}^m$ and weight ω* the network $\Gamma = (V, E, c)$ obtained by attaching the networks Γ_i, $i = 1, \ldots, m$, to Γ_0 by identifying the vertices $x_i \in V_i$ and $x_i \in V_0$; that is, the network whose conductance is given by $c(x, y) = c_i(x, y)$ for any $x, y \in V_i$, $i = 0, \ldots, m$, and by $c(x, y) = 0$ otherwise. This network will be denoted by $\Gamma = \Gamma_0\{\Gamma_1, \ldots, \Gamma_m\}$, see Fig. 3.

We do not exclude the case $V_i = \{x_i\}$ for some $i = 1, \ldots, m$, and it is worth mentioning that when this happens the attachment of Γ_i at $\{x_i\}$ does not alter the base network.

Cluster networks are highly relevant in chemistry applications since all composite molecules consisting of some amalgamation over a central submolecule can be understood as a generalized cluster network.

For instance, they can be used to understand some issues on metal–metal interaction in some molecules, see [12], since a cluster network structure can be easily found. We can observe that fact in Fig. 4, which is a representation of the $[\mathrm{Ru(bpz)_3\,\{Fe(CN)_5\}_6}]^{16}$ metal complex that clearly shows its cluster structure.

Consider, for any $i = 0, \ldots, m$ the value $\sigma_i = \left(\sum_{x\in V_i} \omega^2(x)\right)^{\frac{1}{2}}$. Then, given $i = 0, \ldots, m$, if for any $x \in V_i$ we define $\omega_i(x) = \sigma_i^{-1}\omega(x)$, it is clear that $\omega_i \in \Omega(V_i)$. Moreover, for any $i = 0, \ldots, m$, we identify $\mathcal{C}(V_i)$ with the subspace of $\mathcal{C}(V)$ formed by the functions that are null on $V \setminus V_i$. On the other hand, if $u \in \mathcal{C}(V)$, the restriction of u to V_i is also denoted by u. Observe that if $u \in \mathcal{C}(V_i)$ and $v \in \mathcal{C}(V)$, then $\langle u, v\rangle = \sum_{x\in V_i} u(x)v(x)$ and, in particular, $\langle u, v\rangle = 0$ when $v \in \mathcal{C}(V_j)$ with $j \neq i$ and $i, j = 1, \ldots, m$ and $\langle u, v\rangle = u(x_i)v(x_i)$ when $v \in \mathcal{C}(V_0)$ with $i \neq 0$. Observe that $\sum_{j=1}^m \sigma_j^2 = 1$.

From now on, $\mathcal{L}$ will denote the combinatorial Laplacian of the cluster network Γ and for $i = 0, \dots, m$, $\mathcal{L}^i$ will denote the combinatorial Laplacian of the network Γ_i. Then, we get the following result.

In the following we consider the potential on Γ given by q_ω and its corresponding semi-definite and singular Schrödinger operator $\mathcal{L}_{q_\omega}$. In addition, for any $i = 0, \dots, m$, we also consider the potential on Γ_i given by q_{ω_i} and its corresponding semi-definite and singular Schrödinger operator $\mathcal{L}^i_{q_{\omega_i}}$.

Proposition 2.1. *For any $u \in \mathcal{C}(V)$, it holds that*

$$\mathcal{L}_{q_\omega}(u)(x) = \mathcal{L}^i_{q_{\omega_i}}(u)(x) + \mathcal{L}^0_{q_{\omega_0}}(u)(x_i)\varepsilon_{x_i}(x) \quad \text{for all } x \in V_i,\ i = 1, \dots, m.$$

Proof. It suffices to observe that if $x \in V_i$, $i = 1, \dots, m$, then

$$\mathcal{L}(u)(x) = \mathcal{L}^i(u)(x) + \mathcal{L}^0(u)(x_i)\varepsilon_{x_i}(x) \quad \text{and} \quad q_\omega(x) = q_{\omega_i}(x) + q_{\omega_0}(x_i)\varepsilon_{x_i}(x). \quad \square$$

2.1. Green's function and effective resistances

The main objective in this section is to obtain Green's function of the cluster network in terms of Green's functions of the satellites. As a by-product we also obtain the effective resistances and the Kirchhoff index, with respect to a weight, in terms of the corresponding parameters of each factor by applying Proposition 1.4.

Throughout the section, $\mathcal{G}^i_{q_{\omega_i}}$ denotes Green's operator for $\mathcal{L}^i_{q_{\omega_i}}$ on Γ_i, $i = 0, \dots, m$. It will be useful to consider, for any $f \in \mathcal{C}(V)$, the function $g_f \in \mathcal{C}(V_0)$ defined as

$$g_f(x_j) = \frac{\sigma_j\langle f, \omega_j\rangle}{\omega(x_j)} = \frac{\langle f, \omega_j\rangle}{\omega_j(x_j)}, \quad j = 1, \dots, m,$$

which is motivated by the following result.

Lemma 2.2. *Let $f \in \mathcal{C}(V)$ such that $\langle \omega, f\rangle = 0$ and consider u a solution of the Poisson equation $\mathcal{L}_{q_\omega}(u) = f$ on V, then*

$$\mathcal{L}^0_{q_{\omega_0}}(u) = g_f \quad \text{on } V_0.$$

Proof. From Proposition 2.1 we get that u is a solution of $\mathcal{L}_{q_\omega}(u) = f$ on V iff u satisfies $\mathcal{L}^j_{q_{\omega_j}}(u) = f - \mathcal{L}^0_{q_{\omega_0}}(u)(x_j)\varepsilon_{x_j}$ on V_j. Keeping in mind the Fredholm Alternative and that, for fixed $j = 1, \dots, m$, $\mathcal{L}^j_{q_{\omega_j}}(\omega_j) = 0$ we get that

$$0 = \langle f, \omega_j\rangle - \mathcal{L}^0_{q_{\omega_0}}(u)(x_j)\omega_j(x_j)$$

and hence the result follows. $\square$

Proposition 2.3. *Let $f \in \mathcal{C}(V)$ such that $\langle \omega, f\rangle = 0$ and consider the Poisson equation on V, $\mathcal{L}_{q_\omega}(u) = f$. Then, the function*

$$u = \sum_{i=1}^m \mathcal{G}^i_{q_{\omega_i}}(f - g_f\varepsilon_{x_i}) + \sum_{i=1}^m \frac{1}{\omega_i(x_i)}\left(\mathcal{G}^i_{q_{\omega_i}}(f - g_f\varepsilon_{x_i})(x_i) - \mathcal{G}^0_{q_{\omega_0}}(g_f)(x_i)\right)\left[\sigma_i\omega - \omega_i\right]$$

is the unique solution of the Poisson equation such that $\langle \omega, u\rangle = 0$.

Proof. From Proposition 2.1 we get that u is a solution of $\mathcal{L}_{q_\omega}(u) = f$ on V iff u satisfies $\mathcal{L}^j_{q_{\omega_j}}(u) = f - g_f\varepsilon_{x_j}$ on V_j for all $j = 1, \dots, m$ and, hence, there exist $\beta_0, \beta_j \in \mathbb{R}$, such that

$$u = \mathcal{G}^0_{q_{\omega_0}}(g_f) + \beta_0\omega_0 \quad \text{on } V_0,$$
$$u = \mathcal{G}^j_{q_{\omega_j}}\left(f - g_f\varepsilon_{x_j}\right) + \beta_j\omega_j \quad \text{on } V_j.$$

Consequently, if we define $u_j = \mathcal{G}^j_{q_{\omega_j}}\left(f - g_f\varepsilon_{x_j}\right) + \beta_j\omega_j \in \mathcal{C}(V_j)$, then $u = \sum_{j=1}^m u_j$ satisfies $\mathcal{L}_{q_\omega}(u) = f$ and

$$\langle u, \omega\rangle = \sum_{j=1}^m \langle \omega, u_j\rangle = \sum_{j=1}^m \sigma_j\langle \omega_j, u_j\rangle = \sum_{j=1}^m \sigma_j\beta_j.$$

Therefore, $\langle u, \omega\rangle = 0$ if and only if $\sum_{j=1}^m \beta_j\sigma_j = 0$. On the other hand, keeping in mind that $x_j \in V_j \cap V_0$ and that $\omega_j(x_j) = \frac{\sigma_0\omega_0(x_j)}{\sigma_j}$, we get that

$$\beta_j = \frac{\mathcal{G}^0_{q_{\omega_0}}(g_f)(x_j) - \mathcal{G}^j_{q_{\omega_j}}(f - g_f\varepsilon_{x_j})(x_j)}{\omega_j(x_j)} + \beta_0\frac{\sigma_j}{\sigma_0}, \quad j = 1, \dots, m.$$

Therefore,

$$\beta_0 = \sigma_0 \sum_{j=1}^{m} \frac{\sigma_j}{\omega_j(x_j)} \left(\mathcal{G}^j_{q_{\omega_j}}(f - g_f \varepsilon_{x_j})(x_j) - \mathcal{G}^0_{q_{\omega_0}}(g_f)(x_j) \right),$$

and the result follows. □

Theorem 2.4. *The Green function of the cluster network $\Gamma_0\{\Gamma_1, \dots, \Gamma_m\}$ on $V_i \times V_j$ is given by*

$$\begin{aligned} G_{q_\omega} = G^j_{q_{\omega_j}} &+ \frac{1}{\omega_i(x_i)} \left[\left(\sigma_j \sigma_i G^i_{q_{\omega_i}}(\cdot, x_i) - G^j_{q_{\omega_j}}(\cdot, x_j) \right) \otimes \omega_j \right] \\ &+ \frac{1}{\omega_j(x_j)} \left[\omega_i \otimes \left(\sigma_j \sigma_i G^j_{q_{\omega_j}}(\cdot, x_j) - G^i_{q_{\omega_i}}(\cdot, x_i) \right) \right] + g_{ij} \omega_i \otimes \omega_j \end{aligned}$$

where

$$\begin{aligned} g_{ij} = \; & \frac{\sigma_i \sigma_j}{2\sigma_0^2} \left[\sum_{k=1}^{m} \sigma_k^2 \left(R^0_{\omega_0}(x_k, x_j) + R^0_{\omega_0}(x_k, x_i) \right) \right] - \frac{\sigma_i \sigma_j}{2\sigma_0^2} \left[\sum_{k=1}^{m} \sum_{\ell=1}^{m} \sigma_k^2 \sigma_\ell^2 R^0_{\omega_0}(x_k, x_\ell) - R^0_{\omega_0}(x_i, x_j) \right] \\ & + \sigma_i \sigma_j \left[\sum_{k=1}^{n} \sigma_k^2 r^k_{\omega_k}(x_k) - r^i_{\omega_i}(x_i) - r^j_{\omega_j}(x_j) \right] + r^i_{\omega_i}(x_j). \end{aligned}$$

Proof. For any $y \in V$, $u = G_{q_\omega}(\cdot, y)$ is the unique solution of $\mathcal{L}_{q_\omega}(u) = \varepsilon_y - \omega(y)\omega$ such that $\langle u, \omega \rangle = 0$. Then, applying Proposition 2.3 to $f = \varepsilon_y - \omega(y)\omega$, the explicit expression of G_{q_ω} is deduced if we reduce the problem to some cases: as $y \in V_j$ for a given $j \in \{1, \dots, m\}$, then $f = \varepsilon_y - \sigma_j \omega_j(y)\omega$ and therefore

$$\begin{aligned} & \langle f, \omega_j \rangle = \omega_j(y)(1 - \sigma_j^2) \\ & \mathcal{G}^0_{q_{\omega_0}} \left(\frac{\langle f, \omega_j \rangle}{\omega_j(x_j)} \right) = \frac{\omega_j(y)}{\omega_j(x_j)} G^0_{q_{\omega 0}}(x_j, x_j) - \sigma_j \omega_j(y) \sum_{l=1}^{m} \frac{\sigma_l G^0_{q_{\omega_0}}(x_j, x_l)}{\omega_l(x_l)} \\ & \mathcal{G}^j_{q_{\omega_j}}(f)(x_j) = G^j_{q_{\omega_j}}(x_j, y) \\ & \mathcal{G}^j_{q_{\omega_j}}(f)(x) = G^j_{q_{\omega_j}}(x, y), \quad x \in V_j \\ & \langle f, \omega_k \rangle = -\sigma_j \sigma_k \omega_j(y), \quad k \neq j \\ & \mathcal{G}^0_{q_{\omega_0}} \left(\frac{\langle f, \omega_k \rangle}{\omega_k(x_k)} \right) = \frac{\omega_j(y)}{\omega_j(x_j)} G^0_{q_{\omega 0}}(x_k, x_j) - \sigma_j \omega_j(y) \sum_{l=1}^{m} \frac{\sigma_l G^0_{q_{\omega_0}}(x_k, x_l)}{\omega_l(x_l)}, \quad k \neq j \\ & \mathcal{G}^k_{q_{\omega_k}}(f)(x_k) = 0, \quad k \neq j \\ & \mathcal{G}^k_{q_{\omega_k}}(f)(x) = 0, \quad x \in V_k, \ k \neq j. \end{aligned}$$

Applying Proposition 2.3, keeping in mind that $G_{q_\omega}(x, y) = u(x)$, and taking into account that x can be either in V_j or in V_k, for $k \neq j$, we get the result using the formulae corresponding to each case. □

The following result gives the expression of the Kirchhoff index and the effective resistances with respect to a weight in terms of the corresponding parameters of the satellites. These expressions follow directly from the formulae for the Kirchhoff index and for the effective resistances given in Proposition 1.4. In all expressions, the superscript i in a parameter stands for the corresponding parameter on the network Γ_i.

Proposition 2.5. *The Kirchhoff index of the cluster network $\Gamma_0\{\Gamma_1, \dots, \Gamma_m\}$ is given by*

$$\mathsf{k}(\omega) = \sum_{i=1}^{m} \mathsf{k}^i(\omega_i) + \frac{1}{2\sigma_0^2} \sum_{i=1}^{m} \sum_{j=1}^{m} \sigma_i^2 \sigma_j^2 R^0_{\omega_0}(x_i, x_j) + \sum_{i=1}^{m} (1 - \sigma_i^2) r^i_{\omega_i}(x_i).$$

Moreover, the cluster total resistances and effective resistances with respect to ω are given, for all $x, y \in V$, by

$$\begin{aligned} r_\omega(x) = \; & r^j_{\omega_j}(x) - r^j_{\omega_j}(x_j) - \frac{(\sigma_j^2 - 1)}{\sigma_j^2} R^j_{\omega_j}(x, x_j) + \frac{1}{\sigma_0^2} \sum_{k,\ell=1}^{m} \sigma_k^2 \sigma_\ell^2 \left[R^0_{\omega_0}(x_k, x_j) - R^0_{\omega_0}(x_k, x_\ell) \right] \\ & + \sum_{k=1}^{n} \sigma_k^2 r^k_{\omega_k}(x_k), \quad x \in V_j, \end{aligned}$$

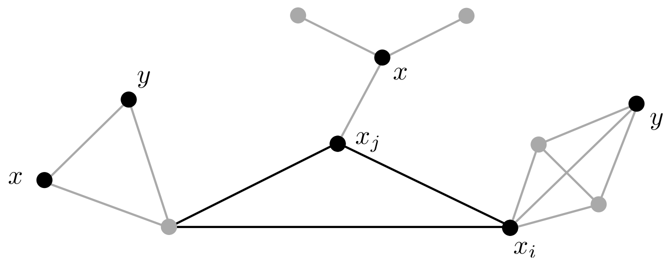

Fig. 5. The resistance between purple x and y is the (weight-adapted) resistance in the satellite network whereas the resistance between brown x and yellow y is the (weight-adapted) sum of the resistance between x and x_j on Γ_j, the resistance between x_j and x_i on Γ_0 and the resistance between x_i and y on Γ_i. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

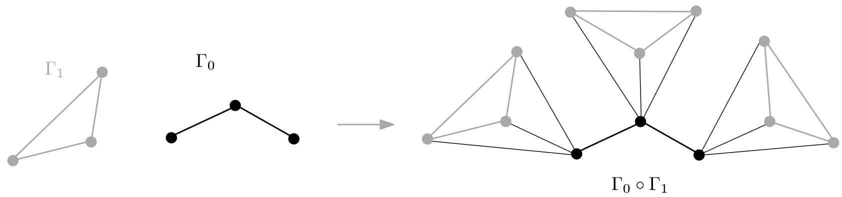

Fig. 6. The corona $\Gamma = \Gamma_0 \circ \Gamma_1$ of two graphs Γ_0 and Γ_1.

$$R_\omega(x, y) = \frac{R^j_{\omega_j}(x, y)}{\sigma_j^2}, \quad x, y \in V_j,$$

$$R_\omega(x, y) = \frac{R^i_{\omega_i}(x, x_i)}{\sigma_i^2} + \frac{R^0_{\omega_0}(x_i, x_j)}{\sigma_0^2} + \frac{R^j_{\omega_j}(x_j, y)}{\sigma_j^2}, \quad x \in V_i, y \in V_j, \ i \neq j.$$

We can say now that the result about effective resistances could have been deduced from the observation of the cluster network structure: the resistance between two vertices belonging to two different satellite networks is the (weight-adapted) sum of the resistances between each one of the vertices and the corresponding vertex in the base network and the resistance between these two vertices on the base network, see Fig. 5.

For the standard case, that is, when all the satellites are the same and the weight and the conductance equal 1, the last result was obtained in [15].

3. Corona networks

The corona $\Gamma = \Gamma_0 \circ \Gamma_1$ of two graphs Γ_0 and Γ_1 with disjoint vertex sets V_0 and V_1 and edge sets E_0 and E_1 is the graph set up by Γ_0 and $m = |V_0|$ copies of Γ_1. The edges are the ones connecting each vertex from a copy of Γ_1 with one vertex of Γ_0, in addition to the ones on the original graphs, as shown in Fig. 6.

For this composite graph the structure of the Kirchhoff index has been also studied (see [15]).

In this section we consider the generalization of the corona graph to the case of $m+1$ different networks and we proceed to study the same parameters and operators as has been done in the previous section for cluster networks.

Let $\Gamma_0 = (V_0, E_0, c_0)$ be a connected network with vertex set given by $V_0 = \{x_1, \dots, x_m\}$ and let $\Gamma_i = (V_i, E_i, c_i)$, $i = 1, \dots, m$, be m connected networks. Let $V = V_0 \cup \coprod_{i=1}^m V_i$ be the disjoint union of all vertex sets. We also consider $\omega \in \Omega(V)$.

We call *corona network with base* Γ_0, *weight* ω *and conductances* $\{a_i\}_{i=1}^m$ the network $\Gamma = (V, E, c)$, denoted by $\Gamma_0 \circ (\Gamma_1, \dots, \Gamma_m)$, where the conductances are given, for every pair $x, y \in V$, by

$$c(x, y) = \begin{cases} c_i(x, y), & x, y \in V_i, \ i = 1, \dots, m \\ a_i \omega(x)\omega(x_i), & x \in V_i, y = x_i \\ 0, & \text{otherwise}, \end{cases}$$

see Fig. 7.

Remark 3.1. $\Gamma_0 \circ (\Gamma_1, \dots, \Gamma_m) = \Gamma_0 \{(\Gamma_1 + \{x_1\}), \dots, (\Gamma_m + \{x_m\})\}$, where $\Gamma_i + \{x_i\}$ is the generalized join network with join conductance a_i, $i = 1, \dots, m$, see [3]. This equivalence is shown graphically in Fig. 8.

Consider, for any $i = 0, \dots, m$, the values $\sigma_i = \left(\sum_{x \in V_i} \omega^2(x)\right)^{\frac{1}{2}}$. Moreover, we define $\omega_i(x) = \sigma_i^{-1}\omega(x)$ for all $x \in V_i$, $i = 0, \dots, m$. It is clear that $\omega_i \in \Omega(V_i)$. Observe that if $u \in \mathcal{C}(V_i)$ and $v \in \mathcal{C}(V)$, then $\langle u, v \rangle = \sum_{x \in V_i} u(x)v(x)$ and, in particular, $\langle u, v \rangle = 0$ when $v \in \mathcal{C}(V_j)$ with $j \neq i$ and $i, j = 0, \dots, m$. Observe that $\sum_{j=1}^m \sigma_j^2 = 1 - \sigma_0^2$.

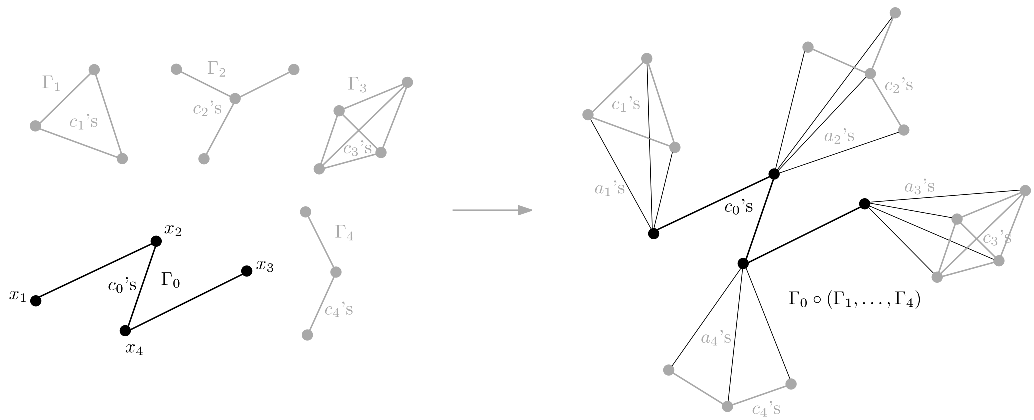

Fig. 7. Generalized corona.

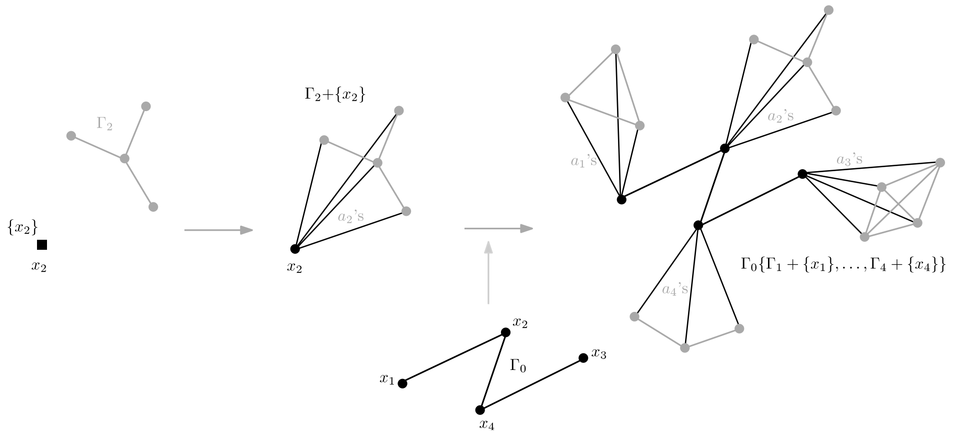

Fig. 8. Equivalence between corona networks and a cluster of join networks.

If $\mathcal{L}$ is the combinatorial Laplacian of the corona network Γ and, for $i = 0, \ldots, m$, $\mathcal{L}^i$ denotes the combinatorial Laplacian of the network Γ_i. Moreover, we consider the potential on Γ given by q_ω and its corresponding semi-definite and singular Schrödinger operator $\mathcal{L}_{q_\omega}$. In addition, for any $i = 0, \ldots, m$, we also consider the potential on Γ_i given by q_{ω_i} and its corresponding semi-definite and singular Schrödinger operator $\mathcal{L}^i_{q_{\omega_i}}$.

Proposition 3.2. *For any $u \in \mathcal{C}(V)$ and for all $j = 1, \ldots, m$,*

$$\mathcal{L}_{q_\omega}(u)(x_j) = \mathcal{L}^0_{q_{\omega_0}}(u)(x_j) + a_j\sigma_j\Big(\sigma_j u(x_j) - \sigma_0\omega_0(x_j)\langle\omega_j, u\rangle\Big),$$
$$\mathcal{L}_{q_\omega}(u)(x) = \mathcal{L}^j_{p_j}(u)(x) - a_j\sigma_0\omega_0(x_j)u(x_j)\sigma_j\omega_j(x), \quad \textit{with } x \in V_j,$$

where $p_j = q_{\omega_j} + \gamma_j$ and $\gamma_j = a_j\sigma_0^2\omega_0^2(x_j)$.

Proof. It suffices to observe that for any $u \in \mathcal{C}(V)$ and for all $j = 1, \ldots, m$,

$$\mathcal{L}(u)(x_j) = \mathcal{L}^0(u)(x_j) + \sigma_0\omega_0(x_j)a_j\sigma_j\big(u(x_j)\langle\omega_j, 1\rangle - \langle\omega_j, u\rangle\big)$$
$$\mathcal{L}(u)(x) = \mathcal{L}^j(u)(x) + a_j\sigma_j\omega_j(x)\omega_0(x_j)\big(u(x) - u(x_j)\big), \quad \text{with } x \in V_j$$

and hence

$$q_\omega(x_j) = q_{\omega_0}(x_j) + a_j\sigma_j\big(\sigma_j - \sigma_0\omega_0(x_j)\langle\omega_j, 1\rangle\big),$$
$$q_\omega(x) = q_{\omega_j}(x) - a_j\sigma_0\omega_0(x_j)\big(\sigma_j\omega_j(x) - \sigma_0\omega_0(x_j)\big), \quad \text{with } x \in V_j. \quad \Box$$

3.1. Green's function and effective resistances

The main objective in this section is to obtain Green's function of the corona network, as well as the effective resistances and the Kirchhoff index, using the techniques developed in the previous section for the case of the cluster network.

From now on, $\mathcal{G}^0_{q_{\omega_0}}$ will denote the Green operator for $\mathcal{L}^0_{q_{\omega_0}}$ on Γ_0 and $\mathcal{G}^i_{p_i}$ will stand for the Green operator of $\mathcal{L}^i_{p_i}$ on Γ_i, $i = 1, \dots, m$. It will be useful to consider, for any $f \in \mathcal{C}(V)$, the function $h_f \in \mathcal{C}(V_0)$ defined as

$$h_f(x_j) = \frac{\sigma_j \langle f, \omega_j \rangle}{\omega(x_j)}, \quad j = 1, \dots, m.$$

In particular, $h_\omega(x_j) = \frac{\sigma_j^2}{\omega(x_j)}$. The definition of h_f is motivated by the following result.

Lemma 3.3. *Let $f \in \mathcal{C}(V)$ such that $\langle \omega, f \rangle = 0$ and consider u a solution of the Poisson equation $\mathcal{L}_{q_\omega}(u) = f$ on V, then*

$$\mathcal{L}^0_{q_{\omega_0}}(u) = f + h_f \quad \text{on } V_0.$$

Proof. From Proposition 3.2 we get that u is a solution of $\mathcal{L}_{q_\omega}(u) = f$ on V iff it satisfies, for every $j = 1, \dots, m$ and for any $x \in V_j$

$$\mathcal{L}^0_{q_{\omega_0}}(u)(x_j) = f(x_j) - a_j\sigma_j\big(\sigma_j u(x_j) - \sigma_0\omega_0(x_j)\langle \omega_j, u \rangle\big),$$
$$\mathcal{L}^j_{p_j}(u)(x) = f(x) + a_j\sigma_0\omega_0(x_j)u(x_j)\sigma_j\omega_j(x).$$

As $\mathcal{L}^j_{p_j}$ is self-adjoint on $\mathcal{C}(V_j)$ and $\mathcal{L}^j_{p_j}(\omega_j) = a_j\sigma_0^2\omega_0^2(x_j)\omega_j$, we get that

$$a_j\sigma_0^2\omega_0^2(x_j)\langle \omega_j, u \rangle = \langle \mathcal{L}^j_{p_j}(\omega_j), u \rangle = \langle \omega_j, \mathcal{L}^j_{p_j}(u) \rangle = \langle \omega_j, f \rangle + a_j\sigma_j\sigma_0\omega_0(x_j)u(x_j).$$

Therefore, $u(x_j)$ can be expressed in terms of the unknown $\langle \omega_j, u \rangle$ and some of the parameters. Specifically, we get that

$$u(x_j) = \frac{\sigma_0\omega_0(x_j)\langle \omega_j, u \rangle}{\sigma_j} - \frac{h_f(x_j)}{a_j\sigma_j^2}$$

and the result follows. $\square$

Proposition 3.4. *Let $f \in \mathcal{C}(V)$ such that $\langle \omega, f \rangle = 0$ and consider the Poisson equation on V, $\mathcal{L}_{q_\omega}(u) = f$. Then, the function*

$$u = \mathcal{G}^0_{q_{\omega_0}}(f + h_f) - \left(\langle \mathcal{G}^0_{q_{\omega_0}}(f + h_f), g_\omega \rangle + \sum_{k=1}^m \frac{h_f(x_k)}{a_k\omega(x_k)} \right)\omega \quad \text{on } V_0$$
$$u = \mathcal{G}^j_{p_j}(f) + \frac{u(x_j)}{\omega(x_j)}\omega \quad \text{on } V_j,\ j = 1, \dots, m,$$

is the unique solution of the Poisson equation such that $\langle \omega, u \rangle = 0$.

Proof. From Proposition 3.2 and the above lemma, u is a solution of the equation $\mathcal{L}_{q_\omega}(u) = f$ on V iff it satisfies, for every $j = 1, \dots, m$ and for any $x \in V_j$

$$\mathcal{L}^0_{q_{\omega_0}}(u) = f + h_f, \quad \text{on } V_0,$$
$$\mathcal{L}^j_{p_j}(u) = f + a_j\sigma_0\omega_0(x_j)u(x_j)\sigma_j\omega_j, \quad \text{on } V_j$$

where

$$u(x_j) = \frac{\sigma_0\omega_0(x_j)\langle \omega_j, u \rangle}{\sigma_j} - \frac{h_f(x_j)}{a_j\sigma_j^2}.$$

Applying the Green operators of the factors we get that

$$u - \langle u, \omega_0 \rangle \omega_0 = \mathcal{G}^0_{q_{\omega_0}}(f + h_f), \quad \text{on } V_0,$$

and for all $j = 1, \dots, m$

$$u = \mathcal{G}^j_{p_j}(f) + \frac{\sigma_j u(x_j)}{\sigma_0\omega_0(x_j)}\omega_j \quad \text{on } V_j.$$

Taking into account that $\langle \omega, u \rangle = 0$ iff

$$\langle \omega_0, u \rangle = -\frac{1}{\sigma_0}\sum_{j=1}^m \sigma_j \langle \omega_j, u \rangle$$

and the two expressions obtained for u on V_0, we get that

$$\frac{\sigma_0\omega_0(x_j)}{\sigma_j}\langle\omega_j,u\rangle+\omega_0(x_j)\sum_{i=1}^m\frac{\sigma_i}{\sigma_0}\langle\omega_i,u\rangle=\mathcal{G}^0_{q_{\omega_0}}(f+h_f)(x_j)+\frac{h_f(x_j)}{a_j\sigma_j^2}.$$

This linear system is equivalent to

$$\sigma_0^2\langle\omega_j,u\rangle+\sum_{i=1}^m\sigma_j\sigma_i\langle\omega_i,u\rangle=\frac{\sigma_0\sigma_j}{\omega_0(x_j)}\left(\mathcal{G}^0_{q_{\omega_0}}(f+h_f)(x_j)+\frac{h_f(x_j)}{a_j\sigma_j^2}\right),$$

whose coefficient matrix is $\mathsf{H}=\sigma_0^2\,\mathsf{I}+\sigma\otimes\sigma$, where I is the identity matrix and $\sigma=(\sigma_1,\dots,\sigma_m)$. Therefore, $\mathsf{H}^{-1}=\sigma_0^{-2}\left[\mathsf{I}-\sigma\otimes\sigma\right]$, which implies that

$$\langle u,\omega_j\rangle=\frac{\sigma_j}{\omega(x_j)}\left(\mathcal{G}^0_{q_{\omega_0}}(f+h_f)(x_j)+\frac{h_f(x_j)}{a_j\sigma_j^2}\right)-\sigma_j\sum_{i=1}^m\frac{\sigma_i^2}{\omega(x_i)}\left(\mathcal{G}^0_{q_{\omega_0}}(f+h_f)(x_i)+\frac{h_f(x_i)}{a_i\sigma_i^2}\right),$$

for all $j=1,\dots,m$, and the results follow by substituting this expression on u. $\square$

Theorem 3.5. *The Green function of the corona network Γ is given by*

$$G_{q_\omega}=G^0_{q_\omega}-\left[\omega\otimes\mathcal{G}^0_{q_{\omega_0}}(h_\omega)+\mathcal{G}^0_{q_{\omega_0}}(h_\omega)\otimes\omega\right]+g_{00}\,\omega\otimes\omega\quad\text{on } V_0\times V_0,$$

$$G_{q_\omega}=\frac{1}{\omega(x_k)}\mathcal{G}^0_{q_{\omega_0}}(\varepsilon_{x_k})\otimes\omega-\mathcal{G}^0_{q_{\omega_0}}(h_\omega)\otimes\omega+\left(g_{00}-\frac{\mathcal{G}^0_{q_{\omega_0}}(h_\omega)(x_k)}{\omega(x_k)}-\frac{1}{\gamma_k}\right)\omega\otimes\omega\quad\text{on } V_0\times V_k,$$

$$G_{q_\omega}=\frac{1}{\omega(x_k)}\omega\otimes\mathcal{G}^0_{q_{\omega_0}}(\varepsilon_{x_k})-\omega\otimes\mathcal{G}^0_{q_{\omega_0}}(h_\omega)+\left(g_{00}-\frac{\mathcal{G}^0_{q_{\omega_0}}(h_\omega)(x_k)}{\omega(x_k)}-\frac{1}{\gamma_k}\right)\omega\otimes\omega\quad\text{on } V_k\times V_0,$$

$$G_{q_\omega}=G^k_{p_k}+\left(g_{00}-\frac{\mathcal{G}^0_{q_{\omega_0}}(h_\omega)(x_k)}{\omega(x_k)}-\frac{\mathcal{G}^0_{q_{\omega_0}}(h_\omega)(x_j)}{\omega(x_j)}\right)\omega\otimes\omega+\left(-\frac{1}{\gamma_k}-\frac{1}{\gamma_j}+\frac{\mathcal{G}^0_{q_{\omega_0}}(x_k,x_j)}{\omega(x_k)\omega(x_j)}\right)\omega\otimes\omega\quad\text{on } V_k\times V_j$$

where $g_{00}=\langle h_\omega,\mathcal{G}^0_{q_{\omega_0}}(h_\omega)\rangle+\sum_{i=1}^m\frac{\sigma_i^2}{\gamma_i}$.

Proof. It suffices to note that if $y\in V$, then $u=G_{q_\omega}(\cdot,y)=\mathcal{G}_{q_\omega}(\varepsilon_y)$ is the unique solution to the equation $\mathcal{L}_{q_\omega}(u)=\varepsilon_y-\omega(y)\omega$ such that $\langle\omega,u\rangle=0$. Then, applying Proposition 3.4 with $f=\varepsilon_y-\omega(y)\omega$, the explicit expression of G_{q_ω} is deduced if we reduce the problem to some cases:

1. If $y=x_j\in V_0$ for a given $j\in\{1,\dots,m\}$, then $f=\varepsilon_{x_j}-\sigma_0\omega_0(x_j)\omega$ and therefore, for all $i=1,\dots,m$,

$$\mathcal{G}^0_{q_{\omega_0}}(f)(x_i)=G^0_{q_{\omega_0}}(x_i,x_j)$$
$$\langle f,\omega_i\rangle=-\sigma_0\sigma_i\omega_0(x_j)$$
$$\mathcal{G}^0_{q_{\omega_0}}\left(\frac{\sigma_i\langle f,\omega_i\rangle}{\omega_0(x_i)}\right)=-\sigma_0\omega_0(x_j)\mathcal{G}^0_{q_{\omega_0}}\left(\frac{\sigma_i^2}{\omega_0(x_i)}\right)$$
$$\mathcal{G}^i_{p_i}(f)(x)=-\frac{\sigma_i\omega_0(x_j)\omega_i(x)}{a_i\sigma_0\omega_0^2(x_i)},\quad x\in V_i.$$

Applying Proposition 3.4, keeping in mind that $G_{q_\omega}(x,y)=u(x)$, and taking into account that x can be either in V_0 or in V_i, where $i\in\{1,\dots,m\}$, we get the result using the formulae corresponding to each case.

2. If $y\in V_j$ for a given $j\in\{1,\dots,m\}$, then $f=\varepsilon_y-\sigma_j\omega_j(y)\omega$ and therefore, for all $i=1,\dots,m$,

$$\langle f,\omega_i\rangle=\omega_i(y)\chi_{V_i}(y)-\sigma_j\sigma_i\omega_j(y)$$
$$\mathcal{G}^0_{q_{\omega_0}}(f)(x_i)=0$$
$$\mathcal{G}^0_{q_{\omega_0}}\left(\frac{\sigma_i\langle f,\omega_i\rangle}{\omega_0(x_i)}\right)=\frac{\sigma_j\omega_j(y)}{\omega_0(x_j)}G^0_{q_{\omega_0}}(x_i,x_j)-\sigma_j\omega_j(y)\mathcal{G}^0_{q_{\omega_0}}\left(\frac{\sigma_i^2}{\omega_0(x_i)}\right)$$
$$\mathcal{G}^i_{p_i}(f)(x)=G^i_{p_i}(x,y)\chi_{V_i}(y)-\frac{\sigma_i\sigma_j\omega_i(x)\omega_j(y)}{a_i\sigma_0^2\omega_0^2(x_i)},\quad x\in V_i.$$

Applying Proposition 3.4, keeping in mind that $G_{q_\omega}(x,y)=u(x)$, and taking care of the fact that x can be either $x=x_i\in V_0$ or $x\in V_i$, where $i\in\{1,\dots,m\}$, we get the result using the formulae corresponding to each x-case for this y. $\square$

The following result gives the expression of the Kirchhoff index and the effective resistances with respect to a weight in terms of the corresponding parameters of the networks that form the corona network. In all the expressions, the superscript i in the parameters stands for the corresponding parameters of the network Γ_i if $i = 0, \dots, m$.

Proposition 3.6. *The Kirchhoff index of the corona network is given by*

$$\mathsf{k}(\omega) = \mathsf{k}_0(\omega_0) + \sum_{i=1}^{m} \mathsf{k}_i(\gamma_i, \omega_i) + \sum_{i=1}^{m} \frac{(1-\sigma_i^2)}{\gamma_i} + \sum_{i=1}^{m} {\sigma_i}^2 r^0_{\omega_0}(x_i) + \frac{1}{2\sigma_0^2} \sum_{i,j=1}^{m} {\sigma_i}^2 {\sigma_j}^2 R^0_{\omega_0}(x_i, x_j).$$

Moreover, for any $i = 1, \dots, m$, it is satisfied that

$$r_\omega(x_i) = r^0_{\omega_0}(x_i) - \sum_{k=1}^{m} \sigma_k^2 r^0_{\omega_0}(x_k) - \frac{1}{2\sigma_0^2} \sum_{k,j=1}^{m} \sigma_k^2 \sigma_j^2 R^0_{\omega_0}(x_k, x_j) + \frac{1}{\sigma_0^2} \sum_{j=1}^{m} {\sigma_j}^2 R^0_{\omega_0}(x_i, x_j) + \sum_{j=1}^{m} \frac{\sigma_j^2}{\gamma_j}$$

$$r_\omega(x) = \frac{r^i_{\gamma_i,\omega_i}(x)}{\sigma_i^2} - \frac{1}{\sigma_i^2 \gamma_i} + r_\omega(x_i), \quad x \in V_i,$$

and for any $i, j = 1, \dots, m$

$$R_\omega(x_i, x_j) = \frac{R^0_{\omega_0}(x_i, x_j)}{\sigma_0^2},$$

$$R_\omega(x_i, y) = \frac{R^0_{\omega_0}(x_i, x_j)}{\sigma_0^2} + \frac{r^j_{\gamma_j,\omega_j}(x)}{\sigma_j^2} + \frac{1}{\sigma_j^2 \gamma_j}, \quad y \in V_j,$$

$$R_\omega(x, y) = \frac{R^j_{\gamma_j,\omega_j}(x, y)}{\sigma_j^2}, \quad x, y \in V_j,$$

$$R_\omega(x, y) = \frac{R^0_{\omega_0}(x_i, x_j)}{\sigma_0^2} + \frac{r^i_{\gamma_i,\omega_i}(x)}{\sigma_i^2} + \frac{r^j_{\gamma_j,\omega_j}(y)}{\sigma_j^2} + \frac{1}{\sigma_i^2 \gamma_i} + \frac{1}{\sigma_j^2 \gamma_j}, \quad x \in V_i, y \in V_j, i \neq j.$$

Proof. It suffices to apply the formulae given in Proposition 1.4 and that

$$\begin{aligned}
\frac{2\mathcal{G}^0_{\omega_0}(h_\omega)(x_i)}{\omega(x_i)} &= \frac{1}{\sigma_0^2} \sum_{j=1}^{m} {\sigma_j}^2 \frac{2G^0_{\omega_0}(x_i, x_j)}{\omega_0(x_i)\omega_0(x_j)} \\
&= \frac{1}{\sigma_0^2} \sum_{j=1}^{m} {\sigma_j}^2 \left(r^0_{\omega_0}(x_j) + r^0_{\omega_0}(x_i) - R^0_{\omega_0}(x_i, x_j) \right) \\
&= \frac{(1-\sigma_0^2)}{\sigma_0^2} r^0_{\omega_0}(x_i) + \frac{1}{\sigma_0^2} \sum_{j=1}^{m} {\sigma_j}^2 r^0_{\omega_0}(x_j) - \frac{1}{\sigma_0^2} \sum_{j=1}^{m} {\sigma_j}^2 R^0_{\omega_0}(x_i, x_j).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle h_\omega, \mathcal{G}^0_{\omega_0}(h_\omega) \rangle &= \sum_{i=1}^{m} \mathcal{G}^0_{\omega_0}(h_\omega)(x_i) \frac{\sigma_i^2}{\omega(x_i)} \\
&= \frac{1}{2} \sum_{i=1}^{m} \left(\frac{(1-\sigma_0^2)}{\sigma_0^2} r^0_{\omega_0}(x_i) + \frac{1}{\sigma_0^2} \sum_{j=1}^{m} {\sigma_j}^2 r^0_{\omega_0}(x_j) \right) \sigma_i^2 - \frac{1}{2} \sum_{i=1}^{m} \left(\frac{1}{\sigma_0^2} \sum_{j=1}^{m} {\sigma_j}^2 R^0_{\omega_0}(x_i, x_j) \right) \sigma_i^2 \\
&= \frac{(1-\sigma_0^2)}{\sigma_0^2} \sum_{i=1}^{m} r^0_{\omega_0}(x_i) \sigma_i^2 - \frac{1}{2\sigma_0^2} \sum_{i,j=1}^{m} {\sigma_i}^2 {\sigma_j}^2 R^0_{\omega_0}(x_i, x_j). \quad \square
\end{aligned}$$

For the standard case, that is, for a standard corona with unnormalized constant weight $\omega = 1$ and constant conductances 1, the last proposition was obtained in [15].

4. Conclusions

We remark that the results on this paper justify the definition of the effective resistances with respect to a non-negative value, since they naturally appear when we want to calculate the effective resistance of a composite network in terms of its satellite networks. Moreover, it justifies the generalization of the concept of Kirchhoff index with respect to a value $\lambda \geq 0$ and a weight $\omega \in \Omega(V)$.

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