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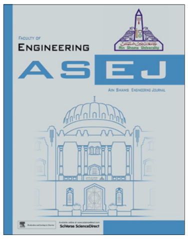

MECHANICAL ENGINEERING

Similarity solution of mixed convective boundary layer slip flow over a vertical plate

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KEYWORDS

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Slip boundary condition

Abstract Effects of slip at the boundary on the mixed convective boundary layer flow over a flat plate are investigated. Similarity transformations are employed to transform the governing partial differential equations into ordinary ones, which are then solved numerically by shooting method. The results are presented for different values of the governing parameters. Comparison with available results for certain cases is excellent. For increasing slip parameter, velocity increases at first but decreases after a point and thermal boundary layer becomes thinner in this case. Velocity overshoot and temperature overshoot are noted near the plate for increasing mixed convection parameter as well as both velocity and thermal slip parameters and also for Prandtl number.

1. Introduction

The thermal buoyancy generated due to heating/cooling of a vertical plate has a large impact on flow and heat transfer characteristics. Convection heat transfer in the fluid flow is a phenomenon of great interest from both theoretical and practical point of views because of its vast applications in many engineering and geophysical fields.

Blasius [1] first considered the steady laminar boundary layer flow of viscous incompressible fluid over a flat plate. Howarth [2] numerically studied various aspects of the Blasius flat-plate flow problem. Combined forced and natural convection over a flat plate has been widely studied from both theoretical and experimental standpoint over the past few decades. The effect of a magnetic field on free convection heat transfer on isothermal vertical plate was discussed by Sparrow and Cess [3]. Gupta [4] studied laminar free convection flow of an electrically conducting fluid past a vertical plate with uniform surface heat flux and variable wall temperature in presence of a magnetic field. Afzal and Hussain [5] discussed the mixed convection over a horizontal plate. Yao [6] investigated the two-dimensional mixed convection along a flat plate. Hossain and Takhar [7] determined the effect of radiation on mixed convection along a vertical plate with uniform free stream velocity and surface temperature. Aydin and Kaya [8] analyzed the mixed convection flow of a viscous dissipating fluid about a vertical flat plate.

The non-adherence of the fluid to a solid boundary, known as velocity slip, is a phenomenon that has been observed under

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certain circumstances. When fluid in microelectro mechanical systems are encountered, the no-slip condition at the solid–fluid interface is abandoned in favour of a slip flow model which represents more accurately the non-equilibrium region near the interface. In all of the above mentioned investigations, the no-slip condition at the boundary was assumed. Even in literature, the study of the slip flow over a flat plate is not sufficiently available. Martin and Boyd [9] considered the momentum and heat transfer in a laminar boundary layer flow over a flat plate with slip boundary condition. Cao and Baker [10] presented local non-similar solutions to the boundary layer equations for mixed convection over a vertical isothermal plate. Also, velocity slip and thermal jump boundary conditions were considered in their problem. Harris et al. [11] studied the mixed convection boundary-layer stagnation point flow on a vertical surface in a porous medium with slip. Recently, Aziz [12] studied the boundary layer slip flow over a flat plate with constant heat flux condition at the surface and in this paper the local similarity was appeared in the slip boundary condition. Bhattacharyya et al. [13,14] discussed the MHD slip flow over a flat plate and the steady slip flow on a flat plate in porous medium. Very recently, a numerical investigation of unsteady mixed convection boundary-layer flow near the two-dimensional stagnation point on a vertical permeable surface embedded in a fluid-saturated porous medium with thermal slip was reported by Rohni et al. [15]. The effects of slip boundary condition on the flow of Newtonian fluid due to a stretching sheet were explained by Andersson [16] and Wang [17].

To the best of authors' knowledge, no previous study has been undertaken to examine the simultaneous effects of velocity slip and thermal slip upon mixed convection presenting similarity solutions to the boundary layer equations for the prediction of flow and heat transfer behaviour. In the present paper, we investigate the slip effect on boundary layer mixed convection flow from a vertical plate. The slip model of Andersson [16] is taken in some modified form [13,14]. A self-similar set of equations is obtained and then those are solved numerically using shooting method. Computed numerical results are plotted and the characteristics of the flow and heat transfer are analyzed in detail.

2. Mathematical formulation of the problem

Let us consider the steady two-dimensional laminar mixed convective flow of a viscous incompressible fluid over a vertical plate. Using boundary layer approximation, the equations for the flow and the temperature are written in usual notation as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}+g\beta^*(T-T_\infty), \tag{2}$$

$$\text{and } u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

where x and y are axes along and perpendicular to the plate respectively, u and v are velocity components in x- and y-directions respectively, ρ is the fluid density, μ is the coefficient of fluid viscosity, $\upsilon\ (=\mu/\rho)$ is the kinematic fluid viscosity, β^* is the volumetric coefficient of thermal expansion, g is the acceleration due to gravity, T is the temperature, T_∞ is the free stream temperature, κ is the thermal conductivity of the fluid and c_p is the specific heat.

The appropriate boundary conditions for the velocity components with velocity slip and the temperature with thermal slip are given by:

$$u=L_1(\partial u/\partial y), v=0 \text{ at } y=0; \quad u\to U_\infty \text{ as } y\to\infty, \tag{4}$$

$$\text{and } T=T_w+D_1(\partial T/\partial y)\, at\ y=0; T\to T_\infty \text{ as } y\to\infty. \tag{5}$$

Here $L_1=L(Re_x)^{1/2}$ is the velocity slip factor and $D_1=D(Re_x)^{1/2}$ is the thermal slip factor with L and D being the initial values of velocity and thermal slip factors having same dimension of length and Re_x being the local Reynolds number and $Re_x=U_\infty x/\upsilon$, U_∞ is the free stream velocity, $T_w=T_\infty+T_0/x$ is the variable temperature of the plate and T_0 is a constant that measures the rate of temperature increase along the plate.

Now, the following similarity transformations are introduced:

$$\psi=\sqrt{U_\infty\upsilon x}f(\eta),\ T=T_\infty+(T_w-T_\infty)\theta(\eta) \text{ and } \eta=y\sqrt{\frac{U_\infty}{\upsilon x}}, \tag{6}$$

where ψ is the stream function defined in the usual notation as $u=\frac{\partial\psi}{\partial y}$ and $v=-\frac{\partial\psi}{\partial x}$ and η is the similarity variable.

Now, the Eq. (1) is identically satisfied and the Eqs. (2) and (3) reduce to the following self-similar equations:

$$f'''+\frac{1}{2}ff''+\lambda\theta=0, \tag{7}$$

$$\text{and } \theta''+Pr\left(\frac{1}{2}f\theta'+f'\theta\right)=0, \tag{8}$$

where $\lambda=g\beta^*T_0/U_\infty^2$ is the mixed convection parameter and $Pr=\mu c_p/\kappa$ is the Prandtl number.

The boundary conditions (4) and (5) reduce to the following forms:

$$f(\eta)=0, f'(\eta)=\delta f''(\eta) \text{ at } \eta=0; f'(\eta)\to 1 \text{ as } \eta\to\infty, \tag{9}$$

$$\text{and } \theta(\eta)=1+\beta\theta'(\eta) \text{ at } \eta=0;\ \theta(\eta)\to 0 \text{ as } (\eta)\to\infty, \tag{10}$$

where $\delta=LU_\infty/\upsilon$ is the velocity slip parameter and $\beta=DU_\infty/\upsilon$ is the thermal slip parameter.

3. Numerical method for solution

The nonlinear coupled differential Eqs. (7) and (8) along with the boundary conditions (9) and (10) form a two point boundary value problem (BVP) and are solved using shooting method [18–27], by converting into an initial value problem (IVP). In this method it is necessary to choose a suitable finite value of $\eta\to\infty$, say η_∞. The following first-order system is set:

$$f'=p, p'=q, q'=-\frac{1}{2}fq-\lambda\theta, \tag{11}$$

$$\text{and } \theta'=z, z'=-Pr\left(\frac{1}{2}fz+p\theta\right), \tag{12}$$

with the boundary conditions

$$f(0) = 0, p(0) = \delta q(0), \theta(0) = 1 + \beta z(0). \qquad (13)$$

To solve (11) and (12) with (13) as an IVP the values for $q(0)$ i.e. $f''(0)$ and $z(0)$ i.e. $\theta'(0)$ are most needed but no such values are given. The initial guess values for $f''(0)$ and $\theta'(0)$ are chosen and applying fourth order Runge–Kutta method the solutions are obtained. Then the calculated values of $f'(\eta)$ and $\theta(\eta)$ at $\eta_\infty(=20)$ are compared with the given boundary conditions $f'(\eta_\infty)=1$ and $\theta(\eta_\infty)=0$ and the values of $f''(0)$ and $\theta'(0)$ are adjusted using "secant method" to give better approximation for the solution. The step-size is taken as $\Delta\eta = 0.01$. The process is repeated until we get the results correct up to the desired accuracy of 10^{-6} level.

4. Results and discussion

The numerical computations are executed for several values of dimensionless parameters involved in the equations viz. the mixed convection parameter (λ), the velocity slip parameter (δ), the thermal slip parameter (β) and the Prandtl number (Pr). To illustrate the computed results, some figures are plotted and physical explanations are given.

Firstly, to assure the accuracy of applied numerical method we compare our derived results corresponding to the velocity and shear stress profiles for forced convection flow for no-slip condition at the boundary (i.e. $\lambda = 0$, $\delta = 0$ and $\beta = 0$) with the given results of Howarth [2] in Fig. 1 and those are found in excellent agreement.

The variations of the dimensionless velocity, shear stress and temperature profiles for different values of the mixed convection parameter λ in presence of slip and in the absence of slip at the boundary are represented in Figs. 2a–2c respectively. For the buoyancy aiding flow ($\lambda > 0$), increase in mixed convection parameter will increase the velocity inside the boundary layer due to favourable buoyancy effects in both slip and no-slip cases (Fig. 2a) and consequently heat transfer rate from the plate will increase. The shear stress profile $f''(\eta)$ (Fig. 2b) though initially increases with λ but it decreases for large η. From Fig. 2c, it is found that for the increase of λ the temperature distribution is suppressed in case of slip as well as no-slip condition and consequently the thermal boundary layer thickness becomes thinner. Velocity overshoot and temperature overshoot are noticed near the plate. Physically $\lambda > 0$ means heating of the fluid or cooling of the surface of the plate (assisting flow). An increase in the values of λ can lead to an increase in the influence of temperature distribution in the velocity field which causes an enhancement of the velocity $f'(\eta)$ due to enhanced convection currents, thus an increase in the boundary layer thickness. An increase in the value of λ results in a decrease in the thermal boundary thickness and this results in an increase in the magnitude of the wall temperature gradient. This in turn produces an increase in the surface heat transfer rate. On the other hand, if the opposing flow ($\lambda < 0$) is considered then the effects become opposite.

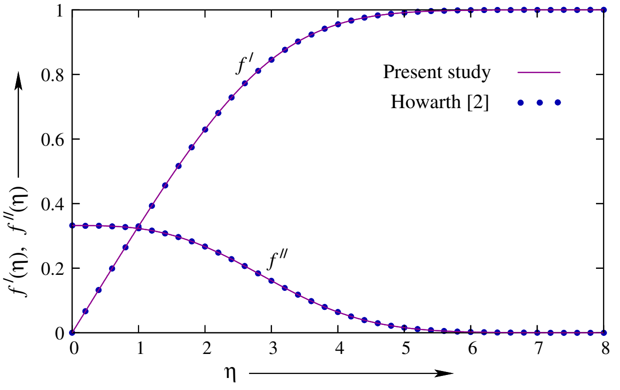

Fig. 1 Velocity profile $f'(\eta)$ and shear stress profiles $f''(\eta)$ for $\lambda = 0$, $\delta = 0$, $\beta = 0$.

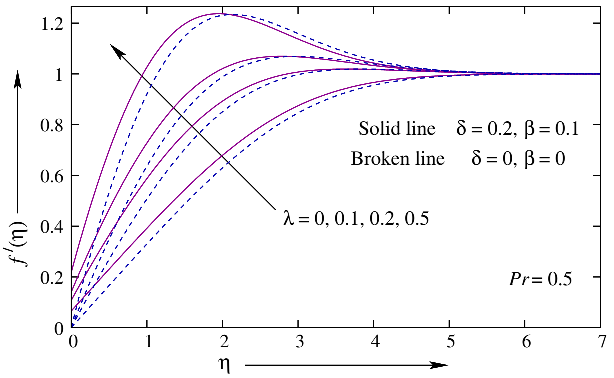

Fig. 2a Velocity profiles $f'(\eta)$ for several values of λ with slip and without slip.

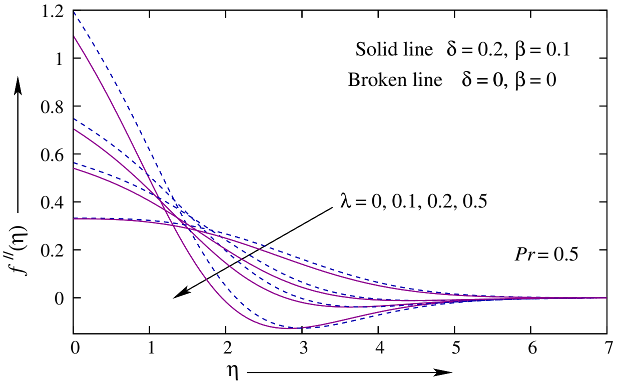

Fig. 2b Shear stress profiles $f''(\eta)$ for several values of λ with slip and without slip.

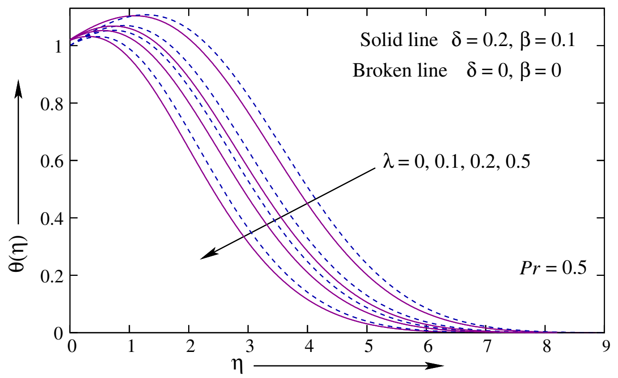

Fig. 2c Temperature profiles $\theta(\eta)$ for several values of λ with slip and without slip.

Next, we discuss the characteristics of the velocity, the shear stress and the temperature profiles when slip occurs at the boundary. For various values of slip parameter δ the velocity, shear stress and temperature profiles are depicted in Figs. 3a–3c respectively. Both, the velocity $f'(\eta)$ and shear stress $f''(\eta)$ profiles exhibit opposite character before and after some points. With increasing values of δ, the velocity increases up to $\eta \approx 2.72$ and then decreases. Also, the dimensionless shear stress decreases up to $\eta \approx 3.52$ and after that it increases. As the slip parameter increases in magnitude, permitting the more fluid to slip past the plate, the flow gets accelerated at distances close to the plate, for distances away from the plate the flow gets decelerated. Opposite is the case for shear stress which is obvious. Such type of behaviour is due to the combined effects of mixed convection and velocity slip parameters. From Fig. 3c, it is observed that the temperature decreases significantly with the increase in slip parameter δ and also the thickness of the thermal boundary layer reduces. Same result has been reported by Bhattacharyya et al. [13].

Due to mixed convection flow, the Prandtl number Pr also affects the velocity and shear stress profiles in addition to the temperature distribution. In this regard, Figs. 4a–4c

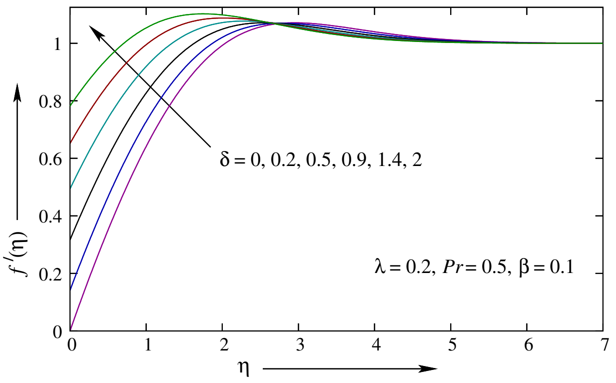

Fig. 3a Velocity profiles $f'(\eta)$ for several values of δ.

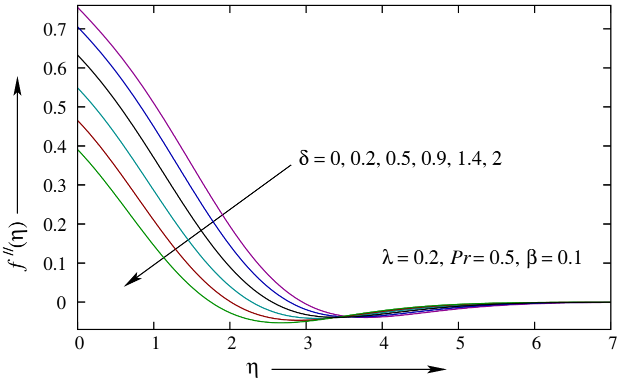

Fig. 3b Shear stress profiles $f''(\eta)$ for several values of δ.

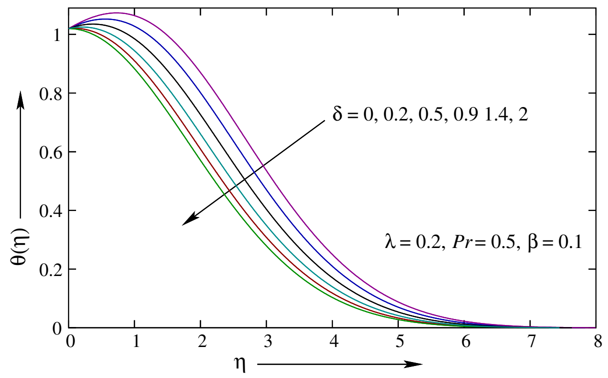

Fig. 3c Temperature profiles $\theta(\eta)$ for several values of δ.

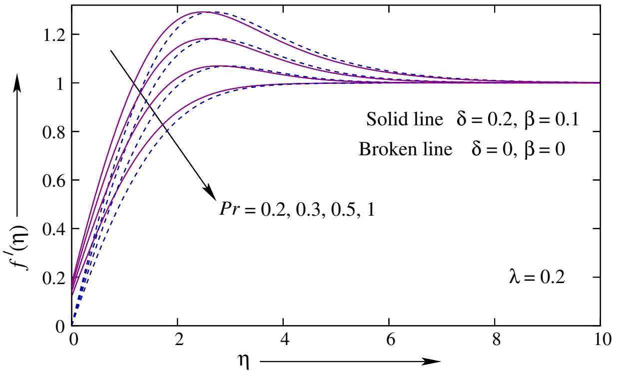

Fig. 4a Velocity profiles $f'(\eta)$ for several values of Pr with slip and without slip.

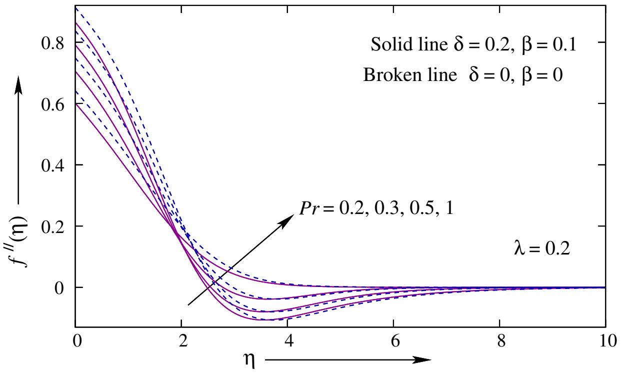

Fig. 4b Shear stress profiles $f''(\eta)$ for several values of Pr with slip and without slip.

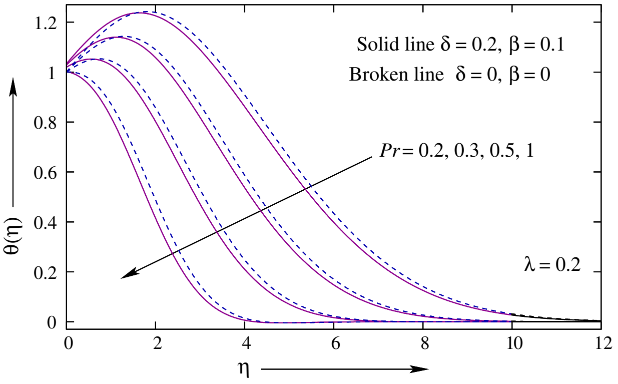

Fig. 4c Temperature profiles $\theta(\eta)$ for several values of Pr with slip and without slip.

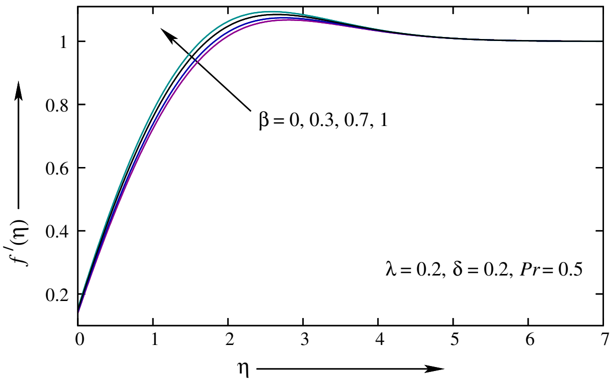

Fig. 5a Velocity profiles $f'(\eta)$ for several values of β.

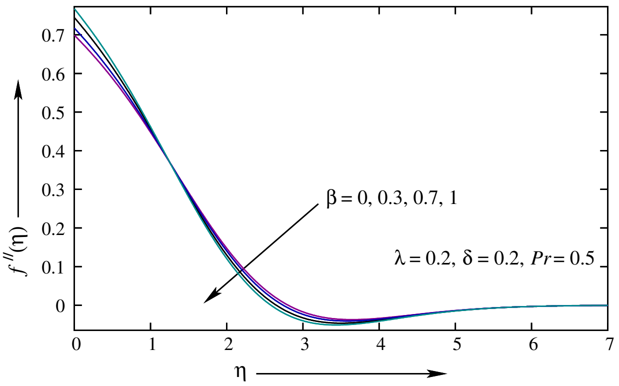

Fig. 5b Shear stress profiles $f''(\eta)$ for several values of β.

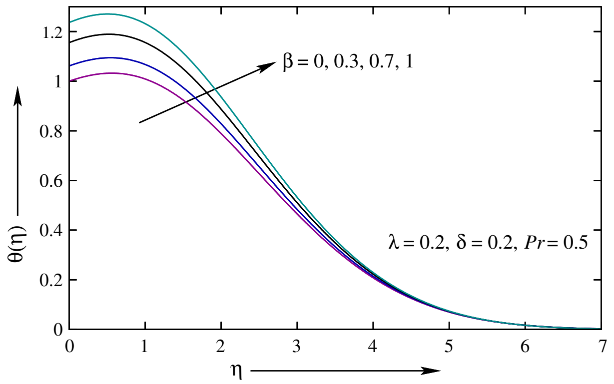

Fig. 5c Temperature profiles $\theta(\eta)$ for several values of β.

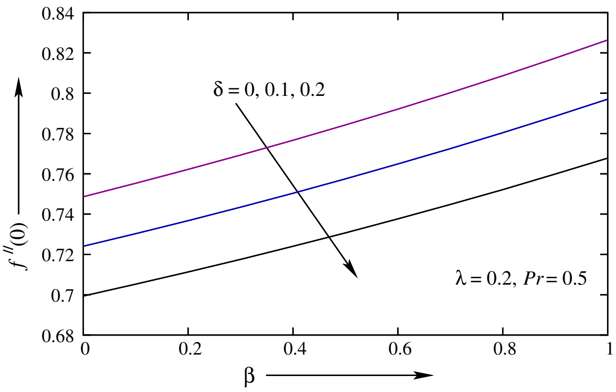

Fig. 6a Skin-friction coefficient $f''(0)$ against β for various values of δ.

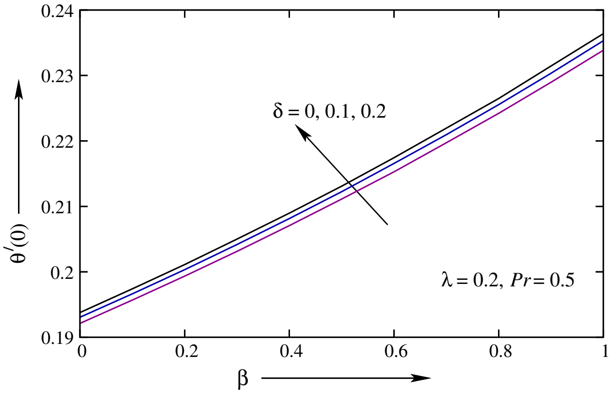

Fig. 6b Temperature gradient at the plate $\theta'(0)$ against β for various values of δ.

demonstrate the effects of the Pr to the velocity, shear stress and temperature distributions. The velocity $f'(\eta)$ along the plate decreases with increase in Pr for both slip and no-slip cases and the profile $f''(\eta)$ decreases up to a point, then increases. In both cases, as Prandtl number increases, the temperature at every location in the thermal boundary layer decreases. The thickness of the boundary layer decreases as Prandtl number increases which is usual case of an isothermal flat plate with no-slip. An increase in Prandtl number means an increase of fluid viscosity which causes a decrease in the flow velocity and the temperature decreases.

Variations of velocity, shear stress and temperature due to thermal slip parameter are presented in Figs. 5a–5c. As the thermal slip parameter increases, the velocity increases and the shear stress at first increases and then after a point ($\eta \approx 1.26$) it decreases. This happens due to the combined effects of mixed convection and velocity slip. From Fig. 5c we observed that with the increasing thermal slip, the temperature rises above the plate temperature T_w (i.e. temperature overshoot is noted) before it decays to the ambient temperature T_∞. In these cases the plate temperature becomes lower as the distance x from the origin increases and the heat transfer is therefore directed from the fluid to the plate, contrary to in the common case direction from the plate to the fluid i.e. $\theta'(0) > 0$ in these cases.

Figs. 6a and 6b demonstrate the effects of velocity slip and thermal slip on skin friction coefficient and temperature gradient at the plate. Though the skin friction coefficient increases with increasing thermal slip parameter but it decreases with increasing velocity slip (Fig. 6a). Plate temperature gradient is found to increase with the increasing thermal slip parameter as well as with the increasing velocity slip parameter. As the thermal slip (parameter) increases an amount of heat is transferred from the fluid to the plate and the temperature gradient at the plate increases in this case. The heat transfer is augmented than that in the no-slip case as a result of the velocity slip.

5. Conclusions

A systematic study on mixed convection over a vertical plate with velocity slip and temperature slip at the boundary is presented. Using similarity variable, the self-similar equations corresponding to momentum and energy equations are obtained and then solved with the help of numerical shooting method. The parameter involved in this study significantly affect the flow and heat transfer. The following conclusions can be drawn as a result of the computations:

(i) Due to increasing mixed convection parameter, the velocity increases and the temperature decreases.
(ii) The velocity shows the increasing and decreasing nature before and after a point as the velocity slip increases and the thermal boundary layer thickness becomes thinner for velocity slip.
(iii) The increase of Prandtl number reduces the velocity along the plate as well as the temperature.
(iv) Velocity and temperature are found to increase with the increasing thermal slip parameter.

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References

[1] Blasius H. Grenzschichten in Flüssigkeiten mit kleiner Reibung. Z Mathematik Physik 1908;56:1–37.
[2] Howarth L. On the solution of the laminar boundary layer equations. Proc Roy Soc Lond A 1938;164:547–79.
[3] Sparrow EM, Cess RD. The effect of a magnetic field on free convection heat transfer. Int J Heat Mass Transfer 1961;3:267–74.
[4] Gupta AS. Laminar free convection flow of an electrically conducting fluid from a vertical plate with uniform surface heat flux and variable wall temperature in the presence of a magnetic field. Z Angew Math Phys 1963;13:324–33.
[5] Afzal N, Hussain T. Mixed convection over a horizontal plate. ASME J Heat Transfer 1984;106:240–1.
[6] Yao LS. Two-dimensional mixed convection along a flat plate. ASME J Heat Transfer 1987;190:440–5.
[7] Hossain MA, Takhar HS. Radiation effects on mixed convection along a vertical plate with uniform surface temperature. Heat Mass Transfer 1996;31:243–8.
[8] Aydin O, Kaya A. Mixed convection of a viscous dissipating fluid about a vertical flat plate. Appl Math Model 2007;31:843–53.
[9] Martin MJ, Boyd ID. Momentum and heat transfer in laminar boundary layer with slip flow. J Thermophys Heat Transfer 2006;20:710–9.
[10] Cao K, Baker J. Slip effects on mixed convective flow and heat transfer from a vertical plate. Int J Heat Mass Transfer 2009;52:3829–41.
[11] Harris SD, Ingham DB, Pop I. Mixed convection boundary-layer flow near the stagnation point on a vertical surface in a porous medium: Brinkman model with slip. Transp Porous Media 2009;77:267–85.
[12] Aziz A. Hydrodynamic and thermal slip flow boundary layers over a flat plate with constant heat flux boundary condition. Commun Nonlinear Sci Numer Simul 2010;15:573–80.
[13] Bhattacharyya K, Mukhopadhyay S, Layek GC. MHD boundary layer slip flow and heat transfer over a flat plate. Chin Phys Lett 2011;28:024701.
[14] Bhattacharyya K, Mukhopadhyay S, Layek GC. Steady boundary layer slip flow and heat transfer over a flat porous plate embedded in a porous media. J Petrol Sci Eng 2011;78:304–9.
[15] Rohni AM, Ahmad S, Pop I, Merkin JH. Unsteady mixed convection boundary-layer flow with suction and temperature slip effects near the stagnation point on a vertical permeable surface embedded in a porous medium. Transp Porous Media 2012;92:1–14.
[16] Andersson HI. Slip flow past a stretching surface. Acta Mech 2002;158:121–5.
[17] Wang CY. Flow due to a stretching boundary with partial slip – an exact solution of the Navier-Stokes equations. Chem Eng Sci 2002;57:3745–7.
[18] Bhattacharyya K, Layek GC. Effects of suction/blowing on steady boundary layer stagnation-point flow and heat transfer towards a shrinking sheet with thermal radiation. Int J Heat Mass Transfer 2011;54:302–7.
[19] Bhattacharyya K, Mukhopadhyay S, Layek GC. Slip effects on boundary layer stagnation-point flow and heat transfer towards a shrinking sheet. Int J Heat Mass Transfer 2011;54:308–13.
[20] Mukhopadhyay S, Bhattacharyya K, Layek GC. Steady boundary layer flow and heat transfer over a porous moving plate in presence of thermal radiation. Int J Heat Mass Transfer 2011;54:2751–7.
[21] Bhattacharyya K. Dual solutions in boundary layer stagnation-point flow and mass transfer with chemical reaction past a stretching/shrinking sheet. Int Commun Heat Mass Transfer 2011;38:917–22.
[22] Bhattacharyya K, Vajravelu K. Stagnation-point flow and heat transfer over an exponentially shrinking sheet. Commun Nonlinear Sci Numer Simul 2012;17:2728–34.
[23] Bhattacharyya K. Boundary layer flow and heat transfer over an exponentially shrinking sheet. Chin Phys Lett 2011;28:074701.
[24] Bhattacharyya K. Dual solutions in unsteady stagnation-point flow over a shrinking sheet. Chin Phys Lett 2011;28:084702.
[25] Bhattacharyya K, Mukhopadhyay S, Layek GC, Pop I. Effects of thermal radiation on micropolar fluid flow and heat transfer over a porous shrinking sheet. Int J Heat Mass Transfer 2012;55:2945–52.
[26] Bhattacharyya K. Mass transfer on a continuous flat plate moving in parallel or reversely to a free stream in the presence of a chemical reaction. Int J Heat Mass Transfer 2012;55:3482–7.
[27] Bhattacharyya K. Heat transfer in unsteady boundary layer stagnation-point flow towards a shrinking sheet. Ain Shams Eng J 2012. http://dx.doi.org/10.1016/j.asej.2012.07.002.

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