Ain Shams Engineering Journal (2015) 6, 605–611

Ain Shams University
Ain Shams Engineering Journal
www.elsevier.com/locate/asej
www.sciencedirect.com

ENGINEERING PHYSICS AND MATHEMATICS

Numerical computation of nonlinear shock wave equation of fractional order

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Received 5 July 2014; revised 26 September 2014; accepted 19 October 2014
Available online 8 December 2014

KEYWORDS

Laplace transform method;
Homotopy analysis method;
Homotopy analysis transform method;
Maple code;
Fractional nonlinear shock wave equation

Abstract The main aim of the present paper was to present a user friendly approach based on homotopy analysis transform method to solve a time-fractional nonlinear shock wave equation arising in the flow of gases. The proposed technique presents a procedure of constructing the set of base functions and gives the high-order deformation equations in a simple form. The auxiliary parameter $\hbar$ in the homotopy analysis transform method solutions has provided a convenient way of controlling the convergence region of series solutions. The method is not limited to the small parameter, such as in the classical perturbation method. The technique gives an analytical solution in the form of a convergent series with easily computable components, requiring no linearization or small perturbation. The numerical solutions obtained by the proposed approach indicate that the approach is easy to implement and computationally very attractive.

1. Introduction

The mathematical model of shock waves arose in connection with problems of the motion of gases and compressible fluids around the second half of the 19th century. The pioneering work in this area was conducted by Earnshaw [1], Riemann [2], Rankine [3] and Hugoniot [4]. The common features of individual theories of continuum physics include conservation laws. The laws are supplemented by constitutive relations which characterize the particular medium in question by relating the values of the main vector field u to the flux, f. The laws are supplemented by constitutive relations which characterize the particular medium in question by relating the values of the main vector field u to the flux f.

Here we assume that these relations are represented by smooth forms, and as a result the conservation laws lead to nonlinear hyperbolic partial differential equations, which are given in simplest form as

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Peer review under responsibility of Ain Shams University.

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http://dx.doi.org/10.1016/j.asej.2014.10.015
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$$u_t(x,t)+f(u(x,t))_x=0, \quad x\in R,\ t>0 \tag{1}$$

with the initial condition

$$u(x,0)=u_0(x), \quad x\in R. \tag{2}$$

Eq. (1) occurs in a model for a diverse range of physical phenomena from shock waves to three-phase flow in porous media. Shock waves find their applications in explosions, traffic flow, glacier waves, airplanes breaking the sound barrier and so on. They are formulated by nonlinear hyperbolic partial differential equations [5–7]. The shock wave and wave equations have been studied using Adomian decomposition method (ADM) [8] and homotopy perturbation method (HPM) [9]. The classical shock wave and wave equations are very handy predictive equations, since solutions are easily obtained. In this article, we present a fractional generalization of shock wave Eq. (1), which can be written in the following form

$$D_t^\alpha u(x,t)+f(u(x,t))_x=0, \quad x\in R,\ t>0 \tag{3}$$

where α is a parameter describing the order of time-derivative. The derivative is considered in the Caputo sense. The general response expression contains a parameter describing the order of fractional shock wave equation that can be varied to obtain various responses. In the case of $\alpha = 1$, the fractional shock wave equation reduces to the classical shock wave equation. The most important advantage of using fractional differential equations in mathematical modeling is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus has become more and more popular [10–18]. There are several methods to solve the fractional shock wave and wave equations, such as the HPM [19]. In 2014, Singh et al. [20] considered the fractional models of shock wave and wave equations and proposed an analytical algorithm based on homotopy perturbation transform method (HPTM). The homotopy analysis method (HAM) was first proposed and developed by Liao [22–25] based on homotopy, a fundamental concept in topology and differential geometry. The HAM has been successfully applied by many research workers to study the physical problems [26–31]. The Laplace transform is a powerful technique for solving various linear partial differential equations having considerable significance in various fields such as engineering and applied sciences. Coupling of semi-analytical methods with Laplace transform giving time consuming consequences and less C.P.U time (Processor 2.65 GHz or more and RAM-1 GB or more) for solving nonlinear problems. Recently, many researchers have paid attention for solving the linear and nonlinear partial differential equations using various methods combined with the Laplace transform. Among these are Laplace decomposition method (LDM) [32–34], homotopy perturbation transform method [35–37] and homotopy analysis transform method [38–40].

In this article, we implement the HATM to find the analytical and numerical solutions of the nonlinear fractional shock wave and wave equations with time-fractional derivatives. The proposed technique solves the nonlinear problems without using Adomian polynomials and He's polynomials which can be considered as a clear advantage of this algorithm over decomposition and the homotopy perturbation transform methods. The plan of our paper is as follows: The basic definitions of fractional calculus are given in Section 2. The basic idea of HATM is presented in Section 3. In Section 4, numerical examples is given to illustrate the applicability of the considered method. Section 5 is dedicated to numerical results and discussion. Conclusions are presented in Section 6.

2. Basic definitions of fractional calculus

In this section, we mention the following basic definitions of fractional calculus.

Definition 1.1. The Laplace transform of the function $f(t)$ is defined by

$$F(s)=L[f(t)]=\int_0^\infty e^{-st}f(t)dt. \tag{4}$$

Definition 1.2. The Laplace transform of the Riemann–Liouville fractional integral is defined as [12]:

$$L\left[I_t^\alpha u(x,t)\right]=s^{-\alpha}L[u(x,t)]. \tag{5}$$

Definition 1.3. The Laplace transform of the Caputo fractional derivative is defined as [12]:

$$L\left[D_t^\alpha u(x,t)\right]=s^\alpha L[u(x,t)]-\sum_{k=0}^{n-1}s^{(\alpha-k-1)}u^{(k)}(x,0), \quad n-1<\alpha\leqslant n. \tag{6}$$

3. Analysis of the method

The HAM was first proposed by Liao [21] based on homotopy, a fundamental concept in topology. The HAM is based on construction of homotopy which continuously deform an initial guess approximation to the exact solution of the given problem. An auxiliary linear operator is chosen to construct the homotopy and an auxiliary linear parameter is used to control the region of convergence of the solution series. The HATM is a combined form of the Laplace transform method and the HAM. We apply the HATM to the following general fractional nonlinear nonhomogeneous partial differential equation of the form:

$$D_t^\alpha u(x,t)+Ru(x,t)+Nu(x,t)=g(x,t), \quad n-1<\alpha\leqslant n \tag{7}$$

where $D_t^\alpha u(x,t)$ is the Caputo fractional derivative of the function $u(x,t)$, R is the linear differential operator, N represents the general nonlinear differential operator and $g(x,t)$, is the source term.

Applying the Laplace transform on both sides of Eq. (7), we get

$$L\left[D_t^\alpha u\right]+L[Ru]+L[Nu]=L[g(x,t)]. \tag{8}$$

Using the differentiation property of the Laplace transform, we have

$$s^\alpha L[u]-\sum_{k=0}^{n-1}s^{\alpha-k-1}u^{(k)}(x,0)+L[Ru]+L[Nu]=L[g(x,t)]. \tag{9}$$

On simplifying

$$L[u]-\frac{1}{s^{\alpha}}\sum_{k=0}^{n-1}s^{\alpha-k-1}u^{(k)}(x,0)+\frac{1}{s^{\alpha}}[L[Ru]+L[Nu]-L[g(x,t)]]=0. \quad (10)$$

We define the nonlinear operator

$$N[\phi(x,t;q)]=L[\phi(x,t;q)]-\frac{1}{s^{\alpha}}\sum_{k=0}^{n-1}s^{\alpha-k-1}\phi^{(k)}(x,0;q)+\frac{1}{s^{\alpha}}\times[L[R\phi(x,t;q)]+L[N\phi(x,t;q)]-L[g(x,t)]], \quad (11)$$

where $q \in [0, 1]$ and $\phi(x,t;q)$ are real functions of x, t and q. We construct a homotopy as follows

$$(1-q)L[\phi(x,t;q)-u_0(x,t)] = \hbar q H(x,t)N[u(x,t)], \quad (12)$$

where L denotes the Laplace transform, $q \in [0, 1]$ is the embedding parameter, $H(x, t)$ denotes a nonzero auxiliary function, $\hbar \neq 0$ is an auxiliary parameter, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $\phi(x,t;q)$ is an unknown function. Obviously, when the embedding parameter $q = 0$ or $q = 1$, it holds

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t), \quad (13)$$

respectively. Thus, as q increases from 0 to 1, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x, t)$ to the solution $u(x, t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to q, we have

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty}u_m(x,t)q^m, \quad (14)$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}|_{q=0}. \quad (15)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter ħ, and the auxiliary function are properly chosen, the series (14) converges at $q = 1$, then we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty}u_m(x,t), \quad (16)$$

which must be one of the solutions of the original nonlinear equations. According to the definition (16), the governing equation can be deduced from the zero-order deformation (12).

Define the vectors

$$\vec{u}_m = \{u_0(x,t), u_1(x,t), \ldots, u_m(x,t)\}. \quad (17)$$

Differentiating the zeroth-order deformation Eq. (12) m-times with respect to q and then dividing them by $m!$ and finally setting $q = 0$, we get the following mth-order deformation equation:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar q H(x,t)\Re_m(\vec{u}_{m-1}). \quad (18)$$

Applying the inverse Laplace transform, we have

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[qH(x,t)\Re_m(\vec{u}_{m-1})], \quad (19)$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}|_{q=0}, \quad (20)$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \quad (21)$$

4. Numerical examples

In this section, we discuss the implementation of our numerical method and investigate its accuracy and stability by applying it to numerical examples on time-fractional shock wave and wave equations.

Example 1. Consider the following time-fractional shock wave equation

$$D_t^{\alpha}u - \left(\frac{1}{c_0} - \frac{\gamma+1}{2}\frac{u}{c_0^2}\right)D_x u = 0, \quad (x,t) \in R \times [0,T], \quad 0 < \alpha \leqslant 1, \quad (22)$$

where c_0, γ are constants and γ is the specific heat. For studying the case under consideration, we take $c_0 = 2$ and $\gamma = 1.5$, which corresponds to the flow of air, subject to the initial condition

$$u(x,0) = \exp\left(-\frac{x^2}{2}\right). \quad (23)$$

Applying the Laplace transform on both sides of Eq. (22) subject to the initial condition (23), we have

$$L[u(x,t)] - \frac{1}{s}\exp\left(-\frac{x^2}{2}\right) - \frac{1}{s^{\alpha}}L\left[\frac{1}{c_0}D_x u - \frac{\gamma+1}{2}\frac{u}{c_0^2}D_x u\right] = 0. \quad (24)$$

We define a nonlinear operator as

$$N[\phi(x,t;q)] = L[\phi(x,t;q)] - \frac{1}{s}\exp\left(-\frac{x^2}{2}\right) - \frac{1}{s^{\alpha}}L\left[\frac{1}{c_0}D_x\phi(x,t;q) - \frac{\gamma+1}{2c_0^2}\phi(x,t;q)D_x\phi(x,t;q)\right] = 0. \quad (25)$$

and thus

$$\Re_m(\vec{u}_{m-1}) = L[u_{m-1}] - (1-\chi_m)\frac{1}{s}\exp\left(-\frac{x^2}{2}\right) - \frac{1}{s^{\alpha}}L\left[\frac{1}{c_0}D_x u_{m-1} - \frac{\gamma+1}{2c_0^2}\sum_{r=0}^{m-1}u_r D_x u_{m-1-r}\right]. \quad (26)$$

The mth-order deformation equation is given by

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \Re_m(\vec{u}_{m-1}). \quad (27)$$

Applying the inverse Laplace transform, we have

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[\Re_m(\vec{u}_{m-1})]. \quad (28)$$

We start with the initial condition $u_0(x,t) = u(x,0) = \exp\left(-\frac{x^2}{2}\right)$, use the iterative scheme (28) and obtain the various iterates

$$u_1(x,t) = \hbar\left[\frac{1}{c_0} - \frac{\gamma+1}{2c_0^2}\exp\left(-\frac{x^2}{2}\right)\right]x\exp\left(-\frac{x^2}{2}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \quad (29)$$

$$u_2(x,t) = \hbar(1+\hbar)\left[\frac{1}{c_0} - \frac{\gamma+1}{2c_0^2}\exp\left(-\frac{x^2}{2}\right)\right]x\exp\left(-\frac{x^2}{2}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \hbar^2\exp\left(-\frac{x^2}{2}\right)\left[-\frac{1}{c_0^2} + \frac{x^2}{c_0^2} + \frac{(\gamma+1)}{c_0^3}\exp\left(-\frac{x^2}{2}\right) - \frac{2(\gamma+1)}{c_0^3}x^2\exp\left(-\frac{x^2}{2}\right) - \frac{(\gamma+1)^2}{4c_0^4}\exp(-x^2) + \frac{3(\gamma+1)^2}{4c_0^4}x^2\exp(-x^2)\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \quad (30)$$

Proceeding in the same manner the rest of components of the HATM solution can be obtained. Finally, we have the series solution

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots. \tag{31}$$

However, most of the results given by the HPM and HPTM converge to the corresponding numerical solutions in a rather small region. But, different from those two methods, the HATM provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter ħ. Choosing $\hbar = -1$ in (31) we can recover all the results obtained by HPM [19] and HPTM [20].

Example 2. Next, consider the following time-fractional wave equation

$$D_t^\alpha u + u D_x u - D_{xxt} u = 0, \tag{32}$$

with the initial condition

$$u(x,0) = 3 \ \sec\ h^2\left(\frac{x-15}{2}\right). \tag{33}$$

Operating the Laplace transform on both sides of Eq. (32) subject to the initial condition (33), we have

$$L[u(x,t)] - \frac{3}{s} \sec\ h^2\left(\frac{x-15}{2}\right) + \frac{1}{s^\alpha} L[u D_x u - D_{xxt} u] = 0. \tag{34}$$

We define a nonlinear operator as

$$N[\phi(x,t;q)] = L[\phi(x,t;q)] - \frac{3}{s} \sec\ h^2\left(\frac{x-15}{2}\right) + \frac{1}{s^\alpha} L[\phi(x,t;q) D_x \phi(x,t;q) - D_{xxt}\phi(x,t;q)] = 0. \tag{35}$$

and thus

$$\Re_m(\vec{u}_{m-1}) = L[u_{m-1}] - (1-\chi_m)\frac{3}{s} \sec\ h^2\left(\frac{x-15}{2}\right) + \frac{1}{s^\alpha} L\left[\sum_{r=0}^{m-1} u_r D_x u_{m-1-r} - D_{xxt} u_{m-1}\right]. \tag{36}$$

The mth-order deformation equation is given by

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \Re_m(\vec{u}_{m-1}). \tag{37}$$

Applying the inverse Laplace transform, we have

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[\Re_m(\vec{u}_{m-1})]. \tag{38}$$

Using the initial approximation $u_0(x,t) = u(x,0) = 3 \ \sec\ h^2\left(\frac{x-15}{2}\right)$ and the iterative scheme (38), obtain the various components of series solution

$$u_1(x,t) = -9\hbar \ \sec\ h^4\left(\frac{x-15}{2}\right) \tanh\left(\frac{x-15}{2}\right) \frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{39}$$

$$\begin{aligned} u_2(x,t) = {} & -9\hbar(1+\hbar) \sec\ h^4\left(\frac{x-15}{2}\right) \tanh\left(\frac{x-15}{2}\right) \frac{t^\alpha}{\Gamma(\alpha+1)} \\ & + \hbar^2\left\{\left[\frac{189}{2} \sec\ h^6\left(\frac{x-15}{2}\right) \tanh^2\left(\frac{x-15}{2}\right) - \frac{27}{2} \sec\ h^6\left(\frac{x-15}{2}\right)\right] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right. \\ & + \left[\frac{135}{2} \sec\ h^4\left(\frac{x-15}{2}\right) \tanh^3\left(\frac{x-15}{2}\right) \right. \\ & \left.\left. - \frac{63}{2} \sec\ h^4\left(\frac{x-15}{2}\right) \tanh\left(\frac{x-15}{2}\right)\right] \frac{\Gamma(\alpha+1) t^{2\alpha-1}}{\Gamma(2\alpha)}\right\}. \end{aligned} \tag{40}$$

In the similar way, the rest of components of the HATM solution can be obtained. Thus the series solution $u(x,t)$ of the Eq. (32) is given as:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots. \tag{41}$$

If we set $\hbar = -1$ and $\alpha = 1$ in (41), then the obtained solution converges to the exact solution

$$u(x,t) = 3 \ \sec\ h^2\left(\frac{x-15-t}{2}\right). \tag{42}$$

We can also recover all the results obtained by HPM [19] and HPTM [20], just by choosing $\hbar = -1$ in (41).

5. Numerical results and discussion

In this section, we calculate the numerical solutions of the probability density function $u(x,t)$ for different time-fractional Brownian motions $\alpha = \frac{1}{4}, \frac{1}{2}$ and also the standard motion $\alpha = 1$.

The numerical values $u(x,t)$ vs. time t and also those vs. x at $\alpha = 1$ for both the two examples are calculated for different particular cases and are depicted through Figs. 1–8. During numerical computation only two iterations are considered. It is evident that by using more terms, the accuracy of the results can be dramatically improved and the errors converge to zero.

The time-fractional shock wave equation considered in Example 1 is graphically represented through Figs. 1–4. The numerical results of the probability density function $u(x,t)$ for fractional Brownian motion $\alpha = \frac{1}{4}$ with $\gamma = 1.5$ and $c_0 = 2$, is shown through Fig. 1 and those for different values of t and α at $x = 1$ are depicted in Fig. 2. The numerical values $u(x,t)$ vs. time t and also those vs. x at $\alpha = 1$, with $\gamma = 1.5$ and $c_0 = 2$, are depicted through Figs. 3 and 4. It is observed from Fig. 2 that as the value of α increase, the value of $u(x,t)$ increases but afterward its nature is opposite. It is also seen from Fig. 3 that $u(x,t)$ increases with the decrease in t. We can also observe from Fig. 4 that $u(x,t)$ is highest at $x = 0$ and decreases as numeric value of x increases.

The wave equation with time-fractional derivative considered in Example 2 is described through Figs. 5–8. It is observed from Fig. 5 that $u(x,t)$ increases with the increase in x and decreases with the increase in t for $\alpha = 1/4$.

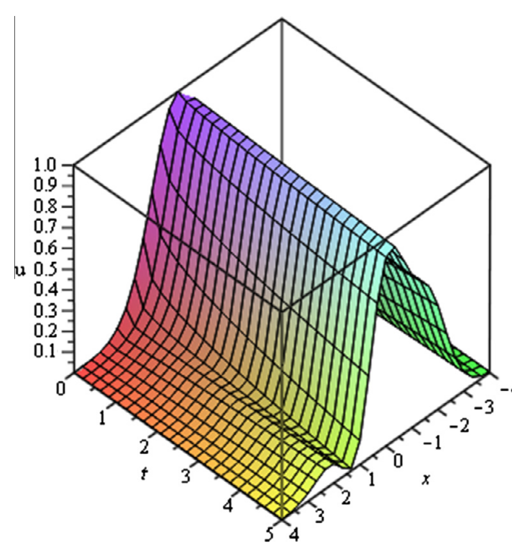

Figure 1 Plot of $u(x,t)$ w.r.t. to x and t at $\alpha = 1/4$ for Example 1.

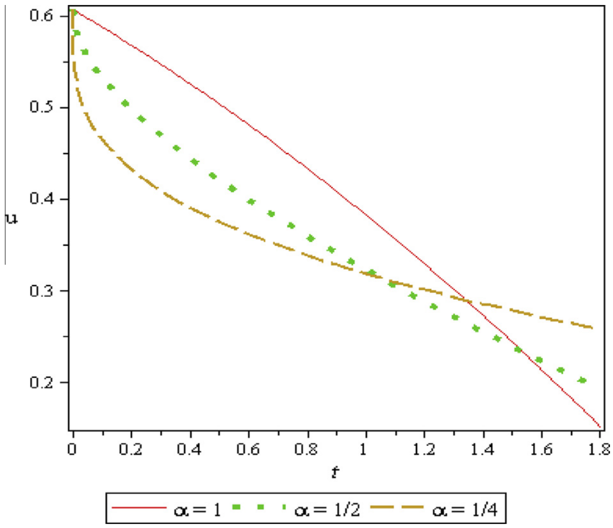

Figure 2 Plots of $u(x,t)$ vs. t at $x = 1$ for different values of α for Example 1.

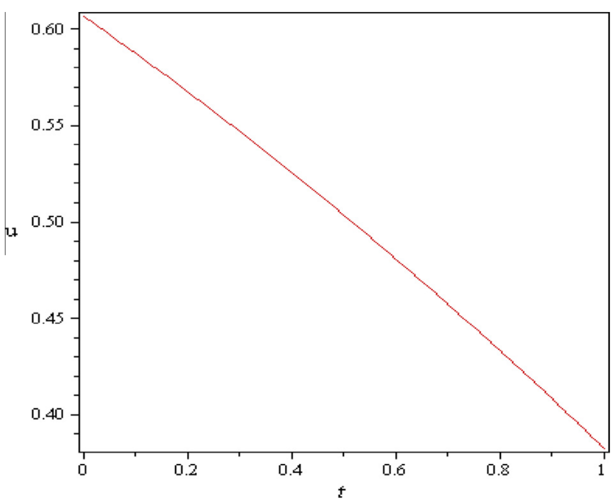

Figure 3 Plot of $u(x,t)$ vs. t at $x = 1$ and $\alpha = 1$ for Example 1.

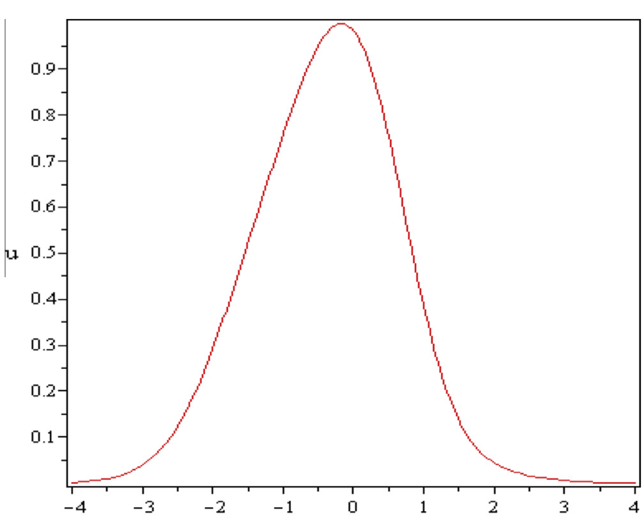

Figure 4 Plot of $u(x,t)$ vs. x at $t = 1$ and $\alpha = 1$ for Example 1.

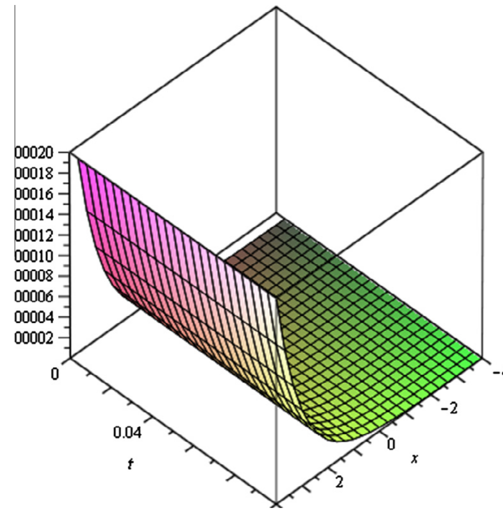

Figure 5 Plot of $u(x, t)$ w.r. to x and t at $\alpha = 1/4$ for Example 2.

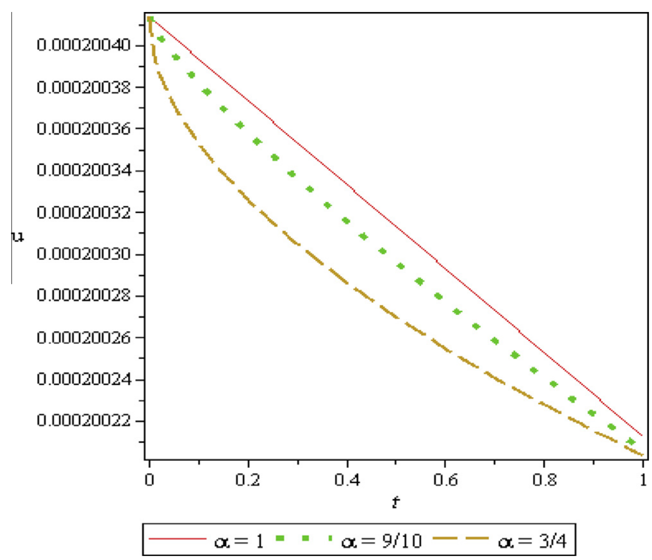

Figure 6 Plots of $u(x,t)$ vs. t at $x = 1$ for different values of α for Example 2.

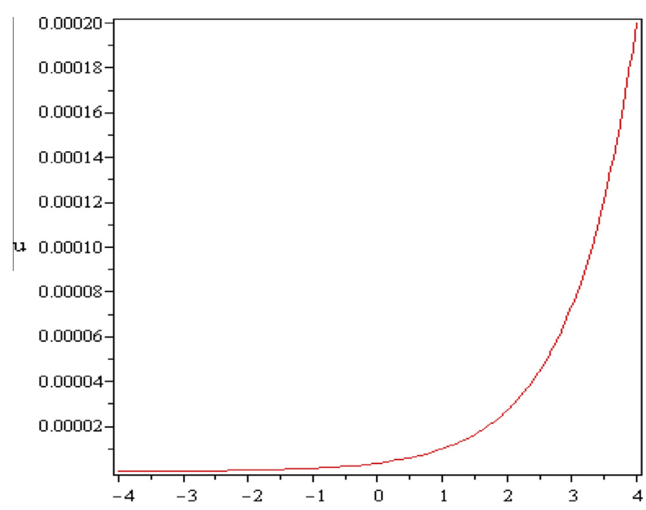

Figure 7 Plot of $u(x,t)$ vs. x at $t = 1$ and $\alpha = 1$ for Example 2.

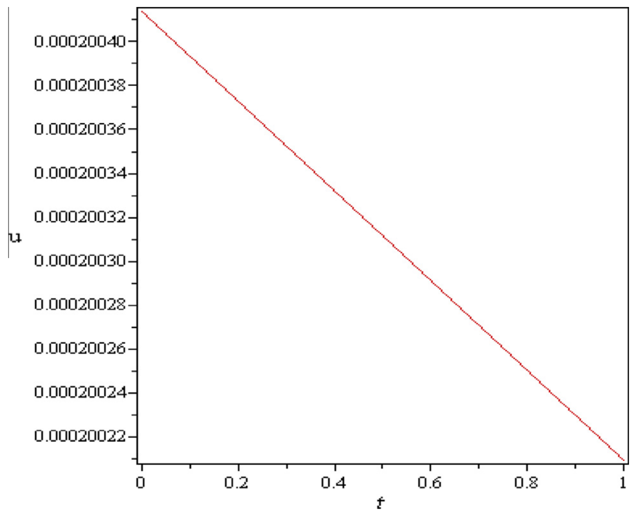

Figure 8 Plot of $u(x,t)$ vs. t at $x = 4$ and $\alpha = 1$ for Example 2.

It is observed from Fig. 6 that as the value of α increase, the value of $u(x, t)$ increases but afterward its nature is opposite. It can be seen from the Figs. 7 and 8 that $u(x, t)$ increases with the increase in x and decreases with the increase in t.

6. Concluding remarks

In this paper, our main concern has been to study the time-fractional shock wave and wave equations arising in flow of gases. The HATM and symbolic calculations have been used to obtain the approximate analytic solutions of the time-fractional shock wave and wave equations. The results obtained by using the scheme presented here agree well with the analytical solutions and the numerical results obtained by HPM [19] and HPTM [20]. It provides us with a simple way to adjust and control the convergence region of solution series by choosing proper values for auxiliary parameter $\hbar$. The proposed method provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. Thus, it can be concluded that the HATM is powerful and efficient in finding analytical as well as numerical solutions for wide classes of nonlinear fractional differential equations arising in science, engineering and finance.

Acknowledgment

The authors are extending their heartfelt thanks to the reviewers for their valuable suggestions for the improvement of the article.

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