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Some characterizations of spheres and elliptic paraboloids

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ABSTRACT

We establish a characterization of spheres in $\mathbb{E}^3$ with respect to a surface area property of regions with the aid of a new meaning of Gaussian curvature. Furthermore, with respect to a volume property of regions, we characterize elliptic paraboloids in arbitrary dimensional Euclidean spaces.

1. Introduction

Consider a sphere $S^2(a)$ of radius a in the Euclidean space $\mathbb{E}^3$. Then by an elementary calculus, it is easy to show that for any two parallel planes with distance h both of which intersect $S^2(a)$, the surface area of the region of $S^2(a)$ between the planes is $2\pi ah$.

In fact, Archimedes proved the above area property of $S^2(a)$ [8, p. 78]. For a differential geometric proof, see Archimedes' Theorem [6, pp. 116–118].

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Conversely, it is natural to ask the following question:

Question 1. Are there any other surfaces which satisfy the above area property?

In Section 2, we prove the following:

Theorem 2. *Let M be a closed and convex surface in the 3-dimensional Euclidean space* $\mathbb{E}^3$*. If M satisfies the condition:*
(C) for any two parallel planes with distance h both of which intersect M, the surface area of the region of M between the planes is a nonnegative function $\phi(h)$*, which depends only on h.*

Then M is an Euclidean sphere.

To establish Theorem 2, first of all, using co-area formula, we prove a lemma (Lemma 6) about a new meaning of Gaussian curvature of M at a point $p \in M$.

A paraboloid of rotation in the 3-dimensional Euclidean space $\mathbb{E}^3$ has an interesting volume property which is originally due to Archimedes. Consider a region of a paraboloid of rotation cut off by a plane not necessarily perpendicular to its axis. Let p be the point of contact of the tangent plane parallel to the base. The line through p, parallel to the axis of the paraboloid meets the base at a point v. Archimedes shows that the volume of the section is $3/2$ times the volume of the cone with the same base and vertex p [8, Chapter 7 and Appendix A].

In fact, in a long series of propositions, Archimedes proves the following [8, p. 66 and Appendices A and B].

Proposition 3. *The volume of such a region of a paraboloid of rotation in the 3-dimensional Euclidean space* $\mathbb{E}^3$ *is proportional to* $\|p - v\|^2$*, where the ratio depends only on the paraboloid.*

This proposition implies directly Archimedes' results (See Remark 8).

Conversely, it is natural to ask the following question:

Question 4. Which surfaces satisfy the above volume property?

In Section 3, we prove the following:

Theorem 5. *Let M be a smooth convex hypersurface in the* $(n + 1)$*-dimensional Euclidean space* $\mathbb{E}^{n+1}$*. Then M is an elliptic paraboloid if and only if there exists a line L for which M satisfies the condition:*
(L) *for any point p on M and any hyperplane section of M parallel to the tangent plane of M at p, let v denote the point where the line through p parallel to L meets the hyperplane. Then the volume of the region of M between these two parallel hyperplanes is a times* $\|p - v\|^{(n+2)/2}$ *for some constant a which depends only on the hypersurface M.*

To complete the proof of Theorem 5, first of all, we get a formula for the Gauss–Kronecker curvature of M at a point $p \in M$ (Lemma 7).

Throughout this article, all objects are smooth and connected, otherwise mentioned.

2. Spheres

Suppose that M is a closed and convex surface in the 3-dimensional Euclidean space $\mathbb{E}^3$. Then the Gaussian curvature K is non-negative. For a fixed point $p \in M$ and for a sufficiently small $h > 0$, consider a plane Φ parallel to the tangent plane Ψ of M at p with distance h which intersects M. We denote by $M_p(h)$ the surface area between the two planes Φ and Ψ.

We introduce a coordinate system (x, y, z) of $\mathbb{E}^3$ with the origin p, the tangent plane of M at p is $z = 0$, and $M = \text{graph(f)}$ for a non-negative convex function $f : \mathbb{R}^2 \to \mathbb{R}$. Then we have

$$M_p(h) = \iint_{f(X)<h} \sqrt{1+|\nabla f|^2}dX, \tag{2.1}$$

where $X = (x, y)$, $dX = dxdy$ and ∇f denote the gradient vector of f.
First of all, we prove

Lemma 6. *If the Gaussian curvature $K(p)$ of M at p is positive, then we have*

$$M_p'(0) = \frac{2\pi}{\sqrt{K(p)}}.$$

Proof. Consider the Taylor expansion of $f(X)$ as follows:

$$f(X) = X^tAX + f_3(X), \tag{2.2}$$

where A is a symmetric 2×2 matrix and $f_3(X)$ is an $O(|X|^3)$ function. Then the Hessian matrix of f at the origin is given by

$$D^2f(0) = 2A.$$

Hence we see that

$$K(p) = \det D^2f(0) = 4\det A. \tag{2.3}$$

Since $K(p) > 0$ and f is non-negative, we see that the matrix A is positive definite. Thus there exists a non-singular matrix B satisfying

$$A = B^tB, \tag{2.4}$$

where B^t denotes the transpose of B. Therefore we obtain

$$f(X) = |BX|^2 + f_3(X). \tag{2.5}$$

In order to compute $M_p'(0)$, we use the decomposition of $M_p(h)$ as follows:

$$\begin{aligned} M_p(h) &= Q(h) + N(h), \\ Q(h) &= \iint_{f(X)<h} 1dX, \\ N(h) &= \iint_{f(X)<h} (\sqrt{1+|\nabla f|^2} - 1)dX. \end{aligned} \tag{2.6}$$

Then we have

$$N(h) \leq \iint_{f(X)<h} |\nabla f| dX.$$

Hence, by the co-area formula [2, p. 86] we get

$$\frac{N(h)}{h} \leq \frac{1}{h}\int_{t=0}^{h}\left(\int_{f^{-1}(t)} 1ds_t\right)dt, \tag{2.7}$$

where ds_t denotes the line element of the curve $f^{-1}(t)$, which shows that the integrand is nothing but the length $L(t)$ of $f^{-1}(t)$. By the fundamental theorem of calculus, we see that

$$\begin{aligned} N'(0) &= \lim_{h\to 0} \frac{N(h)}{h} \\ &= L(0) \\ &= 0. \end{aligned} \tag{2.8}$$

Now we let $h = \epsilon^2$ and $X = \epsilon Y$, then (2.6) gives

$$\begin{aligned} \frac{Q(h)}{h} &= \frac{1}{h}\iint_{f(X)<h} 1dX \\ &= \iint_{|BY|^2+\epsilon g_3(Y)<1} 1dY, \end{aligned} \tag{2.9}$$

where $g_3(Y)$ is an $O(|Y|^3)$ function. As $\epsilon \to 0$, it follows from (2.9) that

$$Q'(0) = \iint_{|BY|^2<1} 1dY. \tag{2.10}$$

If we let $W = BY$, then from (2.10) we get

$$Q'(0) = \frac{1}{\det B}\iint_{|W|<1} 1dW = \frac{\pi}{\det B}. \tag{2.11}$$

Hence it follows from (2.3) and (2.4) that

$$Q'(0) = \frac{2\pi}{\sqrt{K(p)}}. \tag{2.12}$$

Thus together with (2.8) and (2.12), (2.6) completes the proof of Lemma 6.

Now we give a proof of Theorem 2. Since M is closed, there exists a point p where $K(p) > 0$. Hence we see that $U = \{p \in M | K(p) > 0\}$ is nonempty. Together with Condition (C), Lemma 6 implies that at every point $p \in U$, we have $K(p) = 4\pi^2/\phi'(0)^2$, which is independent of $p \in M$. Thus, continuity of K shows that $U = M$, and hence we have $K = 4\pi^2/\phi'(0)^2$ on M. This completes the proof of Theorem 2. □

3. Elliptic paraboloids

Suppose that M is a smooth convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$. For a fixed point $p \in M$ and for a sufficiently small $t > 0$, consider a hyperplane Φ parallel to the tangent hyperplane Ψ of M at p with distance t which intersects M.

We denote by $S_p(t)$ (respectively, $R_p(t)$) the volume of the region bounded by the hypersurface and the hyperplane Φ (respectively, of the cylinder with base $\Phi \cap M$ and height t). Then $R_p(t)$ is $(n+1)$ times the volume of the cone with the same base and the vertex p.

Now we may introduce a coordinate system $(x, z) = (x_1, x_2, \ldots, x_n, z)$ of $\mathbb{E}^{n+1}$ with the origin p, the tangent plane of M at p is $z = 0$. Furthermore, we may assume that M is locally the graph of a non-negative convex function $f : \mathbb{R}^n \to \mathbb{R}$. Then we have

$$\begin{aligned} R_p(t) &= t\int_{f(x)<t} 1dx, \\ S_p(t) &= \int_{f(x)<t} \{t - f(x)\}dx, \end{aligned} \tag{3.1}$$

where $dx = dx_1 dx_2 \cdots dx_n$.

Note that we also have

$$\begin{aligned} S_p(t) &= \int_{f(x)<t} \{t - f(x)\} dx \\ &= \int_{z=0}^{t} \{\int_{f(x)<z} 1 dx\} dz. \end{aligned} \tag{3.2}$$

Hence together with the fundamental theorem of calculus, (3.2) shows that

$$tS_p'(t) = t \int_{f(x)<t} 1 dx = R_p(t). \tag{3.3}$$

First of all, we prove the following.

Lemma 7. *If the Gauss–Kronecker curvature $K(p)$ of M at p is positive, then we have*

$$\lim_{t\to 0} \frac{1}{t^{(n+2)/2}} R_p(t) = \frac{2^{n/2}\omega_n}{\sqrt{K(p)}}, \tag{3.4}$$

where ω_n denotes the volume of the n-dimensional unit ball.

Proof. Consider the Taylor expansion of $f(x)$ as follows:

$$f(x) = x^t A x + f_3(x), \tag{3.5}$$

where x denotes the column vector $(x_1, x_2, \ldots, x_n)^t$, A is a symmetric $n \times n$ matrix, and $f_3(x)$ is an $O(|x|^3)$ function.

Then the Hessian matrix of f at the origin is given by

$$D^2 f(0) = 2A.$$

Hence we see that

$$K(p) = \det D^2 f(0) = 2^n \det A. \tag{3.6}$$

Since $K(p) > 0$ and f is non-negative, we see that the matrix A is positive definite. Thus there exists a nonsingular matrix B satisfying

$$A = B^t B, \tag{3.7}$$

where B^t denotes the transpose of B. Therefore we obtain

$$f(x) = |Bx|^2 + f_3(x). \tag{3.8}$$

Now we let $t = \epsilon^2$ and $x = \epsilon y$. Then (3.3) gives

$$\frac{1}{t^{(n+2)/2}} R_p(t) = \frac{1}{t^{n/2}} \iint_{f(x)<t} 1 dx = \int_{|By|^2+\epsilon g_3(y)<1} 1 dy, \tag{3.9}$$

where $g_3(y)$ is an $O(|y|^3)$ function. As $\epsilon \to 0$, it follows from (3.9) that

$$\lim_{t\to 0} \frac{1}{t^{(n+2)/2}} R_p(t) = \int_{|By|^2<1} 1 dy. \tag{3.10}$$

If we let $w = By$, then from (3.10) we get

$$\lim_{t\to 0} \frac{1}{t^{(n+2)/2}} R_p(t) = \frac{1}{|\det B|} \iint_{|w|<1} 1dw = \frac{\omega_n}{|\det B|}, \tag{3.11}$$

where ω_n denotes the volume of the n-dimensional unit ball. Hence it follows from (3.6) and (3.7) that

$$\lim_{t\to 0} \frac{1}{t^{(n+2)/2}} R_p(t) = \frac{2^{n/2}\omega_n}{\sqrt{K(p)}}. \tag{3.12}$$

This completes the proof of Lemma 7.

Now we give a proof of the if part of Theorem 5. We may assume that the line L is the z-axis and the convex hypersurface M is given locally by $z = f(x)$ for some convex function $f : \mathbb{R}^n \to \mathbb{R}$. For a fixed point $p = (x, f(x))$ in M with positive Gauss–Kronecker curvature $K(p)$ and $t > 0$, consider the region of M cut off by the hyperplane parallel to the tangent hyperplane to M at p with distance $t > 0$. Then the hypothesis shows that $S_p(t) = a\|p - v\|^{(n+2)/2}$ for all $t \in \mathbb{R}$, where a is a constant.

If we let

$$N = \frac{1}{W}(-f_{x_1}, -f_{x_2}, \ldots, -f_{x_n}, 1), \tag{3.13}$$

where $W = \{1 + f_{x_1}^2 + f_{x_2}^2 + \cdots + f_{x_n}^2\}^{1/2}$, then N is an upward unit normal vector. Let θ be the angle between N and $\overrightarrow{pv}$, then we have $\cos\theta = 1/W$.

Hence we get $\|p - v\| = tW$, which shows that

$$S_p(t) = aW^{(n+2)/2} t^{(n+2)/2}. \tag{3.14}$$

Thus (3.3) yields

$$R_p(t) = \frac{n+2}{2} aW^{(n+2)/2} t^{(n+2)/2}. \tag{3.15}$$

Therefore, Lemma 7 shows that

$$K(p) = \frac{2^{n+2}\omega_n^2}{(n+2)^2 a^2} \frac{1}{W^{n+2}}. \tag{3.16}$$

Since the Gauss–Kronecker curvature $K(p)$ of M at p is given by [9, p. 93]

$$K(p) = \frac{\det D^2 f(x)}{W^{n+2}}, \tag{3.17}$$

it follows from (3.16) that the determinant $\det D^2 f(x)$ of the Hessian of $f(x)$ is a positive constant. The continuity of $\det D^2 f(x)$ shows that it is a positive constant on the whole space $\mathbb{R}^n$. Thus $f(x)$ is a globally defined quadratic polynomial [3,5]. This completes the proof of the if part of Theorem 5.

Finally, consider an elliptic paraboloid $M : z = \Sigma_{i=1}^n a_i^2 x_i^2$, $a_i > 0$, a hyperplane Σ intersecting M, a point $p \in M$ where the tangent plane of M is parallel to Σ, and a point v where the line through p parallel to the z-axis meets Σ. Then the linear mapping

$$T_1(x_1, x_2, \ldots, x_n, z) = (a_1x_1, a_2x_2, \ldots, a_nx_n, z) \tag{3.18}$$

transforms M onto a paraboloid of revolution $M' : z = x_1^2 + x_2^2 + \cdots + x_n^2$, Σ to a hyperplane Σ', $p \in M$ to a point of tangency $p' \in M'$, and v to a point v' where the line through p' parallel to the z-axis meets Σ' (cf. [8, Appendix A]).

Let's consider the affine mapping defined by

$$\begin{aligned} x_1' &= tx_1 + h_1, \ldots, x_n' = tx_n + h_n, \\ z' &= 2th_1x_1 + \cdots + 2th_nx_n + t^2z + h_1^2 + \cdots + h_n^2. \end{aligned} \tag{3.19}$$

Then for any constants, $t, h_1, \ldots, h_n$, where t is not 0, the affine mapping takes the paraboloid M' into itself.

Suppose that the equation of Σ' is given by

$$z' = p_1x_1' + p_2x_2' + \cdots + p_nx_n' + d. \tag{3.20}$$

Then we denote by T_2 the affine mapping defined by (3.19) with

$$h_1 = p_1/2, \ldots, h_n = p_n/2, t = \sqrt{(p_1^2 + \cdots + p_n^2 + 2d)/2}. \tag{3.21}$$

The inverse mapping T_2^{-1} of T_2 takes Σ' into $\Sigma'' : z' = 1$, p' to $p'' = 0$, the origin, and v' to $v'' = (0, 0, \ldots, 0, 1)$.

It follows from (3.18) and (3.19) that T_1 (resp., T_2^{-1}) magnifies volumes by the factor $a_1a_2 \cdots a_n$ (resp., $t^{-(n+2)}$) and segments parallel to the z-axis by the factor 1 (resp., t^{-2}). Hence, if we denote by $VS(M)$ the volume of the region of M cut off by Σ and so on, we obtain

$$\begin{aligned} \frac{VS(M)}{\|p - v\|^{(n+2)/2}} &= \frac{VS(M')}{a_1a_2 \cdots a_n\|p' - v'\|^{(n+2)/2}} \\ &= \frac{t^{n+2}VS(M'')}{a_1a_2 \cdots a_n(t^2\|p'' - v''\|)^{(n+2)/2}} \\ &= \frac{VS(M'')}{a_1a_2 \cdots a_n} \\ &= \frac{2\sigma_{n-1}}{n(n+2)a_1a_2 \cdots a_n}, \end{aligned} \tag{3.22}$$

where σ_{n-1} denotes the surface area of the $(n-1)$-dimensional unit sphere. This completes the proof of the only if part of Theorem 5. □

Remark 8. Consider a region of an elliptic paraboloid M cut off by a hyperplane Σ. Then with the same notations as above, (3.22) shows that $VS(M) = c\|p - v\|^{(n+2)/2}$. Since $\|p - v\| = tW$, it follows from (3.3) that $R_p(t) = \frac{n+2}{2}VS(M)$. Therefore the volume $VC(M)$ of the cone with the vertex p becomes $\frac{n+2}{2(n+1)}VS(M)$. For $n = 2$, this gives Archimedes' results.

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