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The Numerical Solution of Variational Problems

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I. INTRODUCTION

In two earlier papers [1, 2] we have derived various conditions that must be satisfied by a curve if it is to be a solution of a variational problem. In [1] we considered mainly the classical problem of Lagrange. In [2] we studied the problem of Mayer and investigated the conditions implied by the introduction of inequality constraints on either the shape of the solution curve (decision variable constraints) or the region in which it could lie (state variable constraints). In this paper we shall discuss the numerical solution of variational problems of both the conventional and inequality-constrained types. We shall outline briefly what, until recently, has been the usual approach. Then we shall consider a gradient technique that has proved very successful in practice. This approach is known to a few practitioners of the numerical art such as Kelley [3] and Bryson [4]. Our derivation, using essentially the same concepts and techniques of dynamic programming that we employed in ref. 1 and 2, is both new and simple. We conclude this paper with a discussion of the numerical solution of a variant of the classical brachistochrone problem where a state variable inequality constraint has been introduced but for which the analytic solution is still known.

We shall assume that the reader is familiar with such dynamic programming terms as "state variable," "decision variable," "optimal return function," etc., that were introduced in our two previous papers and also with the different Lagrange multiplier relationships that hold on and off the boundary, as derived in [2].

II. THE CONVENTIONAL APPROACH

Analysis of variational problems invariably leads to sets of simultaneous nonlinear differential equations (the Euler equations and the kinematic equations of the problem). Almost invariably one is faced with the problem

of finding a solution satisfying mixed boundary conditions—some variables (usually state variables) being known initially and others (usually a mixture of state variables and Lagrange multipliers) being known at the final point.

The usual approach, see for example [5], involves a process of guessing the unknown initial values of the Lagrange multipliers and numerically integrating the set of nonlinear differential equations. When the final values of the variables are not as specified, the guessed initial values are adjusted with the aim of correcting the wrong final values. If all goes well, each incorrect guess contributes information about the effects of initial multipliers on the final values and the process converges to the solution.

One can think of this process as one involving a successive approximation to the problem, since at each step one determines the optimal solution to an unwanted problem—a solution with the wrong final values for states and multipliers. The solutions do represent optimal trajectories to the final state variables that they yield. Optimistically we suggest the name "successive approximation to the problem" for this technique, in the hope that its "conventional" title is transient and will die out in time. This name also has the merit of accentuating the contrast with the approach we shall advocate below and call "successive approximation to the solution."

Experience, rather than mathematical analysis, seems to condemn the problem-approximation approach. Many attempts in a variety of problem areas have uncovered unsuspected numerical instabilities which lead to great sensitivity to initial value changes and have stymied convergence. Minute changes in the initial value of guessed multipliers lead to wildly fluctuating solution curves, and successive improvement of the solution rapidly gives way to successive, but not often successful, modification of the procedure. At best, the technique must be classified as art rather than science, and one readily turns to a computational algorithm that avoids the instabilities that stem from the simultaneous integration of the physical equations and the Euler equations.

III. The Constrained Problem

If one considers constrained state variable problems of the type discussed in [2], one is confronted by conditions necessary for optimality specified not only at the beginning and end of the solution curve but also at intermediate points where the solution joins or leaves a boundary.

Consequently, one is faced with the problem of guessing many multipliers that yield a curve satisfying many conditions specified at various points of the trajectory. But the manner in which these numbers must be guessed makes matters even worse. One would have to discover experimentally first of all the subset of initial multipliers that leads to curves that satisfy the corner

conditions at the point of intersection of optimal curve and boundary. Then one would have to search this subset for that set of initial multipliers that, in conjunction with more unknowns guessed at the point of departure from the boundary, satisfied the final conditions of the problem.

This procedure would then yield a relative extremal, if such exists, consisting of a free interior curve, a portion of the boundary, and then a free curve. If the optimal curve contained several different segments of the boundary the process would be even more complex. This hierarchical guessing game is clearly unplayable in an unstable environment of equations that stymies numerical solution of even unconstrained problems.

Consequently, we discard the concept of successive approximation to the problem and turn to a technique of successive approximation to the solution.

IV. A General Problem

To make precise what will follow, let us now state a particular variational problem. We shall try to steer a course between triviality and complete generality. The reader will immediately realize that the technique we shall present can be generalized in many ways.

Suppose that we wish to minimize a function

$$\phi(y_1, \cdots, y_n, t) \tag{4.1}$$

of the state variables $y_1, \cdots, y_n$ and time t at some unspecified future time T, where T is the first time that a certain "stopping relation"

$$\psi(y_1, \cdots, y_n, t) = 0 \tag{4.2}$$

is satisfied. The y's are determined by their given initial values

$$y_{1_0}, \cdots, y_{n_0} \tag{4.3}$$

and the difference equations

$$y_i(t + \Delta) = y_i(t) + g_i(y_1, \cdots, y_n, t, z)\,\Delta \qquad i = 1, \cdots, n \tag{4.4}$$

where $z(t)$ is the decision variable to be chosen optimally.

That is, we wish to choose the sequence of numbers $\{z_k\}$ where $z_k = z(k\Delta)$ such that the state variables y_i developed with time so that we encounter the stopping relation $\psi = 0$ with minimum ϕ.

This is a problem of the Mayer type. We have taken the kinematic equations (4.4) to be difference, rather than differential, equations since digital computers operate discretely anyway. Minimum time problems can be

considered to be special cases of the above problem where $\phi = t$. Also, if the final time T is required explicitly to equal T_F, then $\psi = t - T_F = 0$ is the stopping condition. The important case where auxiliary conditions $\theta_j = 0$ are required at the endpoint (hence $\psi = 0$ is just one of several final conditions) will be considered subsequently. Finally, to avoid treating t as a special variable in what is to follow we shall, with no loss of generality, let

$$y_{n+1} = t \tag{4.5}$$

and

$$g_{n+1} = 1. \tag{4.6}$$

V. Successive Approximation to the Solution

Our program for solution is the following. We shall guess a presumably nonoptimal decision sequence $\{z_k\}$. By simple reasoning we shall derive a set of recurrence relations that can be used to evaluate the effect of a small change in the decision sequence. We shall then use this information about the effect of decision changes to generate a new, improved, sequence of decisions. We then evaluate the effect of changes in the new sequence and continue this iterative process until no further improvement is possible.

Each successive solution we obtain will be feasible for the problem, but not optimal. This is in contrast with the conventional approach which generates a sequence of optimal, but not feasible, curves.

We begin by guessing a presumably nonoptimal sequence of decisions $\{z_k\}$ where $z_k = z(k\Delta)$ and compute the curve generated by these decisions in conjunction with Eq. (4.3) and (4.4).

We define the nonoptimal return function

$$f(y_1, \cdots, y_{n+1}) = \text{The value of } \phi \text{ at stopping condition } \psi = 0 \text{ where we start in state } y_1, \cdots, y_{n+1} \text{ and use the guessed policy } \{z_k\}. \tag{5.1}$$

The function f is immediately seen to satisfy the recurrence relation

$$f(y_1, \cdots, y_{n+1}) = f(y_1 + g_1\Delta, \cdots, y_{n+1} + g_{n+1}\Delta) \tag{5.2}$$

where the g's are evaluated using the guessed $\{z_k\}$ and associated trajectory.

In order to discover the first order effect of a change in the decision variable at time t we seek to evaluate $\partial f/\partial z|_t$ where this notation means $\partial f/\partial z$ evaluated in terms of the state and decision variables at time t.

By partial differentiation of (5.2) with respect to z we see that

$$\frac{\partial f}{\partial z}\Big|_t = \left\{ \sum_{i=1}^{n+1} \left(\frac{\partial f}{\partial y_i}\Big|_{t+\Delta} \right) \left(\frac{\partial g_i}{\partial z}\Big|_t \right) \right\} \Delta. \tag{5.3}$$

To evaluate this expression we see that we need to know $\partial f/\partial y_j|_{t+\Delta}$. A recurrence relation for that quantity is obtained by partial differentiation with respect to y_j of (5.2)

$$\frac{\partial f}{\partial y_j}\Big|_t = \left\{\sum_{i=1}^{n+1}\left(\frac{\partial f}{\partial y_i}\Big|_{t+\Delta}\right)\left(\frac{\partial g_i}{\partial y_j}\Big|_t\right)\right\}\Delta + \frac{\partial f}{\partial y_j}\Big|_{t+\Delta} \qquad j = 1, \cdots, n+1. \quad (5.4)$$

Both Eq. (5.3) and (5.4) have obvious verbal interpretations. Equation (5.3) states that the rate of change of f with respect to z at time t equals the rate at which the state of the system at time $t + \Delta$ changes as z varies multiplied by the rate at which f changes as the state of the system changes at time $t + \Delta$. Equation (5.4) adds the change in f due to the effect of a change in y_i on the g_i to the direct effect of the change in $y_i(t)$ on $y_i(t + \Delta)$ to obtain the net change in f.

Equation (5.4) is seen to be the discrete analogue of the Multiplier Rule

$$\frac{d}{dt}\frac{\partial f}{\partial y_j} = -\sum_{i=1}^{n+1}\frac{\partial f}{\partial y_i}\frac{\partial g_i}{\partial y_j} \qquad j = 1, \cdots, n+1 \quad (5.5)$$

derived in [1]. If $\partial f/\partial z = 0$ no improvement is possible and the nominal curve is optimal. This observation leads to the optimality condition derived in [1].

We now have two ordinary recurrence relations, (5.3) and (5.4), that permit us to evaluate the effect of a change in z at any time upon the final objective function ϕ. We determine the boundary conditions for the recurrence relations by observing that a change in a state variable at the final time T has two effects, the immediate change in ϕ and the change in ϕ due to the change in the final time determined by $\psi = 0$. Applying this reasoning we have

$$\frac{\partial f}{\partial y_j}\Big|_T = \frac{\partial \phi}{\partial y_j}\Big|_T - \frac{\dot{\phi}}{\dot{\psi}}\Big|_T \frac{\partial \psi}{\partial y_j}\Big|_T \qquad j = 1, \cdots, n+1. \quad (5.6)$$

We have derived, by two simple differentiations, expressions evaluating the first order effect of a decision change at any time upon the final value of ϕ. These results can be thought of as "influence functions," or adjoint equations, and are usually derived from theorems concerning the representation of solutions of linear differential equations.

The manner in which these results can be used most efficiently for the successive improvement of a nonoptimal solution is largely an experimental matter in the realm of numerical analysis. In the following section we shall present one seemingly very efficient way of using these results due to Bryson.

VI. The Means of Improvement

We postulate the rule

$$z_{\text{new}}(t) = z_{\text{old}}(t) + \delta z(t) \tag{6.1}$$

for adjusting z and seek an expression for δz. We start by adopting the reasonable policy (there are alternatives) of changing z at each time t proportionally to the quantity

$$\sum_{i=1}^{n+1} \frac{\partial f}{\partial y_i} \frac{\partial g_i}{\partial z}$$

evaluated at time t. That is, where the potential payoff rate $\partial f/\partial z$ is greater we will act more decisively. Writing

$$\delta z = K \left\{ \sum_{i=1}^{n+1} \frac{\partial f}{\partial y_i} \Big|_{t+\Delta} \frac{\partial g_i}{\partial z} \Big|_t \right\}, \tag{6.2}$$

where K is an as yet undetermined constant of proportionality, and recognizing that since we are considering only first order effects, the total change in ϕ is the sum of the changes during each time interval,

$$\Delta\phi = \sum_{t=0}^{T} \frac{\partial f}{\partial z} \delta z = K \sum_{t=0}^{T} \left\{ \sum_{i=1}^{n+1} \left(\frac{\partial f}{\partial y_i} \Big|_{t+\Delta} \frac{\partial g_i}{\partial z} \Big|_t \right) \right\}^2 \Delta \tag{6.3}$$

where $\Delta\phi$ is the change in the final value of ϕ due to the changes of $\delta z(t)$ in $z(t)$ at all times $0 \leq t \leq T$. The summand in the expression for $\Delta\phi$ is easily computable along a given trajectory by means of the recurrence relations (5.3) and (5.4).

If we desire an improvement of $\overline{\Delta\phi}$ in the value of ϕ, we choose

$$K = \frac{\overline{\Delta\phi}}{\sum_{t=0}^{T} \left\{ \sum_{i=1}^{n+1} \left(\frac{\partial f}{\partial y_i} \Big|_{t+\Delta} \frac{\partial g_i}{\partial z} \Big|_t \right) \right\}^2 \Delta} \tag{6.4}$$

and use, for the next iteration, the new decision function given by

$$z_{\text{new}}(t) = z_{\text{old}}(t) + \frac{\left\{ \sum_{i=1}^{n+1} \frac{\partial f}{\partial y_i} \Big|_{t+\Delta} \frac{\partial g_i}{\partial z} \Big|_t \right\} \overline{\Delta\phi}}{\sum_{t=0}^{T} \left\{ \sum_{i=1}^{n+1} \left(\frac{\partial f}{\partial y_i} \Big|_{t+\Delta} \frac{\partial g_i}{\partial z} \Big|_t \right) \right\}^2 \Delta}. \tag{6.5}$$

We would be well advised to seek only a modest improvement $\overline{\Delta\phi}$ at each successive iteration since our analysis is first order and only accurate for small changes.

Let us now introduce some notation and recapitulate results before deriving successive approximation techniques for more complicated problems. We shall write here and throughout $\lambda_{y_i}(\phi)$ for $\partial f/\partial y_i$, remembering that $\lambda_{y_i}(\phi)$ can be interpreted as the effect of a change in y_i on the value of ϕ at the final time. For $\Sigma_{i=1}^{n+1}(\partial f/\partial y_i)(\partial g_i/\partial z)$ we write $\lambda_z(\phi)$. In this notation the technique of successive improvement is

(1) Guess $z(t)$.
(2) Integrate the equations of motion (4.4).
(3) Evaluate $\lambda_{y_i}(\phi)$ at the final time T by means of (5.6).
(4) Determine $\lambda_{y_i}(\phi)$ along the nominal trajectory by backwards recursion of (5.4) and simultaneously compute $\lambda_z(\phi)$ and $\Sigma_t^T(\lambda_z(\phi))^2\Delta$ from (5.3).
(5) Determine z_{new} for a specified small $\overline{\Delta\phi}$ by (6.5).
(6) Return to step (2).

Suppose now that an additional relationship

$$\theta(y_1, \cdots, y_{n+1}) = 0 \tag{6.6}$$

must be satisfied at the stopping time. The same arguments as in the preceding paragraphs allow us to compute the influence of a change in z on the final value of θ by means of the formulas

$$\lambda_{y_j}(\theta)\Big|_t = \left\{\sum_{i=1}^{n+1}\left(\lambda_{y_i}(\theta)\Big|_{t+\Delta}\right)\left(\frac{\partial g_i}{\partial y_j}\Big|_t\right)\right\}\Delta + \lambda_{y_j}(\theta)\Big|_{t+\Delta} \qquad j = 1, \cdots, n+1 \tag{6.7}$$

$$\lambda_z(\theta)\Big|_t = \sum_{i=1}^{n+1}\left(\lambda_{y_i}(\theta)\Big|_{t+\Delta}\right)\left(\frac{\partial g_i}{\partial z}\Big|_t\right) \tag{6.8}$$

$$\lambda_{y_j}(\theta)\Big|_T = \frac{\partial\theta}{\partial y_j}\Big|_T - \frac{\dot\theta}{\dot\psi}\frac{\partial\psi}{\partial y_j}\Big|_T. \qquad j = 1, \cdots, n+1. \tag{6.9}$$

We now let δz take the form

$$\delta z = K_1\lambda_z(\phi) + K_2\lambda_z(\theta) \tag{6.10}$$

and conclude, from linearity, that

$$\Delta\phi = \sum_{t=0}^{T}\lambda_z(\phi)\,\Delta\,\delta z \tag{6.11}$$

$$\Delta\theta = \sum_{t=0}^{T}\lambda_z(\theta)\,\Delta\,\delta z. \tag{6.12}$$

If the nominal trajectory, due to either numerical roundoff, nonlinearity, or difficulties in finding an initial feasible trajectory, does not satisfy the auxiliary condition (6.6), we choose $\overline{\Delta\theta}$ as minus the deviation for the desired final condition. If the nominal trajectory is feasible, $\overline{\Delta\theta}$ is taken to be zero.

We now solve the simultaneous linear equations

$$\overline{\Delta\phi} = \left[\sum_{t=0}^{T} \lambda_z(\phi)^2 \, \Delta\right] K_1 + \left[\sum_{t=0}^{T} \lambda_z(\phi) \, \lambda_z(\theta) \, \Delta\right] K_2 \tag{6.13}$$

$$\overline{\Delta\theta} = \left[\sum_{t=0}^{T} \lambda_z(\phi) \, \lambda_z(\theta) \, \Delta\right] K_1 + \left[\sum_{t=0}^{T} \lambda_z(\theta)^2 \, \Delta\right] K_2 \tag{6.14}$$

for K_1 and K_2 to be used in Eq. (6.10) to achieve an improvement $\overline{\Delta\phi}$ in the objective function and a correction $\overline{\Delta\theta}$ in the final value of the subsidiary condition.

The above device can be used to include any reasonable number of auxiliary final conditions.

In an unpublished paper, Bryson has extended the above technique so that one can ask for the maximum improvement in the objective function given a specified value for

$$\sum_{t=0}^{T} (\delta z)^2 \tag{6.15}$$

and has also developed techniques that prevent one from asking for incompatible changes in the objective and auxiliary conditions. The analysis, however, is beyond the scope of this discussion.

VII. Inequality Constraints

Having fixed the basic ideas of numerical solution, let us see what modifications are necessary if the problem involves state variable inequality constraints.

For concreteness let us assume that the solution is known to have the structure:

(1) An interior curve of free variation connects the specified initial point to the boundary given by equality in the constraint.

(2) A boundary segment where, it is assumed, no variation is permitted.

(3) A curve of free variation leading from the boundary to the partially specified final point.

When we speak of a nominal initial trajectory we shall mean

(1) A guessed decision sequence yielding a curve from the initial point to the boundary satisfying any continuity conditions that may be stipulated at the point of intersection (see [2]).

(2) A guessed time at which the nominal curve is to leave the boundary.

(3) A guessed decision sequence that yields a curve from the boundary to the final point. (By the argument of Section VI it is seen that all final conditions need not be satisfied by the nominal curve.)

It should ne noted that since the decision variable interior to the boundary is specified as a function of t, in subsequent analysis a change in t implies a change in z and also a change in the point where the curve leaves the boundary.

After guessing a trajectory we perform an analysis upon it in the manner of the previous sections. As discussed in [2], along the boundary segment we have a reduced set of multipliers and different equations for their time derivatives. When performing the backwards recursion of (5.3) and (5.4), there are always sufficient conditions to reclaim the full set of multipliers along the segment connecting the initial point to the boundary. Also, our analysis gives us sufficient information to determine the effect of a change in the time off the boundary on the final objective. Hence we use the method of the above sections to adjust the decision sequences for the free parts of the curve and some new analysis presented below to adjust the time off the boundary. When a trajectory is found with no further improvement possible from either source, we have at least a relative extremum for the problem.

In the following section we shall note the equations necessary for the solution of a particular problem. Then we shall discuss the numerical solution of that problem.

VIII. The Bounded Brachistochrone Problem

The classical brachistochrone problem, the study of which gave considerable impetus to the development of the calculus of variations in the eighteenth century, is stated:

Find that path down which a particle, under the influence of gravity alone, would slide in order to reach a final destination in minimum time.

We shall augment the traditional problem by specifying that the solution path should remain in a particular region of space.

The analytic solution to the unconstrained problem is a cycloid. It can be shown that for a problem constrained by a sloping straight line the solution consists of the cycloid through the initial point and tangent to the line, a

segment of the boundary line, and a cycloid through the final point and tangent to the boundary. If the final value of x, x_f, is specified but the final y value is not, the second cycloid will be tangent to the boundary and perpendicular to the line $x = x_f$. It is this problem we shall consider numerically, using the analytic solution only for the evaluation of the accuracy of our numerical solution.

Mathematically, we wish to determine the decision function

$$\gamma(t) \tag{8.1}$$

such that, given

$$\begin{aligned} x(0) &= x_0 & v(0) &= v_0 \\ y(0) &= y_0 & x(T) &= x_f \end{aligned} \tag{8.2}$$

and

$$y(t) \geq ax(t) + b \tag{8.3}$$

for all t, and the kinematic equations

$$\begin{aligned} \dot{x} &= v \cos \gamma \\ \dot{y} &= v \sin \gamma \\ \dot{v} &= -g \sin \gamma, \end{aligned} \tag{8.4}$$

the final time, T, is minimum.

In our previous notation

$$\begin{aligned} y_1 &= x & z &= \gamma & g_4 &= 1 \\ y_2 &= y & g_1 &= v \cos \gamma & \phi &= t \\ y_3 &= v & g_2 &= v \sin \gamma & \psi &= x - x_f = 0. \\ y_4 &= t & g_3 &= -g \sin \gamma \end{aligned} \tag{8.5}$$

Graphically, we have the situation as given in Fig. 1 and wish to connect A and the line $x = x_f$ with the curve of minimum descent time.

We guess a curve of the form ABCD (see Fig. 2).

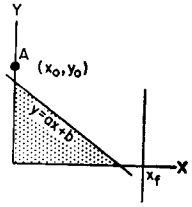

FIG. 1

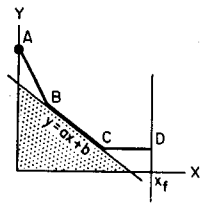

FIG. 2

Along the segment CD we have the multiplier equations

$$\begin{aligned}
\dot{\lambda}_x &= 0 \\
\dot{\lambda}_y &= 0 \\
\dot{\lambda}_v &= -\lambda_x \cos\gamma - \lambda_y \sin\gamma \\
\dot{\lambda}_t &= \lambda_x v \sin\gamma\,\dot{\gamma} - \lambda_y v \cos\gamma\,\dot{\gamma} + \lambda_v g \cos\gamma\,\dot{\gamma} = -\lambda_\gamma \dot{\gamma} \\
\lambda_\gamma &= -\lambda_x v \sin\gamma + \lambda_y v \cos\gamma - \lambda_v g \cos\gamma
\end{aligned} \tag{8.6}$$

where the λ's are the influence functions on the time at the endpoint.

At the endpoint we have, by (5.6),

$$\begin{aligned}
\lambda_x(T) &= -\frac{1}{v\cos\gamma} \\
\lambda_y(T) &= 0 \\
\lambda_v(T) &= 0 \\
\lambda_t(T) &= 1.
\end{aligned} \tag{8.7}$$

These are the initial conditions for the backwards iteration of (5.3) and (5.4).

At the corner C we have

$$\begin{aligned}
\lambda_x^* &= \lambda_x + a\lambda_y \\
\lambda_v^* &= \lambda_v \\
\lambda_t^* &= -\lambda_x^* \dot{x}\,|_{\text{bdy}} - \lambda_v^* \dot{v}\,|_{\text{bdy}}
\end{aligned} \tag{8.8}$$

where x, v, and t are taken as independent state variables along the boundary. (See [2] where the starred-multiplier notation is introduced). The equation for λ_t^* requires some explanation since it is new. Along the boundary a change in time t implies a change in the location of the corner C, since it is specified in terms of time, and no other change. If t is increased by amount Δt, all other state variables being held fixed, the corner C will occur Δt sooner. The result of such a change is a smaller x and v at the corner. The effect of this change is evaluated to obtain λ_t^* in (8.8). The symbol $\dot{x}|_{\text{bdy}}$ means $\dot{x}$ evaluated in terms of the γ on the boundary, not the γ on the free curve just past the boundary. Later we shall use the expression $\dot{x}|_{\text{free}}$ with the obvious interpretation. The multiplier λ_t is discontinuous across C because a "change in t" implies a different effect on either side of C. On CD it implies a change in decision choice, while on BC it results in a different corner followed by an unchanged segment CD.

Along BC

$$\begin{aligned} \dot{\lambda}_x^* &= 0 \\ \dot{\lambda}_v^* &= -\lambda_x^* \cos\gamma \\ \dot{\lambda}_t^* &= 0 \end{aligned} \tag{8.9}$$

At B,

$$\begin{aligned} \lambda_x^* &= \lambda_x + a\lambda_y \\ \lambda_v^* &= \lambda_v \\ \lambda_t^* &= \lambda_t \end{aligned} \tag{8.10}$$

$$\lambda_x \dot{x}\,|_{\text{free}} + \lambda_y \dot{y}\,|_{\text{free}} + \lambda_v \dot{v}\,|_{\text{free}} + \lambda_t = 0.$$

The last equation is a statement that relation (5.2) must hold and allows us to recover λ_y to initiate the backwards solution along AB.

Along AB we have the same differential equations, (8.6), as along CD.

This gives us all necessary equations to compute λ_γ along AB and CD and λ_t^* at C (this is the effect of changing the location of the corner that we noted earlier was needed). Now γ is varied according to the rules of Section VI, and C is varied according to the sign and magnitude of λ_t^*.

When, after several iterations, λ_γ is zero and λ_t^* is one a relative extremal has been found. The reader can verify that, when this occurs, all the necessary conditions of [2] are satisfied.

IX. Numerical Solution

Before presenting numerical results, let us discuss some computational aspects of this problem.

As the optimal solution is approached, the improvement $\overline{\Delta\phi}$ that can be sought naturally must be reduced. If too large a $\overline{\Delta\phi}$ is specified, the resulting trajectory will overshoot the optimal one and may be worse, not better, than the "improved" old one. Consequently, it is useful to store the old decision sequence $\{z_{\text{old}}\}$ in the computer memory, even after $\{z_{\text{new}}\}$ has been computed. Then, if the resulting new trajectory is inferior, $\overline{\Delta\phi}$ is reduced to, say, $\overline{\Delta\phi}/2$ and a new $\{z_{\text{new}}\}$ is computed from $\{z_{\text{old}}\}$. Convergence can be considered attained when the $\overline{\Delta\phi}$ asked for, and not achieved due to overshoot, is smaller than some prespecified ϵ.

It is possible that a longer decision sequence will be required for some segment of a new trajectory than was for the old. If this occurs, for K greater than the old final K value there is no z_K available to be modified. Some provi-

sion must be made for this case. Letting $z_{K+1} = z_K$, i.e., fixing z at its old final value, works satisfactorily.

Note should be made of the last equation of set (8.10). It the free curve is tangent to the boundary this equation is automatically satisfied and contributes nothing to the determination of the missing multiplier λ_y. However λ_y can be determined by considering the limit as $\gamma_{\text{free}} \to \gamma_{\text{bdy}}$ of the equation set. This yields the result

$$\lambda_y = \frac{(\lambda_v^* g \cos\gamma + \lambda_x^* v \sin\gamma)\cos\gamma}{v} \tag{9.1}$$

for the segment AB, a result verified by the numerical computation. Care must be exercised to assure that (8.10) applies for nontangent initial trajectories, but that (9.1) is used as the corrected curve approaches tangency.

All of these possible pitfalls are mentioned to forewarn the reader that, while unconstrained problems can now be considered routine, constrained problems still raise many problems and must be handled with care.

The bounded brachistochrone problem was programmed for an IBM 7090 computer using the following initial and final conditions

$$\begin{aligned} x_0 &= 0 & v_0 &= 1 \\ y_0 &= 6 & x_f &= 6 \end{aligned} \tag{9.2}$$

and the boundary

$$y \geq -\tfrac{1}{2}x + 5. \tag{9.3}$$

The initial guessed decision sequence was

$$\gamma = -0.785 \text{ rad} \tag{9.4}$$

until the boundary is reached,

$$t = 0.7 \text{ sec} \tag{9.5}$$

when the boundary is left, and

$$\gamma = 0 \text{ rad} \tag{9.6}$$

to the end-line $x = 6$.

The time off the boundary was modified only after the procedure had converged to the optimal solution for the given corner (i.e., $\lambda_\gamma = 0$). Then the computation was allowed to converge again for the new corner which was then modified etc., until both the curve and corner were optimal.

The new C-corner position, after two corners had been tried and corresponding λ_t^*'s evaluated, was found automatically by linear extrapolation

which determined the corner position such that λ_t^*'s would have its optimal value, 1. A corner value of 1 implies stationarity since it means that changing the time by Δt on the boundary changes the final time by Δt, with no further modification of final time due to the change in the corner position.

In Fig. 3 we show the initial nonoptimal solution. Descent required 0.7766 sec.

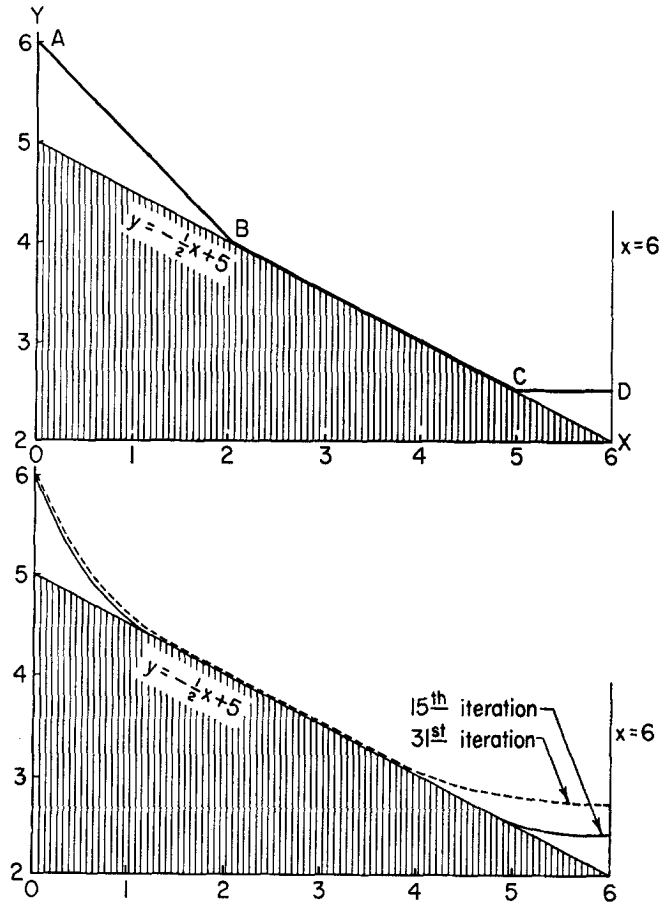

FIG. 3. The initial nonoptimal solution

FIG. 4. Improved solutions

After 15 iterations the curve was almost optimal for the given corner C. This curve is shown in solid line in Fig. 4. Descent time was 0.7445 sec, and λ_t^* equalled 0.963.

The corner was then modified on the basis of λ_t^* to occur at $t = 0.594$ sec. The dotted line in Fig. 4 shows the optimal curve for this corner position,

attained on the 31st iteration, with descent time of 0.7422 sec and $\lambda_t^* = 0.993$.

After a total of 50 iterations and 4 corner modifications requiring 10 min of computer time, the computed optimal curve looked like the solid line in Fig. 5, with descent time of 0.7420 sec, and with $\lambda_t^* = 0.999$.

Finally, Fig. 6 shows the λ_γ functions corresponding to the initial curve and the computed optimal curve. Recall that λ_γ is only defined off the boundary, and that $\lambda_\gamma = 0$ implies optimality, for a specified corner C.

The true optimal solution obtained analytically is shown by a dotted line in Fig. 5. The convergence of the numerical solution was hampered by the extreme flatness of the optimal time curve in the neighborhood of its mini-

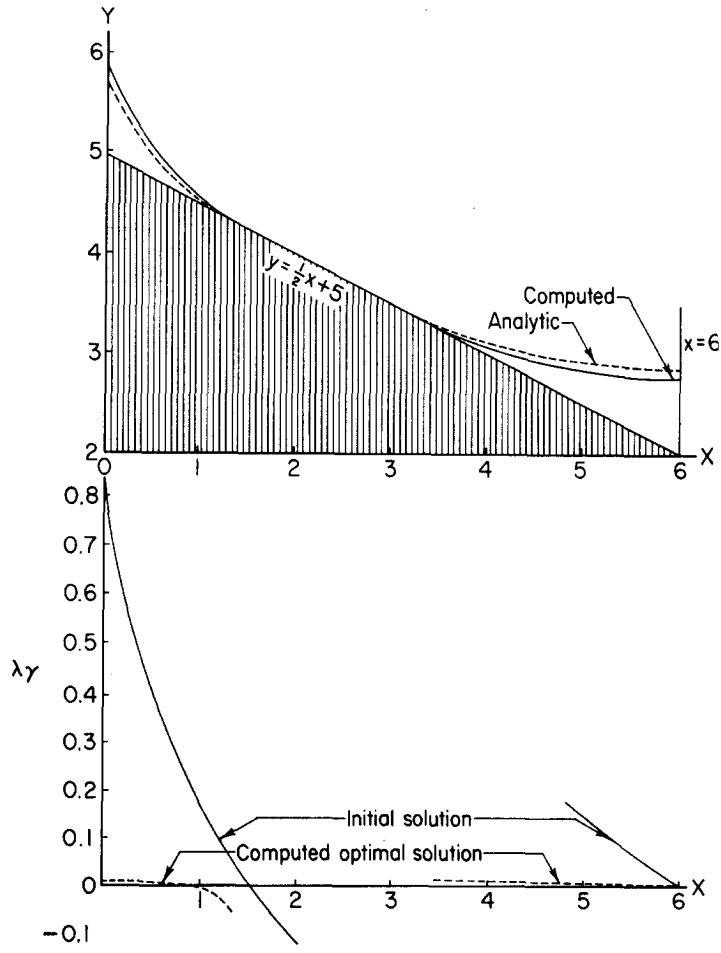

FIG. 5. Optimal solution

FIG. 6. Improvement criterion, λ_γ

mum, as can be verified by comparing the time for the solid line curve shown in Fig. 4 to the time attained 35 iterations later, an improvement of just 0.3%. Difficulties were also introduced by the tangency of the optimal curve. Since, within reason, the shape of the descent curve makes little difference in the time of descent, very accurate integration and a small integration step size ($t = 0.001$ sec) were required. While the solution curve is not quite as accurate as one might hope, the descent time is essentially minimal.

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