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ORIGINAL ARTICLE

Fuzzy Ideals and Fuzzy Interior Ideals in Ordered Semirings

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Abstract The notions of fuzzy ideals and fuzzy interior ideals in ordered semirings are introduced and some of their characterizations are obtained through regularity criterion and with some operations on them.

Keywords Cartesian product · Fuzzy ideal · Fuzzy interior ideal · Homomorphism · Ordered semiring

1. Introduction

Semirings [4] which provide a common generalization of rings and distributive lattices arise naturally in such diverse areas of mathematics as combinatorics, functional analysis, graph theory, automata theory, mathematical modelling and parallel computation systems etc. (for example, see [4, 5]). Semirings have also been proved to be an important algebraic tool in theoretical computer science, see for instance [5], for some detail and example. Many of the semirings, such as $\mathbb{N}$, have an order structure in addition to their algebraic structure and indeed the most interesting results concerning them make use of the interplay between these two structures. Ideals of semirings play an important role in the structure theory of ordered semirings [3, 6, 11] and useful for many purposes. In this paper, like ordered semigroup [8, 10], it is an attempt to show how similar is the theory in terms of fuzzy set which was introduced by Zadeh [14], since nowadays fuzzy research concerns standardization, axiomatization,

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extensions to lattice-valued fuzzy sets, critical comparison of the different so-called soft computing models that have been launched during the past three decennia for the representation and processing of incomplete information [9].

In Section 2, we introduce fuzzy ideals and fuzzy interior ideals in ordered semirings and study some of their related properties. In Section 3, we define intersection, composition and addition in order to investigate the structure theory of fuzzy ideals in ordered semirings. It is shown that the set of all fuzzy ideals of an ordered semiring forms a zerosumfree semiring under addition and composition of fuzzy ideals. We also define and characterize regularity criterion in ordered semiring in terms of fuzzy subsets.

2. Preliminaries

We recall the following definitions for subsequent use.

Definition 2.1 *A semiring is a system consisting of a non-empty set S on which operations addition and multiplication (denoted in the usual manner) have been defined such that $(S, +)$ is a semigroup, $(S, \cdot)$ is a semigroup and multiplication distributes over addition from either side.*

A zero element of a semiring S is an element 0 such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$.

A semiring S is zerosumfree if and only if $s + s^{'} = 0$ implies that $s = s^{'} = 0$.

Definition 2.2 *A left ideal I of semiring S is a nonempty subset of S satisfying the following conditions:*

(i) *If $a, b \in I$, then $a + b \in I$;*

(ii) *If $a \in I$ and $s \in S$, then $sa \in I$.*

A right ideal of S is defined in an analogous manner and an ideal of S is a nonempty subset which is both a left and a right ideal of S.

Definition 2.3 *A nonempty subset A of S is said to be an interior ideal if it is closed under addition, $A^2 \subseteq A$ and $SAS \subseteq A$.*

Definition 2.4 *An ordered semiring is a semiring S equipped with a partial order $\leq$ such that the operation is monotonic and constant 0 is the least element of S.*

Now we recall the definition and example of ordered ideal from [3].

Definition 2.5 *A left (resp. right) ideal I of S is called a left (resp. right) ordered ideal if for any $a \in S$, $b \in I$, $a \leq b$ implies $a \in I$ (i.e., $(I] \subseteq I$). I is called an ordered ideal of S if it is both a left and a right ordered ideal of S.*

Example 1 Let $S = ([0, 1], \vee, \cdot, 0)$ where [0,1] is the unit interval, $a \vee b = \max\{a, b\}$ and $a \cdot b = (a + b - 1) \vee 0$ for $a, b \in [0, 1]$. Then it is easy to verify that S equipped with the usual ordering $\leq$ is an ordered semiring and $I = [0, \frac{1}{2}]$ is an ordered ideal of S.

Definition 2.6 [14] *A fuzzy subset μ of a non-empty set S is defined as a mapping from S to* $[0, 1]$.

Definition 2.7 [14] *Let μ be a fuzzy subset of a set S and $t \in [0, 1]$. The set*

$$\mu_t = \{x \in S \mid \mu(x) \geq t\}$$

is called the level subset of μ. Clearly, $\mu_t \subseteq \mu_s$, whenever $t \geq s$.

Definition 2.8 *The union and intersection of two fuzzy subsets μ and σ of a set S, denoted by $\mu \cup \sigma$ and $\mu \cap \sigma$ respectively, are defined as*

$$(\mu \cup \sigma)(x) = \max\{\mu(x), \sigma(x)\} \text{ for all } x \in S,$$

$$(\mu \cap \sigma)(x) = \min\{\mu(x), \sigma(x)\} \text{ for all } x \in S.$$

3. Structure of Fuzzy Ideals in Ordered Semirings

Throughout this paper, unless otherwise mentioned, S denote the ordered semiring and χ_S denote its characteristic function.

Definition 3.1 *Let μ and ν be two fuzzy subsets of an ordered semiring S and $x, y, z \in S$. We define composition and sum of μ and ν as follows:*

$$\begin{aligned} \mu \circ_1 \nu(x) &= \sup_{x \leq yz}\{\min\{\mu(y), \nu(z))\}\} \\ &= 0, \text{ if } x \text{ cannot be expressed as } x \leq yz \end{aligned}$$

and

$$\begin{aligned} \mu +_1 \nu(x) &= \sup_{x \leq y+z}\{\min\{\mu(y), \nu(z))\}\} \\ &= 0, \text{ if } x \text{ cannot be expressed as } x \leq y + z. \end{aligned}$$

Proposition 3.1 *For any fuzzy subset μ of an ordered semiring S, $(\chi_S \circ_1 \mu)(x) \geq (\chi_S \circ_1 \mu)(y)$ (resp. $(\chi_S +_1 \mu)(x) \geq (\chi_S +_1 \mu)(y)$) $\forall x, y \in S$ with $x \leq y$.*

Proof Let μ be a fuzzy subset of an ordered semiring S and $x, y \in S$ with $x \leq y$. If y cannot be expressed as $y \leq y_1 y_2$ for $y_1, y_2 \in S$, then the proof is trivial, so we omit it.

Let y have such an expression. Then

$$(\chi_S \circ_1 \mu)(y) = \sup_{y \leq y_1 y_2}\{\min\{\chi_S(y_1), \mu(y_2)\}\} = \sup_{y \leq y_1 y_2}\{\mu(y_2)\}.$$

Since $x \leq y \leq y_1 y_2$, we have

$$(\chi_S \circ_1 \mu)(x) = \sup_{x \le x_1 x_2} \{\min\{\chi_S(x_1), \mu(x_2)\}\} \\ \ge \sup_{x \le y_1 y_2} \{\min\{\chi_S(y_1), \mu(y_2)\}\} \\ = \sup_{y \le y_1 y_2} \{\mu(y_2)\} = (\chi_S \circ_1 \mu)(y).$$

Similarly for $x \le y$, we can prove that $(\chi_S +_1 \mu)(x) \ge (\chi_S +_1 \mu)(y)$.

Definition 3.2 *Let μ be a non-empty fuzzy subset of an ordered semiring S (i.e., $\mu(x) \neq 0$ for some $x \in S$). Then μ is called a fuzzy left ideal [resp. fuzzy right ideal] of S if*

(i) $\mu(x + y) \ge \min\{\mu(x), \mu(y)\}$,

(ii) $\mu(xy) \ge \mu(y)$ [*resp.* $\mu(xy) \ge \mu(x)$] *and*

(iii) $x \le y$ *implies* $\mu(x) \ge \mu(y)$

for all $x, y \in S$.
By a fuzzy ideal we mean, it is both a fuzzy left ideal as well as a fuzzy right ideal.

Example 2 Let $S = \{0, a, b, c\}$ with the ordered relation $0 \prec a \prec b \prec c$. Define operations on S by following:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

and

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
c	0	a	a	a

Then $(S, \oplus, \odot)$ forms an ordered semiring.

Now if we define a fuzzy subset μ of S by $\mu(0) = 1$, $\mu(a) = 0.7$, $\mu(b) = 0.4$ and $\mu(c) = 0.2$, then μ will be a fuzzy ideal of S.

Example 3 Let $R = \{0, 1, 2, 3\}$ be a set with the following operations:

$*$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	2
3	3	3	2	2

and

$\diamond$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	1	2	2
3	0	1	3	3

Then $(R, *, \diamond)$ forms an ordered semiring with the relation defined by "$x \le y$" if and only if $x * y * x \diamond y = y$, $x \diamond y = x$ and $x \neq y$.

Let μ be a fuzzy subset of R defined by $\mu(0) = 1$, $\mu(1) = 0.9$, $\mu(2) = 0.4$ and $\mu(3) = 0.6$. Then μ is not fuzzy ideal of R because $\mu(3 * 3) = \mu(2) = 0.4 \ngeq 0.6 = \min\{\mu(3), \mu(3)\}$.

Theorem 3.1 *A fuzzy subset μ of S is a fuzzy ordered ideal if and only if its level subset μ_t, $t \in [0, 1]$ is an ordered ideal of S.*

Proof We prove the theorem only for left ordered ideal. For right ordered ideal, it follows similarly.

To prove this, it is sufficient to consider the case only for ordered semiring and left ordered ideal, since there are several authors who have proved this result in case of semiring.

Let μ be a fuzzy left ordered ideal of S. Suppose $a \in S$ and $b \in \mu_t$ with $a \leq b$. As μ is a fuzzy left ordered ideal of S, $\mu(a) \geq \mu(b) \geq t$ so that $a \in \mu_t$ i.e., μ_t is a left ordered ideal of S.

Conversely, if μ_t is a left ordered ideal of S, then μ is a fuzzy ideal of S. Now suppose $x, y \in S$ with $x \leq y$. We have to show that $\mu(x) \geq \mu(y)$. Let $\mu(x) < \mu(y)$. Then there exists $t_1 \in [0, 1]$ such that $\mu(x) < t_1 < \mu(y)$. Then $y \in \mu_{t_1}$ but $x \notin \mu_{t_1}$ which is a contradiction to the fact that μ_t is a left ordered ideal of S.

Hence the proof.

As a consequence of it, we can obtain the following:

Theorem 3.2 *Let I be a non-empty subset of an ordered semiring S. Then I is a left ordered ideal of S if and only if the characteristic function χ_I is a fuzzy left ordered ideal of S.*

Definition 3.3 *Let μ be a fuzzy subset of an ordered semiring S and $a \in S$. We denote I_a the subset of S defined as follows:*

$$I_a = \{b \in S \mid \mu(b) \geq \mu(a)\}.$$

Proposition 3.2 *Let S be an ordered semiring and μ be a fuzzy right (resp. left) ideal of S. Then I_a is a right (resp. left) ideal of S for every $a \in S$.*

Proof Let μ be a fuzzy right ideal of S and $a \in S$. Then $I_a \neq \phi$ because $a \in I_a$ for every $a \in S$. Let $b, c \in I_a$ and $x \in S$. Since $b, c \in I_a$, $\mu(b) \geq \mu(a)$ and $\mu(c) \geq \mu(a)$. Now

$$\begin{aligned} \mu(b + c) &\geq \min\{\mu(b), \mu(c)\} [\because \mu \text{ is a fuzzy right ideal}] \\ &\geq \mu(a), \end{aligned}$$

which implies $b + c \in I_a$.

Also $\mu(bx) \geq \mu(b) \geq \mu(a)$, i.e., $bx \in I_a$.

Let $b \in I_a$ and $S \ni x \leq b$. Then $\mu(x) \geq \mu(b) \geq \mu(a) \Rightarrow x \in I_a$.

Thus I_a is a right ideal of S.

Similarly, we can prove the result for left ideal also.

The converse of the above proposition is not possible which can be seen by the following example.

Example 4 Let $S = \{0, a, b, c\}$ with the ordered relation $0 \prec c \prec b \prec a$. Define operations on S by following:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	a	a
c	c	a	a	a

and

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

Then $(S, \oplus, \odot)$ forms an ordered semiring.

Now suppose μ be a fuzzy subset of S defined by $\mu(0) = 1$, $\mu(c) = 0.3$, $\mu(b) = 0.2$ and $\mu(a) = 0.1$. Then $I_0 = \{0\}$, $I_c = \{0, c\}$, $I_b = \{0, c, b\}$ and $I_a = \{0, c, b, a\}$ – all are right ideal of S. But μ is not a fuzzy right ideal, since $\mu(b \oplus c) = \mu(a) = 0.1 \not\geq 0.2 = \min\{0.2, 0.3\} = \min\{\mu(b), \mu(c)\}$.

Proposition 3.3 *Intersection of a non-empty collection of fuzzy right (resp. left) ideals is also a fuzzy right (resp. left) ideal of S.*

Proof Let $\{\mu_i \mid i \in I\}$ be a non-empty family of fuzzy right ideals of S and $x, y \in S$. Then

$$\bigcap_{i\in I}\mu_i(x+y) = \inf_{i\in I}\{\mu_i(x+y)\} \geq \inf_{i\in I}\{\min\{\mu_i(x), \mu_i(y)\}\}$$
$$= \min\{\inf_{i\in I}\mu_i(x), \inf_{i\in I}\mu_i(y)\} = \min\{\bigcap_{i\in I}\mu_i(x), \bigcap_{i\in I}\mu_i(y)\}.$$

Again

$$\bigcap_{i\in I}\mu_i(xy) = \inf_{i\in I}\{\mu_i(xy)\} \geq \inf_{i\in I}\{\mu_i(x)\} = \bigcap_{i\in I}\mu_i(x).$$

Suppose $x \leq y$. Then $\mu_i(x) \geq \mu_i(y)$ for all $i \in I$ which implies $\bigcap_{i\in I}\mu_i(x) \geq \bigcap_{i\in I}\mu_i(y)$.

Hence $\bigcap_{i\in I}\mu_i$ is a fuzzy right ideal of S.

Similarly, we can prove the result for fuzzy left ideal also.

Proposition 3.4 *Let $\{\mu_i \mid i \in I\}$ be a family of fuzzy ideals (interior ideals, see Definition* 4.1*) of S such that $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for $i, j \in I$. Then $\cup_{i\in I}\mu_i$ is a fuzzy ideal (interior ideal) of S.*

Proof The proof follows by routine verification.

Proposition 3.5 *Let $f : R \to S$ be a morphism of ordered semirings, i.e., semiring homomorphism satisfying additional condition $a \leq b \Rightarrow f(a) \leq f(b)$. Then if ϕ is a fuzzy left ideal of S, then $f^{-1}(\phi)$* [12] *is also a fuzzy left ideal of R.*

Proof Let $f : R \to S$ be a morphism of ordered semirings and ϕ is a fuzzy left ideal of S.

Now $f^{-1}(\phi)(0_R) = \phi(0_S) \geq \phi(x^{'}) \neq 0$ for some $x^{'} \in S$.

Therefore $f^{-1}(\phi)$ is non-empty.

Now for any $r, s \in R$,

$$\begin{aligned} f^{-1}(\phi)(r+s) &= \phi(f(r+s)) = \phi(f(r)+f(s)) \\ &\geq \min\{\phi(f(r)), \phi(f(s))\} \\ &= \min\{(f^{-1}(\phi))(r), (f^{-1}(\phi))(s)\}. \end{aligned}$$

Again $(f^{-1}(\phi))(rs) = \phi(f(rs)) = \phi(f(r)f(s)) \geq \phi(f(s)) = (f^{-1}(\phi))(s)$.

Also if $r \leq s$, then $f(r) \leq f(s)$. Then

$$(f^{-1}(\phi))(r) = \phi(f(r)) \geq \phi(f(s)) = (f^{-1}(\phi))(s).$$

Thus $f^{-1}(\phi)$ is a fuzzy left ideal of R.

Definition 3.4 [1] *Let μ and ν be fuzzy subsets of X. The cartesian product of μ and ν is defined by $(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}$ for all $x, y \in X$.*

Theorem 3.3 *Let μ and ν be fuzzy left ideals of an ordered semiring S. Then $\mu \times \nu$ is a fuzzy left ideal of $S \times S$.*

Proof Let $(x_1, x_2), (y_1, y_2) \in S \times S$. Then

$$\begin{aligned} (\mu \times \nu)((x_1, x_2) + (y_1, y_2)) &= (\mu \times \nu)(x_1 + y_1, x_2 + y_2) \\ &= \min\{\mu(x_1 + y_1), \nu(x_2 + y_2)\} \\ &\geq \min\{\min\{\mu(x_1), \mu(y_1)\}, \min\{\nu(x_2), \nu(y_2)\}\} \\ &= \min\{\min\{\mu(x_1), \nu(x_2)\}, \min\{\mu(y_1), \nu(y_2)\}\} \\ &= \min\{(\mu \times \nu)(x_1, x_2), (\mu \times \nu)(y_1, y_2)\} \end{aligned}$$

and

$$\begin{aligned} (\mu \times \nu)((x_1, x_2)(y_1, y_2)) &= (\mu \times \nu)(x_1 y_1, x_2 y_2) = \min\{\mu(x_1 y_1), \nu(x_2 y_2)\} \\ &\geq \min\{\mu(y_1), \nu(y_2)\} = (\mu \times \nu)(y_1, y_2). \end{aligned}$$

Also if $(x_1, x_2) \leq (y_1, y_2)$, then

$$(\mu \times \nu)(x_1, x_2) = \min\{\mu(x_1), \nu(x_2)\} \geq \min\{\mu(y_1), \nu(y_2)\} = (\mu \times \nu)(y_1, y_2).$$

Therefore $\mu \times \nu$ is a fuzzy left ideal of $S \times S$.

Theorem 3.4 *Let μ be a fuzzy subset of an ordered semiring S. Then μ is a fuzzy left ideal of S if and only if $\mu \times \mu$ is a fuzzy left ideal of $S \times S$.*

Proof Assume that μ is a fuzzy left ideal of S. Then by Theorem 3.3, $\mu \times \mu$ is a fuzzy left ideal of $S \times S$.

Conversely, suppose that $\mu \times \mu$ is a fuzzy left ideal of $S \times S$. Let $x_1, x_2, y_1, y_2 \in S$.

Then

$$\begin{aligned}\min\{\mu(x_1+y_1),\mu(x_2+y_2)\} &= (\mu\times\mu)(x_1+y_1,x_2+y_2)\\ &= (\mu\times\mu)((x_1,x_2)+(y_1,y_2))\\ &\geq \min\{(\mu\times\mu)(x_1,x_2),(\mu\times\mu)(y_1,y_2)\}\\ &= \min\{\min\{\mu(x_1),\mu(x_2)\},\min\{\mu(y_1),\mu(y_2)\}\}.\end{aligned}$$

Now, putting $x_1 = x,\ x_2 = 0,\ y_1 = y$ and $y_2 = 0$, in this inequality and noting that $\mu(0) \geq \mu(x)$ for all $x \in S$, we obtain $\mu(x+y) \geq \min\{\mu(x),\mu(y)\}$.

Next, we have

$$\begin{aligned}\min\{\mu(x_1y_1),\mu(x_2y_2)\} &= (\mu\times\mu)(x_1y_1,x_2y_2) = (\mu\times\mu)((x_1,x_2)(y_1,y_2))\\ &\geq (\mu\times\mu)(y_1,y_2) = \min\{\mu(y_1),\mu(y_2)\}.\end{aligned}$$

Taking $x_1 = x,\ y_1 = y$ and $y_2 = 0$, we obtain $\mu(xy) \geq \mu(y)$.

Also if $(x_1,x_2) \leq (y_1,y_2)$, then $\min\{\mu(x_1),\mu(x_2)\} \geq \min\{\mu(y_1),\mu(y_2)\}$. Now, putting $x_1 = x,\ x_2 = 0,\ y_1 = y$ and $y_2 = 0$, in this inequality we have $\mu(x) \geq \mu(y)$.

Hence μ is a fuzzy left ideal of S.

Proposition 3.6 *For any three fuzzy subset $\mu_1,\ \mu_2,\ \mu_3$ of an ordered semiring S, $\mu_1 \circ_1 (\mu_2 +_1 \mu_3) = (\mu_1 \circ_1 \mu_2) +_1 (\mu_1 \circ_1 \mu_3)$.*

Proof Let $\mu_1,\ \mu_2,\ \mu_3$ be any three fuzzy subset of an ordered semiring S and $x \in S$. Then

$$\begin{aligned}(\mu_1 \circ_1 (\mu_2 +_1 \mu_3))(x) &= \sup_{x\leq yz}\{\min\{\mu_1(y),(\mu_2 +_1 \mu_3)(z)\}\}\\ &= \sup_{x\leq yz}\{\min\{\mu_1(y),\sup_{z\leq a+b}\{\min\{\mu_2(a),\mu_3(b)\}\}\}\}\\ &= \sup\{\min\{\sup\{\min\{\mu_1(y),\mu_2(a)\}\},\sup\{\min\{\mu_1(y),\mu_3(b)\}\}\}\}\\ &\leq \sup_{x\leq ya+yb}\{\min\{(\mu_1\circ_1\mu_2)(ya),(\mu_1\circ_1\mu_3)(yb)\}\}\\ &\leq ((\mu_1\circ_1\mu_2) +_1 (\mu_1\circ_1\mu_3))(x).\end{aligned}$$

Also

$$\begin{aligned}&((\mu_1\circ_1\mu_2) +_1 (\mu_1\circ_1\mu_3))(x)\\ &= \sup_{x\leq x_1+x_2}\{\min\{(\mu_1\circ_1\mu_2)(x_1),(\mu_1\circ_1\mu_3)(x_2)\}\}\\ &= \sup_{x\leq x_1+x_2}\{\min\{\sup_{x_1\leq c_1d_1}\{\min\{\mu_1(c_1),\mu_2(d_1)\}\},\sup_{x_2\leq c_2d_2}\{\min\{\mu_1(c_2),\mu_3(d_2)\}\}\}\}\\ &\leq \sup_{x\leq x_1+x_2\leq c_1d_1+c_2d_2<(c_1+c_2)(d_1+d_2)}\{\min\{\mu_1(c_1+c_2),\sup\{\min\{\mu_2(d_1),\mu_3(d_2)\}\}\}\}\\ &\leq \sup_{x\leq cd}\{\min\{\mu_1(c),(\mu_2 +_1 \mu_3)(d)\}\}\\ &= (\mu_1\circ_1(\mu_2 +_1 \mu_3))(x).\end{aligned}$$

Therefore $\mu_1 \circ_1 (\mu_2 +_1 \mu_3) = (\mu_1 \circ_1 \mu_2) +_1 (\mu_1 \circ_1 \mu_3)$.

Theorem 3.5 *If $\mu_1,\ \mu_2$ be any two fuzzy ideals of an ordered semiring S, then $\mu_1 +_1 \mu_2$ is also so.*

Proof Assume that $\mu_1,\ \mu_2$ are any two fuzzy ideals of an ordered semiring S and $x,\ y \in S$. Then

$$\begin{aligned}(\mu_1 +_1 \mu_2)(x+y) &= \sup_{x+y\leq c+d}\{\min\{\mu_1(c),\mu_2(d)\}\}\\ &\geq \sup_{x+y\leq(a_1+b_1)+(a_2+b_2)=(a_1+a_2)+(b_1+b_2)}\{\min\{\mu_1(a_1+a_2),\mu_2(b_1+b_2)\}\}\\ &\geq \sup\{\min\{\mu_1(a_1),\mu_1(a_2),\mu_2(b_1),\mu_2(b_2)\}\}\\ &= \min\{\sup_{x\leq a_1+b_1}\{\min\{\mu_1(a_1),\mu_2(b_1)\}\},\sup_{y\leq a_2+b_2}\{\min\{\mu_1(a_2),\mu_2(b_2)\}\}\}\\ &= \min\{(\mu_1 +_1 \mu_2)(x),(\mu_1 +_1 \mu_2)(y)\}.\end{aligned}$$

Now assume $\mu_1,\ \mu_2$ are as fuzzy right ideals and we have

$$\begin{aligned}(\mu_1 +_1 \mu_2)(xy) &= \sup_{xy\leq c+d}\{\min\{\mu_1(c),\mu_2(d)\}\}\\ &\geq \sup_{xy\leq(x_1+x_2)y}\{\min\{\mu_1(x_1y),\mu_2(x_2y)\}\}\\ &\geq \sup_{x\leq x_1+x_2}\{\min\{\mu_1(x_1),\mu_2(x_2)\}\}\\ &= (\mu_1 +_1 \mu_2)(x).\end{aligned}$$

Similarly assuming $\mu_1,\ \mu_2$ are as fuzzy left ideal, we can show that $(\mu_1 +_1 \mu_2)(xy) \geq (\mu_1 +_1 \mu_2)(y)$.

Now suppose $x \leq y$. Then $\mu_1(x) \geq \mu_1(y)$ and $\mu_2(x) \geq \mu_2(y)$,

$$\begin{aligned}(\mu_1 +_1 \mu_2)(x) &= \sup_{x\leq x_1+x_2}\{\min\{\mu_1(x_1),\mu_2(x_2)\}\}\\ &\geq \sup_{x\leq y\leq y_1+y_2}\{\min\{\mu_1(y_1),\mu_2(y_2)\}\}\\ &= \sup_{y\leq y_1+y_2}\{\min\{\mu_1(y_1),\mu_2(y_2)\}\}\\ &= (\mu_1 +_1 \mu_2)(y).\end{aligned}$$

Hence $\mu_1 +_1 \mu_2$ is a fuzzy ideal of S.

Theorem 3.6 *If $\mu_1,\ \mu_2$ be any two fuzzy ideals of an ordered semiring S, then $\mu_1 o_1 \mu_2$ is also so.*

Proof Let $\mu_1,\ \mu_2$ be any two fuzzy ideals of an ordered semiring S and $x,\ y \in S$. Then

$$\begin{aligned}(\mu_1 \circ_1 \mu_2)(x+y) &= \sup_{x+y\leq cd}\{\min\{\mu_1(c),\mu_2(d)\}\}\\ &\geq \sup_{x+y\leq c_1d_1+c_2d_2<(c_1+c_2)(d_1+d_2)}\{\min\{\mu_1(c_1+c_2),\ \mu_2(d_1+d_2)\}\}\\ &\geq \sup\{\min\{\mu_1(c_1),\mu_1(c_2),\mu_2(d_1),\mu_2(d_2)\}\}\\ &\geq \min\{\sup_{x\leq c_1d_1}\{\min\{\mu_1(c_1),\mu_2(d_1)\}\},\sup_{y\leq c_2d_2}\{\min\{\mu_1(c_2),\mu_2(d_2)\}\}\}\\ &= \min\{(\mu_1 \circ_1 \mu_2)(x),(\mu_1 \circ_1 \mu_2)(y)\}.\end{aligned}$$

Now, assume $\mu_1,\ \mu_2$ are as fuzzy right ideals and we have

$$\begin{aligned}(\mu_1 \circ_1 \mu_2)(xy) &= \sup_{xy\le cd}\{\min\{\mu_1(c), \mu_2(d)\}\}\\ &\ge \sup_{xy\le(x_1x_2)y}\{\min\{\mu_1(x_1), \mu_2(x_2y)\}\}\\ &\ge \sup_{x\le x_1x_2}\{\min\{\mu_1(x_1), \mu_2(x_2)\}\}\\ &= (\mu_1 \circ_1 \mu_2)(x).\end{aligned}$$

Similarly, assuming $\mu_1,\ \mu_2$ are as fuzzy left ideal, we can show that $(\mu_1 \circ_1 \mu_2)(xy) \ge (\mu_1 \circ_1 \mu_2)(y)$.

Now suppose $x \le y$. Then $\mu_1(x) \ge \mu_1(y)$ and $\mu_2(x) \ge \mu_2(y)$.

$$\begin{aligned}(\mu_1 \circ_1 \mu_2)(x) &= \sup_{x\le x_1x_2}\{\min\{\mu_1(x_1), \mu_2(x_2)\}\}\\ &\ge \sup_{x\le y\le y_1y_2}\{\min\{\mu_1(y_1), \mu_2(y_2)\}\}\\ &= \sup_{y\le y_1y_2}\{\min\{\mu_1(y_1), \mu_2(y_2)\}\}\\ &= (\mu_1 \circ_1 \mu_2)(y).\end{aligned}$$

Hence $\mu_1 \circ_1 \mu_2$ is a fuzzy ideal of S.

Theorem 3.7 *Let S be an ordered semiring. Then set of all fuzzy ideals of S (in short $FI(S)$) is zerosumfree semiring with infinite element $\mathbf{1}$ under the operations of sum and composition of fuzzy ideals of S.*

Proof Clearly, $\phi \in FI(S)$. Suppose μ_1, μ_2, μ_3 to be three fuzzy ideals of S. Then

(i) $\mu_1 +_1 \mu_2 \in FI(S)$,

(ii) $\mu_1 \circ_1 \mu_2 \in FI(S)$,

(iii) $\mu_1 +_1 \mu_2 = \mu_2 +_1 \mu_1$,

(iv) $\phi +_1 \mu_1 = \mu_1$,

(v) $\mu_1 +_1 (\mu_2 +_1 \mu_3) = (\mu_1 +_1 \mu_2) +_1 \mu_3$,

(vi) $\mu_1 \circ_1 (\mu_2 \circ_1 \mu_3) = (\mu_1 \circ_1 \mu_2) \circ_1 \mu_3$,

(vii) $\mu_1 \circ_1 (\mu_2 +_1 \mu_3) = (\mu_1 \circ_1 \mu_2) +_1 (\mu_1 \circ_1 \mu_3)$,

(viii) $(\mu_2 +_1 \mu_3) \circ_1 \mu_1 = (\mu_2 \circ_1 \mu_1) +_1 (\mu_3 \circ_1 \mu_1)$.

Also $\phi +_1 \mu_1 = \mu_1 +_1 \phi = \mu_1$.

Thus $FI(S)$ is a semiring under the operations of sum and composition of fuzzy ideals of S.

Now $\mathbf{1} \subseteq \mathbf{1} +_1 \mu_1$ for $\mu_1 \in FI(S)$.

Also $(\mathbf{1} +_1 \mu)(x) = \sup_{x\le y+z}\{\min\{\mathbf{1}(y), \mu(z)\} \mid y, z \in S\} \le 1 = \mathbf{1}(x)$ for all $x \in S$.

Therefore $\mathbf{1} +_1 \mu_1 \subseteq \mathbf{1}$ and hence $\mathbf{1} +_1 \mu_1 = \mathbf{1}$ for all $\mu_1 \in FI(S)$.

Thus **1** is an infinite element of $FI(S)$.

Next let $\mu_1 +_1 \mu_2 = \phi$ for $\mu_1, \mu_2 \in FI(S)$.

Then $\mu_1 \subseteq \mu_1 +_1 \mu_2 = \phi \subseteq \mu_1$ and so $\mu_1 = \phi$.

Similarly, it can be shown that $\mu_2 = \phi$.

Hence the semiring $FI(S)$ is zerosumfree.

4. Fuzzy Interior Ideals and Regularity Criterion

Definition 4.1 *A fuzzy subset μ of an ordered semiring S is called fuzzy interior-ideal if for all $x, y, z \in S$, we have*

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$,

(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,

(iii) $\mu(xyz) \geq \mu(y)$,

(iv) $x \leq y \Rightarrow \mu(x) \geq \mu(y)$.

Example 5 The fuzzy subset defined in Example 2 is also an example of fuzzy interior ideal.

Example 6 Let $S = \{0, a, b, c\}$ with the ordered relation $0 \prec a \prec b \prec c$. Define operations on S by following:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

and

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	b	c

Then $(S, \oplus, \odot)$ forms an ordered semiring with the order defined by $x \leq y$ if and only if $x \oplus y = y$ and $x \odot y = x$.

Now if we define a fuzzy subset μ of S by $\mu(0) = 1$, $\mu(a) = 0.8$, $\mu(b) = 0.6$ and $\mu(c) = 0.3$, then μ will be a fuzzy interior ideal of S.

Now by 'transfer principle in fuzzy set theory [7]', we have the following theorems:

Theorem 4.1 *A fuzzy subset μ of S is a fuzzy interior ideal if and only if its level subset μ_t, $t \in [0, 1]$ is an interior ideal of S.*

Theorem 4.2 *Let I be a non-empty subset of an ordered semiring S. Then I is an interior ideal of S if and only if the characteristic function χ_I is a fuzzy interior ideal of S.*

Definition 4.2 *An ordered semiring S is called regular (resp. intra-regular) if for each $a \in S$, there exist $x, y \in S$ such that $a \leq axa$ (resp. $a \leq xa^2y$).*

Proposition 4.1 *Every fuzzy interior ideal of a regular ordered semiring S is a fuzzy ideal of S.*

Proof Assume that μ is a fuzzy interior ideal of a regular ordered semiring S and $a, b \in S$. It is sufficient to prove that $\mu(ab) \geq \mu(a)$ and $\mu(ab) \geq \mu(b)$.

Since S is regular, there exists $x \in S$ such that $a \leq axa$ and hence we have

$$\begin{aligned} \mu(ab) &\geq \mu((axa)b) \text{ [since } ab \leq axab \text{ and } \mu \text{ is an ordered ideal]} \\ &= \mu((ax)ab) \geq \mu(a) \text{ [since } \mu \text{ is an interior ideal].} \end{aligned}$$

Similarly, for $b \in S$, there exists $y \in S$ such that $b \leq byb$ and hence $ab \leq abyb$.
Now $\mu(ab) \geq \mu(ab(yb)) \geq \mu(b)$.
Therefore μ is a fuzzy ideal of S.

Proposition 4.2 *Every fuzzy interior ideal of an intra-regular ordered semiring S is also a fuzzy ideal of S.*

Proof Let μ be a fuzzy interior ideal of an intra-regular ordered semiring S and $a, b \in S$. It is sufficient to prove that $\mu(ab) \geq \mu(a)$ and $\mu(ab) \geq \mu(b)$.

Since S is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y$ and hence we have

$$\begin{aligned} \mu(ab) &\geq \mu(xa^2yb) \text{ [since } ab \leq xa^2yb \text{ and } \mu \text{ is an ordered ideal]} \\ &= \mu((xa)a(yb)) \geq \mu(a) \text{ [since } \mu \text{ is an interior ideal].} \end{aligned}$$

Similarly, we can show that $\mu(ab) \geq \mu(b)$.
Hence μ is a fuzzy ideal of S.

Definition 4.3 *An ordered semiring S is called simple if it does not contain any proper ideal.*

Definition 4.4 *An ordered semiring S is called semisimple if for $a \in S$, there exist $x, y, z \in S$ such that $a \leq xayaz$.*

Proposition 4.3 *Let S be a semisimple ordered semiring. Then every fuzzy interior ideal is also a fuzzy ideal of S.*

Proof Assume that μ is a fuzzy interior ideal of a semisimple ordered semiring S and $a, b \in S$. It is sufficient to prove that $\mu(ab) \geq \mu(a)$ and $\mu(ab) \geq \mu(b)$.

Since S is semisimple, there exist $x, y, z \in S$ such that $a \leq xayaz$ and hence we have

$$\begin{aligned} \mu(ab) &\geq \mu(xayazb) \text{ [since } ab \leq xayazb \text{ and } \mu \text{ is an ordered ideal]} \\ &= \mu((xay)a(zb)) \geq \mu(a) \text{ [since } \mu \text{ is an interior ideal].} \end{aligned}$$

Similarly, we can show that $\mu(ab) \geq \mu(b)$.
Hence the result follows.

Definition 4.5 *An ordered semiring S is called fuzzy simple if for any fuzzy ideal μ of S, we have $\mu(a) \geq \mu(b)$ for all $a, b \in S$.*

Theorem 4.3 *An ordered semiring S is simple if and only if it is fuzzy simple.*

Proof Let μ be a fuzzy ideal of a simple ordered semiring S and $a,\ b \in S$. By Proposition 3.2, I_a is an ideal of S. Since S is simple, we have $I_a = S$ and so $b \in I_a$ from which it follows that $\mu(b) \geq \mu(a)$.

Conversely, suppose S contains proper ideals and let I be such an ideal of S i.e., $I \subseteq S$. We know that χ_I, the characteristic function of I, is a fuzzy ideal of S. Indeed, let $x \in S$. Since S is fuzzy simple $\chi_I(x) \geq \chi_I(b)$ for all $b \in S$. Now let $a \in I$. Then we have $\chi_I(x) \geq \chi_I(a) = 1$ which implies $x \in I$. Thus $S \subseteq I$ and so $S = I$.

Hence the result follows.

Theorem 4.4 *An ordered semiring S is simple if and only if for every fuzzy interior ideal μ of S, $\mu(a) \geq \mu(b)$ for all $a,\ b \in S$.*

Proof Let μ be a fuzzy interior ideal of a simple ordered semiring S and $a,\ b \in S$. Since S is simple and $b \in S$, we have $S = (SbS]$ [3] and also as $a \in S$, $a \in (SbS]$. Then there exist $x,\ y \in S$ such that $a \leq xby$ and so $\mu(a) \geq \mu(xby) \geq \mu(b)$.

Conversely, suppose for every fuzzy interior ideal ν of S, $\nu(a) \geq \nu(b)$ for all $a,\ b \in S$.

Now let μ be a fuzzy ideal of S. Then it is fuzzy interior ideal of S. So by definition S is fuzzy simple.

We end this paper with the following characterization.

Theorem 4.5 *An ordered semiring S is regular if and only if for any fuzzy right ideal μ and fuzzy left ideal ν, we have $\mu \circ_1 \nu = \mu \cap \nu$.*

Proof Suppose S be a regular ordered semiring and $x \in S$. Then there exists $a \in S$ such that $x \leq xax$.
Then

$$\begin{aligned}(\mu \circ_1 \nu)(x) &= \sup_{x \leq yz}\{\min\{\mu(y), \nu(z)\}\} \geq \min_{x \leq xax}\{\mu(xa), \nu(x)\} \\ &\geq \min\{\mu(x), \nu(x)\} = (\mu \cap \nu)(x).\end{aligned} \tag{1}$$

Now since $x \leq yz$, $\mu(x) \geq \mu(yz) \geq \mu(y)$ and $\nu(x) \geq \nu(yz) \geq \nu(z)$ which implies

$$(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\} \geq \min\{\mu(y), \nu(z)\}$$

and this relation is true for all representations of x. Therefore

$$(\mu \cap \nu)(x) \geq \sup_{x \leq yz}\{\min\{\mu(y), \nu(z)\}\} = (\mu \circ_1 \nu)(x). \tag{2}$$

Therefore (1) and (2) imply that $\mu \cap \nu = \mu \circ_1 \nu$.

Conversely, let C and D be respectively right and left ideals of S. Then χ_C and χ_D are respectively fuzzy right ideal and fuzzy left ideal. Moreover, $CD \subseteq C \cap D$.

Let $a \in C \cap D$. Then $\chi_C(a) = 1 = \chi_D(a)$.
Thus $(\chi_C \circ_1 \chi_D)(a) = (\chi_C \cap \chi_D)(a) = \min\{\chi_C(a), \chi_D(a)\} = 1$.

So, $\min\{\chi_C(a_1), \chi_D(a_2)\} = 1$ for some $a_1, a_2 \in S$ satisfying $a \leq a_1a_2$ i.e., $a \in CD$. Hence $C \cap D = CD$ and so S is regular by Proposition 6.35 of [4].

5. Conclusion

In this paper, we consider some operations on fuzzy ideals of an ordered semirings in order to study the structure of fuzzy ordered semirings. After that, we study regularity criterion and obtain some of its characterizations. As a continuation of this paper, we have studied (this work will be submitted with in few days) the generalized form of regularity i.e., for each $x \in S$, there exist $x, y \in S$ such that $x + axa \leq aya$ in the ordered semiring S and characterize that in terms of fuzzy k-ideal [2] (k-bi-ideal, k-interior ideal, k-quasi-ideal) and also in terms of fuzzy h-ideal in ordered semiring with more generalized form of sums and compositions of fuzzy ideals of ordered semiring.

Our future work on this topic will also focus on a generalized form of ordered semirings i.e., ordered Γ-semirings [13].

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