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Quasi-varieties in Abstract Algebraic Institutions

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To provide a formal framework for discussing specifications of abstract data types we restrict the notion of institution due to Goguen and Burstall ("Lecture Notes in Computer Sci.," No. 164, pp. 221–256, Springer, Berlin, 1984) which formalizes the concept of a logical system for writing specifications, and deal with *abstract algebraic institutions*. These are institutions equipped with a notion of submodel which satisfy a number of technical conditions. In this framework we introduce an abstract notion of ground equation which, in turn, determines notions of abstract infinitary conditional equation and inequation. We prove that quasi-varieties (i.e., classes of models closed under submodels and nonempty products) are exactly classes of models definable by abstract infinitary conditional equations and inequations. As a consequence we obtain "syntactic" characterizations of abstract algebraic institutions which guarantee the existence of reachable initial models for any consistent set of axioms, as well as those which guarantee the existence of a free model of a theory generated by any model of a subtheory (with respect to an arbitrary theory morphism). We also show how to specialize these results for abstract algebraic institutions of, respectively, total, partial, and continuous algebras. © 1986 Academic Press, Inc.

1. INTRODUCTION

The study of classes of algebras definable by equations has a well-established tradition in universal algebra. Perhaps the first (and most) important result in this area is Birkhoff's theorem [Bir 35] which states that these classes are exactly varieties, i.e., classes closed under subalgebras, products and quotients. Varieties and equational logic have a number of nice algebraic and proof-theoretic properties (cf., e.g., [GM 81]), not the least important among them being that any variety contains an initial algebra, which is an appropriate quotient of the initial algebra of terms (no junk!).

In the spirit of this tradition the pioneering papers on *algebraic specification* [ADJ 76, Gut 75, Zil 74] proposed to specify abstract data types by giving a signature and a set of equations over this signature, which describes a variety of algebras satisfying the equations. Moreover, the *initial* algebra in this variety was viewed as "the standard" realization of the abstract data type (cf. [ADJ 76]). Today, however, examples of logical systems used in specifications include first-

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order logic (with and without equality), Horn-clause logic, higher-order logic, infinitary logic, temporal logic, and many others. Note that all these logical systems may be (and actually are) considered with or without predicates, admitting partiality of operations or not. This leads to different concepts of signature and of model, perhaps even more obvious in examples like polymorphic signatures, order-sorted signatures, continuous algebras or error algebras.

The informal notion of logical system has been formalized by Goguen and Brustall [GB 84], who introduced for this purpose the notion of *institution* (which generalizes the ideas of "abstract model theory" [Bar 74]). An institution consists of a collection of "abstract signatures" together with for any "signature" Σ collections of Σ-sentences and of Σ-models and a satisfaction relation between Σ-models and Σ-sentences. The only "semantic" requirement ("satisfaction condition") is that when we change signatures, the induced translations of sentences and models preserve the satisfaction relation. This satisfaction condition expresses the intentional independence of the meaning of specifications from the actual notation.

Among standard algebraic institutions (i.e., when only usual algebraic signatures and total many-sorted algebras are considered) the institution of infinitary conditional equations and inequations (infinitary Horn-clauses) has a special place. As proved by Mahr and Makowsky (cf. [MM 84]), this is the most general standard algebraic institution which has the basic property of the institution of equations: any consistent set of axioms has an initial model with no junk (following [MM 84] we say that institutions which satisfy this condition *strongly admit initial semantics*). Moreover, a similar result holds if we require the institution to be *strongly liberal*, i.e., that for any theory and for any model of a subtheory (w.r.t. an arbitrary theory morphism) there is a model of the theory which is free over and generated by this model of the subtheory. The most general strongly liberal standard algebraic institution is the institution of infinitary conditional equations (cf. [Tar 84]).

In [Tar 84a, 85] we partly generalized these results and proved than an *abstract algebraic institution* strongly admits initial semantics if and only if every class definable in it is a quasi-variety and that it is strongly liberal if and only if every class definable in it is a strict quasi-variety.

By an abstract algebraic institution we mean (Sect. 3, cf. [Tar 85]) an institution equipped with a notion of submodel and quotient model. This amounts to the requirement that for every signature Σ the category of Σ-models has a factorization system. Moreover, we require that every ground variety w.r.t. this factorization system is definable in the institution and that the institution satisfies the "abstractness condition" (the satisfaction relation identifies isomorphic models). Finally, we assume that the institution guarantees the existence of a diagram (in the sense of model theory) for any model. Some other restrictions are purely technical.

It should be stressed that in this paper we deal only with *reachable* initial models (and free models which are *generated* by their submodels). Of course, in general initial models do not have to be reachable. For example, in the standard algebraic framework, the quantifier $\exists!$, "there exists a unique" (easily expressible in first-order logic with equality: $\exists! x.\varphi(x)$ stands for $\exists x.(\varphi(x) \,\&\, \forall y.(\varphi(y) \Rightarrow x = y))$) leads to theories with nonreachable initial models. To illustrate this, consider a one-sorted signature with constant *zero* and unary operation *succ*, and the axiom $\exists! x.succ(x)=x$. It is easy to see that the initial model of this axiom consists of (a copy of) the natural numbers with exactly one additional element which is a fixed point of *succ*; of course, this model is not reachable. Although the "no junk" restriction seems to be quite acceptable (many approaches to abstract data types are based on this restriction anyway), it would be very interesting to admit arbitrary initial models in our characterization results. Some recent work by J. Makowsky [Mak 85] addresses this problem in the standard algebraic framework.

In a series of very interesting papers Andreka, Nemeti and Sain (cf. [AN 76, AN 77, AN 79, NS 77], also, e.g., [BH 76] explored classes of morphisms (or, more generally, cones and trees) and the notion of injectivity w.r.t. these classes as categorical generalizations of the notions of, respectively, formulae and their satisfaction in a model. Along this line they obtained several Birkhoff-type characterization theorems which hold in any category satisfying rather mild assumptions and which may be used in the framework of abstract algebraic institutions (cf. [Tar 84a, 85]). We briefly recall those of their results which we use here in Section 2.

The main purpose of this paper is to pursue this line of investigation and to give a Birkhoff-type characterization of quasi-varieties in terms of definability by formulae of a certain *standard* form.

Any abstract algebraic institution determines a semantic notion of "ground equation" (positive elementary sentence)—"ground equations" are exactly the sentences which define ground varieties (Sect. 4). Of course, we cannot expect that a "syntactic" characterization of these "ground equations" may be given without referring to a particular institution. What is possible and what we do in this paper (Sect. 6) is that when this syntactic notion of "ground equation" is given, quasi-varieties and strict quasi-varieties may be characterized in a uniform way, independent from any particular institution, as classes of models definable by (resp.) universally quantified infinitary conditional "equations" and "inequations," and universally quantified infinitary conditional "equations." To formalize this we need, however, an abstract notion of open formula and universal quantification in an arbitrary institution (Sect. 5).

We use the results of Section 6 to obtain "syntactic" characterizations of the most general abstract algebraic institution which strongly admits initial semantics and of the most general abstract algebraic institution which is strongly liberal (Sect. 7). Section 8 contains a brief summary of our results.

Throughout this paper we illustrate introduced definitions and obtained results using three typical notions of model (over standard algebraic signatures): total, partial and continuous algebras. Our presentation of total algebras is based on [ADJ 76], of partial algebras on [Bur 82, Rei 84, and BrW 82], and of continuous algebras on [ANR 84] and [Mes 80] (cf. also [TW 85]).

We assume that the reader is familiar with the basic notions of category theory. See [AM 78, Mac 71, HS 73] for the standard definitions of, e.g., category, functor, pushout, colimit, cocontinuity, etc., which we omit here. Apart from that, the paper is formally self-contained, i.e., it contains all formal definitions and facts proved elsewhere which are necessary to interpret our definitions and results. However, some acquaintance with the basic intuition behind the notions of institution [GB 84], of injectivity w.r.t. cones [AN 76, 77, 79] and of abstract algebraic institution [Tar 85] would be helpful in following the details.

2. Preliminaries

In this section, mainly to fix the notation, we very briefly review basic notions, definitions, and facts used in the rest of the paper.

Let **K** be an arbitrary category.

By a *factorization system* for **K** we mean a pair $\langle \mathbf{E}, \mathbf{M} \rangle$ such that:

(i) **E** is a class of epimorphisms in **K**, **M** is a class of monomorphisms in **K**.

(ii) **E** and **M** are closed under composition and contain all isomorphisms in **K**.

(iii) every morphism in **K** has $\langle \mathbf{E}, \mathbf{M} \rangle$-factorization, i.e., for any morphism f there are $e_f \in \mathbf{E}$ and $m_f \in \mathbf{M}$ such that $f = e_f; m_f$.

(iv) the $\langle \mathbf{E}, \mathbf{M} \rangle$-factorizations are unique up to isomorphism, i.e., for any $e1, e2 \in \mathbf{E}$ and $m1, m2 \in \mathbf{M}$, if $e1; m1 = e2; m2$ then there is an isomorphism i such that $e1; i = e2$ and $i; m2 = m1$.

National Remark. Throughout the paper the composition in any category is denoted by ; (semicolon) and written in diagrammatic order. Identities are denoted by id (with indices, if necessary).

For the rest of this section let us fix an arbitrary category **K** with a factorization system $\langle \mathbf{E}, \mathbf{M} \rangle$. Sometimes we refer to elements of **E** and **M** as *factorization epimorphisms* and *monomorphisms*, respectively. We assume that **K** is **E**-co-well-powered, i.e., see [HS 73, Def. 17.15], for every object $A \in |\mathbf{K}|$ there is a *set* of factorization epimorphisms $E \subseteq \mathbf{E}$ with domain A such that for *every* $e \in \mathbf{E}$ with domain A there is $e' \in E$ and an isomorphism i such that $e = e'; i$. Moreover, we assume that **K** has an initial object $\mathcal{A}$ and all products (of sets of objects).

We say that an object $A \in |\mathbf{K}|$ is *reachable* if every morphism $m \in \mathbf{M}$ with codomain A is an isomorphism, or equivalently (see [Tar 85]) if the unique morphism from $\mathcal{A}$ to A is a factorization epimorphism. More intuitively, A is reachable if it has no proper subobject (where the notion of subobject is determined by the given factorization system). In the standard algebraic case this corresponds to the "no junk" requirement: an algebra is reachable iff it is generated by the empty set.

We now list a few basic properties of factorization systems and reachable objects we rely on throughout this paper.

FACT 2.1. [Tar 85]. (1) If $e \in E$ and $e;f \in \mathbf{M}$ for some f, then e is an isomorphism. If $m \in \mathbf{M}$ and $f; m \in \mathbf{E}$ for some f, then m is an isomorphism.

(2) If $A \in |\mathbf{K}|$ is reachable then for every $B \in |\mathbf{K}|$ there is at most one morphism from A to B.

(3) If $A, B \in |\mathbf{K}|$, B is reachable and $f: A \to B$ then $f \in \mathbf{E}$.

(4) Every object $A \in |\mathbf{K}|$ has a unique (up to isomorphism) reachable subobject.

We say that a class $K \subseteq |\mathbf{K}|$ of objects of $\mathbf{K}$ is *closed under*

(i) *isomorphism* if for any isomorphism i, if the domain of i belongs to K then so does its codomain,

(ii) *products* if for any set $F \subseteq K$, the product of F belongs to K,

(iii) *nonempty products* if for any nonempty set $F \subseteq K$, the product of F belongs to K,

(iv) *subobjects* ("submodels") if for any morphism $m \in \mathbf{M}$, if the codomain of m belongs to K then so does its domain,

(v) *quotients* ("homomorphic images") if for any morphism $e \in E$, if the domain of e belongs to K then so does its codomain,

(vi) *"extensions"* if for any morphism f, if the domain of f belongs to K then so does its codomain.

Throughout the rest of the paper we assume that all classes of objects of any category we talk about are closed under isomorphism.

For any object $A \in |\mathbf{K}|$, $\text{Ext}(A)$ denotes the least class of objects in $\mathbf{K}$ which contains A and is closed under extensions, i.e., $B \in \text{Ext}(A)$ if and only if there is a morphism from A to B.

A class $K \subseteq |\mathbf{K}|$ is called a *variety* (resp. *strict quasi-variety*, *quasi-variety*) if it is closed under quotients, subobjects, and products (resp. under subobjects and products, under subobjects and non-empty products). $K \subseteq |\mathbf{K}|$ is called a *ground variety* if it is of the form $\text{Ext}(A)$ for some reachable object $A \in |\mathbf{K}|$. If $A \in |\mathbf{K}|$ is reachable then it is initial in $\text{Ext}(A)$. Moreover, if $A \in |\mathbf{K}|$ is reachable then $\text{Ext}(A)$ is closed under products, subobjects and quotients, i.e., any ground variety is a variety (see [Tar 85]).

LEMMA 2.1. [Tar 85]. *Any nonempty quasi-variety has a reachable initial object.*

In the rest of this section we briefly recall those Brikhoff-type characterization results formulated in [AN 76, AN 77, AN 79, NS 77], also, e.g., [BH 76] which we directly apply in our framework.

For any morphism $f: A \to B$ and object $M \in |\mathbf{K}|$ we say that M is *injective* w.r.t. f if any morphism $g: A \to M$ factors through f, i.e., $g = f;h$ for some $h: B \to M$. (Actually, as the reader will see in the following, it might be more appropriate to use the expression "M satisfies f" to name this concept—we keep, however, the original terminology of [AN 76].)

By a *cone* in **K** we mean any object $A \in |\mathbf{K}|$ together with a family of morphisms with domain A.

Let $\gamma = \langle A, \{f_\beta: A \to B_\beta\}_{\beta<\alpha}\rangle$ be a cone in **K**. We say that an object $M \in |\mathbf{K}|$ is injective w.r.t. γ if any morphism $g: A \to M$ factors through at least one morphism of γ, i.e., $g = f_\beta; h$ for some $\beta < \alpha$ and $h: B_\beta \to M$. If Γ is a family of cones in **K** then we say that an object $M \in |\mathbf{K}|$ is injective w.r.t. Γ if it is injective w.r.t. any element of Γ. $\mathrm{Inj}(\Gamma) \subseteq |\mathbf{K}|$ denotes the class of all objects which are injective w.r.t. Γ. We say that Γ *defines* $\mathrm{Inj}(\Gamma)$.

Simple examples of how this notion of injectivity relates to the logical satisfaction of sentences in standard categories of algebras will be given at the end of this section; more interesting examples of such a relationship will appear in the sequel (in the proofs of Lemmas 4.1, 4.2 and Theorems 6.1, 6.2). Here let us only state the main result.

THEOREM 2.1 *A class of objects of* **K** *is*

(1) *a quasi-variety iff it is definable by a family of cones of the form* $\langle A, \{e\}\rangle$ *or* $\langle A, \varnothing\rangle$, *where* $e \in \mathbf{E}$.

(2) *a strict quasi-variety iff it is definable by a family of cones of the form* $\langle A, \{e\}\rangle$, *where* $e \in \mathbf{E}$.

(3) *a ground variety iff it is definable by a family of cones of the form* $\langle A, \{e\}\rangle$, *where* $e \in E$ *(and* A *is initial in* **K**).

The proof is given, e.g., in [AN 76, NS 77, BH 76] (under the assumptions adopted here it may be slightly simplified—details in [Tar 85]).

Three Examples.

An *algebraic signature* is a pair $\langle S, \Omega\rangle$ where S is a set (of sort names) and Ω is a family of sets $\{\Omega_{w,s}\}_{w\in S^*, s\in S}$ (of operation names). We write $f: w \to s$ to denote $w \in S^*$, $s \in S$, $f \in \Omega_{w,s}$. An *algebraic signature morphism* $\sigma: \langle S, \Omega\rangle \to \langle S', \Omega'\rangle$ is a pair $\langle \sigma_{\mathrm{sorts}}, \sigma_{\mathrm{opns}}\rangle$ where $\sigma_{\mathrm{sorts}}: S \to S'$ and σ_{opns} is a family of maps $\{\sigma_{w,s}: \Omega_{w,s} \to \Omega'_{\sigma^*(w),\sigma(s)}\}_{w\in S^*, s\in S}$ where $\sigma^*(s1,..., sn)$ denotes $\sigma_{\mathrm{sorts}}(s1),..., \sigma_{\mathrm{sorts}}(sn)$ for $s1,..., sn \in S$. We will write $\sigma(s)$ for $\sigma_{\mathrm{sorts}}(s)$, $\sigma(w)$ for $\sigma^*(w)$ and $\sigma(f)$ for $\sigma_{w,s}(f)$, where $f \in \Omega_{w,s}$.

The category of algebraic signatures AlgSig has algebraic signatures as objects and algebraic signature morphisms as morphisms; the composition of morphisms is the composition of their corresponding components as functions. Category AlgSig is cocomplete.

Let $\Sigma = \langle S, \Omega\rangle$ be an algebraic signature.

We define the categories of, respectively, total, partial and continuous Σ-algebras and their natural factorization systems.

A *partial* Σ*-algebra* A consists of an S-sorted family of carrier sets $|A| =$

$\{|A|_s\}_{s\in S}$ and for each $f\colon s1,\dots, sn \to s$ a *partial* function $f_A\colon |A|_{s1} \times \cdots \times |A|_{sn} \to |A|_s$. A *total* Σ*-algebra* is a partial Σ-algebra in which all these functions are total. A (*weak*) Σ*-homomorphism* from a partial Σ-algebra A to a partial Σ-algebra B, $h\colon A \to B$, is a family of total functions $\{h_s\colon |A|_s \to |B|_s\}_{s\in S}$ such that for any $f\colon s1,\dots, sn \to s$ and $a_1 \in |A|_{s1},\dots, a_n \in |A|_{sn}$

$$f_A(a_1,\dots, a_n) \text{ defined} \Rightarrow f_B(h_{s1}(a_1),\dots, h_{sn}(a_n)) \text{ defined and}$$
$$h_s(f_A(a_1,\dots, a_n)) = f_B(h_{s1}(a_1),\dots, h_{sn}(a_n))$$

([BrW 82] would call this a *total* Σ-homomorphism). If moreover h satisfies the condition

$$f_B(h_{s1}(a_1),\dots, h_{sn}(a_n)) \text{ defined} \Rightarrow f_A(a_1,\dots, a_n) \text{ defined}$$

then h is called a *closed* Σ*-homomorphism*.

Let TAlg(Σ) denote the category of total Σ-algebras and Σ-homomorphisms. TAlg(Σ) has an initial object T_Σ, the algebra of ground Σ-terms, all products of sets of Σ-algebras defined in the usual way and, moreover, a factorization system $\langle \mathbf{TE}_\Sigma, \mathbf{TM}_\Sigma \rangle$, where $\mathbf{TE}_\Sigma$ is the class of all surjective Σ-homomorphisms (epimorphisms in TAlg(Σ)) and $\mathbf{TM}_\Sigma$ is the class of all injective Σ-homomorphisms (monomorphisms in TAlg(Σ)). TAlg(Σ) is $\mathbf{TE}_\Sigma$-co-well-powered.

Let PAlg(Σ) denote the category of partial Σ-algebras and (weak) Σ-homomorphisms. PAlg(Σ) has an initial object ϕ_Σ, the algebra with all carriers empty and so all operations totally undefined, all products of sets of partial Σ-algebras defined in the usual way and, moreover, a factorization system $\langle \mathbf{PE}_\Sigma, \mathbf{PM}_\Sigma \rangle$, where $\mathbf{PE}_\Sigma$ is the class of all epimorphisms in PAlg(Σ) and $\mathbf{PM}_\Sigma$ is the class of all injective closed Σ-homomorphisms. PAlg(Σ) is $\mathbf{PE}_\Sigma$-co-well-powered.

Note that under this factorization system a subobject of a partial Σ-algebra corresponds to a partial subalgebra in the sense of [Grä 79, p. 80]: if B is a partial Σ-algebra then a partial subalgebra A of B has a carrier $|A| \subseteq |B|$ such that $|A|$ is closed under all operations (as defined in B).

Note also that epimorphisms in PAlg(Σ) need not be surjective. A Σ-homomorphism $h\colon A \to B$ is an epimorphism if and only if B has no proper subalgebra containing the (set-theoretic) image of $|A|$ under h.

For any S-sorted set $X = \{X_s\}_{s\in S}$ the (total) algebra of Σ-terms with variables X, denoted by $T_\Sigma(X)$, is defined as usual as "the" initial total $\Sigma(X)$-algebra where $\Sigma(X)$ is the enrichment of Σ by elements of X as constants of the appropriate sorts (see, e.g., [ADJ 76] or [BG 82]). For any partial Σ-algebra A and any S-sorted function $v\colon X \to |A|$ (called a *valuation* of variables X) the *value* of a term $t \in |T_\Sigma(X)|_s$, $s \in S$, in A under v is denoted by $t^A(v)$ (note that $t^A(v)$ may be undefined—see [Bur 82] or [Rei 84] for a precise definition of this notion). We write T_Σ for $T_\Sigma(\phi)$ and refer to terms with no variables as ground terms. For a ground term t we write t^A rather than $t^A(\phi)$.

Note that a Σ-homomorphisms $h: A \to B$ is an epimorphism (in PAlg(Σ)) if and only if any element of $|B|$ is the value of a Σ-term with variables X under a valuation which maps X into the image of $|A|$ under h. In particular, a partial Σ-algebra B is reachable in PAlg(Σ) if and only if every element of $|B|$ is the value in B of a ground Σ-term.

By a continuous Σ-algebra we mean a total Σ-algebra A together with a chain-complete ordering $\leqslant^A \subseteq |A| \times |A|$ (i.e., for $s \in S$, $\leqslant^A_s$ is an ordering on $|A|_s$ such that any countable chain $a_0 \leqslant^A_s a_1 \leqslant^A_s \ldots$, of elements of $|A|_s$ has a least upper bound $\bigsqcup^A_{i \geqslant 0} a_i$ in $|A|_s$) such that all operations in A are continuous (i.e., for any $f: s_1 \times \cdots \times s_n \to s$, for any chains $a_{k,0} \leqslant^A_{s_k} a_{k,1} \leqslant^A_{s_k} \ldots$, in $|A|_{s_k}$ for $k = 1, \ldots, n$, $f_A(\bigsqcup^A_{i \geqslant 0} a_{1,i}, \ldots, \bigsqcup^A_{i \geqslant 0} a_{n,i}) = \bigsqcup^A_{i \geqslant 0} f_A(a_{1,i}, \ldots, a_{n,i})$).

For any continuous Σ-algebras A, B, by a continuous Σ-homomorphism from A to B we mean a (discrete) Σ-homomorphism $h: A \to B$ which is continuous w.r.t. the orderings in A and B, i.e., for any chain $a_0 \leqslant^A a_1 \leqslant^A \ldots$, $h(\bigsqcup^A_{i \geqslant 0} a_i) = \bigsqcup^B_{i \geqslant 0} h(a_i)$. (Note that, whenever possible, we omit the subscripts s in formulae.) We say that a continuous Σ-homomorphism $h: A \to B$ is *full* if for any $a, b \in |A|$, $a \leqslant^A b$ iff $h(a) \leqslant^B h(b)$.

Let CAlg(Σ) denote the category of continuous Σ-algebras and continuous Σ-homomorphisms. CAlg(Σ) has an initial object, which is an initial total Σ-algebra of (finitary) ground terms with the discrete ordering, all products of sets of continuous Σ-algebras defined in the usual way and, moreover, a factorization system $\langle \mathbf{CE}_\Sigma, \mathbf{CM}_\Sigma \rangle$, where (cf. [Mes 80]) $\mathbf{CE}_\Sigma$ is the class of all *strongly dense* epimorphisms in CAlg(Σ) and $\mathbf{CM}_\Sigma$ is the class of all full injective continuous Σ-homomorphism (full monomorphisms in CAlg(Σ)). A continuous Σ-homomorphism $h: A \to B$ is strongly dense if B has no proper continuous subalgebra which contains the set-theoretic image of $|A|$ under h. (Note that the usual notion of a continuous subalgebra is determined by the accepted factorization monomorphisms.) This is equivalent to the requirement that every element of $|B|$ is the least upper bound of a countable chain of least upper bounds of countable chains of … of elements of the set-theoretic image of $|A|$ under h. Fortunately, the length of the iteration represented by the elipsis "…" in the previous statement may be bounded (see [Nel 81, ANR 84]) which shows that CAlg(Σ) is $\mathbf{CE}_\Sigma$-co-well-powered.

Let $|\Sigma| = 2^{\max\{\mathrm{card}(\Sigma), \aleph_0\}}$ and let $|\Sigma|^+$ be the least regular ordinal larger than $|\Sigma|$. Define inductively the (S-sorted) family $\{T_\alpha(\Sigma)\}_{\alpha < |\Sigma|^+}$

(1) $T_0(\Sigma)$ is the (carrier of the) usual discrete initial Σ-algebra of ground finitary Σ-terms,

(2) for any ordinal $\alpha < |\Sigma|^+$, for $s \in S$, $T_{\alpha+1}(\Sigma)_s = T_\alpha(\Sigma)_s \cup \{\bigsqcup_{i \geqslant 0} t_i \mid$ for $i \geqslant 0$ $t_i \in T_\alpha(\Sigma)_s\}$,

(3) for any limit ordinal $\alpha < |\Sigma|^+$, $T_\alpha(\Sigma) = \bigcup_{\beta < \alpha} T_\beta(\Sigma)$.

Let $T^\infty_\Sigma = \bigcup_{\alpha < |\Sigma|^+} T_\alpha(\Sigma)$, the family of ground infinitary Σ-terms. (Note that here $\bigsqcup_{i \geqslant 0} t_i$ is nothing more than just a formal expression; it is not a least upper bound.)

For any continuous Σ-algebra A and term $t \in T_\Sigma^\infty$, we define the value of t in A, t^A, as follows:

(1) for $t \in T_0(\Sigma)$, t^A is defined as in the discrete case above,

(2) for $t \in T_{\alpha+1}(\Sigma)$, $t = \bigsqcup_{i \geqslant 0} t_i$, where for $i \geqslant 0$, $t_i \in T_\alpha(\Sigma)$, t^A is defined if and only if for $i \geqslant 0$, t_i^A is defined and $t_i^A \leqslant^A t_{i+1}^A$, and if this is the case then $t^A = \bigsqcup_{i \geqslant 0}^A t_i^A$.

Now, the above definitions extend in the usual way (as in the discrete case) to define infinitary Σ-terms with variables and their values in a continuous algebra under a valuation of variables.

A continuous Σ-homomorphism $h: A \to B$ is a strongly dense epimorphism if and only if any element of $|B|$ is the value of an infinitary Σ-term with variables X under a valuation which maps X into the image of $|A|$ under h. In particular, a continuous Σ-algebra B is reachable in CAlg(Σ) if and only if every element of $|B|$ is the value in B of a ground infinitary Σ-term.

To conclude this section, let us illustrate the notion of injectivity w.r.t. cones by means of a very simple example (in the framework of total algebras). Let Σ be an algebraic signature with exactly one sort and three constants a, b and c; let A and B be the following Σ-algebras:

A:	∘	∘	B:	∘
	$a=b$	c		$a=b=c$

Finally, let h^A and h be (the unique, by Fact 2.1) Σ-homomorphisms from T_Σ (the initial Σ-algebra) to A and, respectively, from A to B.

Now, for any Σ-algebra C

(a) C is injective w.r.t. $\langle T_\Sigma, \{h^A\}\rangle$ if and only if C satisfies the equation $a=b$ (as by the definition of the initial algebra, the injectivity of C w.r.t. this cone is equivalent to the existence of a Σ-homomorphism from A to C).

(b) C is injective w.r.t. $\langle A, \phi\rangle$ if and only if C does not satisfy the equation $a=b$ (as by the definition, the injectivity of C w.r.t. this cone is equivalent to the fact that there is no Σ-homomorphism from A to C).

(c) C is injective w.r.t. $\langle A, \{h\}\rangle$ if and only if either C does not satisfy the equation $a=b$ or C satisfies the equations $a=b$ and $b=c$ (as by Fact 2.1, the injectivity of C w.r.t. this cone is equivalent to the fact that either there is no Σ-homomorphism from A to C or there is a Σ-homomorphism from B to C).

3. Abstract Algebraic Institutions

Following [GB 84] we introduce institutions to formalize the notion of a logical system for writing specifications. The work of [Bar 74] on abstract model theory is similar in intent to the theory of institutions but the notions used there and the conditions they must satisfy are more restrictive and rule out some of the examples we would like to deal with.

DEFINITION 3.1. [GB 84]. An *institution* INS consists of:

(i) A category $\mathrm{Sign}_{\mathrm{INS}}$ (of signatures).

(ii) A functor $\mathrm{Sen}_{\mathrm{INS}}: \mathrm{Sign}_{\mathrm{INS}} \to \mathrm{Cat}$ (where Cat is the category of all categories[1]) such that for any signature Σ $\mathrm{Sen}_{\mathrm{INS}}(\Sigma)$ is a discrete category. $\mathrm{Sen}_{\mathrm{INS}}$ gives for any signature Σ the class of *Σ-sentences* and for any signature morphism $\sigma: \Sigma \to \Sigma'$ the function $\mathrm{Sen}_{\mathrm{INS}}(\sigma): \mathrm{Sen}_{\mathrm{INS}}(\Sigma) \to \mathrm{Sen}_{\mathrm{INS}}(\Sigma')$ translating Σ-sentences to Σ'-sentences.

(iii) A functor $\mathrm{Mod}_{\mathrm{INS}}: \mathrm{Sign}_{\mathrm{INS}}^{op} \to \mathrm{Cat}$. $\mathrm{Mod}_{\mathrm{INS}}$ gives for any signature Σ the category of *Σ-models* and for any signature morphism $\sigma: \Sigma \to \Sigma'$ the *σ-reduct functor* $\mathrm{Mod}_{\mathrm{INS}}(\sigma): \mathrm{Mod}_{\mathrm{INS}}(\Sigma') \to \mathrm{Mod}_{\mathrm{INS}}(\Sigma)$ translating Σ'-models to Σ-models.

(iv) A satisfaction relation $\models_{\Sigma,\mathrm{INS}} \subseteq |\mathrm{Mod}_{\mathrm{INS}}(\Sigma)| \times |\mathrm{Sen}_{\mathrm{INS}}(\Sigma)|$ for each signature Σ such that the following "satisfaction condition" holds:
For any signature morphism $\sigma: \Sigma \to \Sigma'$ the translations $\mathrm{Mod}_{\mathrm{INS}}(\sigma)$ of models and $\mathrm{Sen}_{\mathrm{INS}}(\sigma)$ of sentences preserve the satisfaction relation, i.e., for any $\varphi \in |\mathrm{Sen}_{\mathrm{INS}}(\Sigma)|$ and $M' \in |\mathrm{Mod}_{\mathrm{INS}}(\Sigma')|$

$$M' \models_{\Sigma',\mathrm{INS}} \mathrm{Sen}_{\mathrm{INS}}(\sigma)(\varphi) \text{ iff } \mathrm{Mod}_{\mathrm{INS}}(\sigma)(M') \models_{\Sigma,\mathrm{INS}} \varphi.$$

Notational conventions:

(a) We omit subscripts (INS, Σ) whenever possible.

(b) For any signature morphism $\sigma: \Sigma \to \Sigma'$, $\mathrm{Sen}(\sigma)$ is denoted just by σ and $\mathrm{Mod}(\sigma)$ is denoted by $\_|_\sigma$ (i.e., for $\varphi \in |\mathrm{Sen}(\Sigma)|$, $\sigma(\varphi)$ stands for $\mathrm{Sen}(\sigma)(\varphi)$, and, e.g., for $M' \in |\mathrm{Mod}(\Sigma')|$, $M'|_\sigma$ stands for $\mathrm{Mod}(\sigma)(M')$).

(c) For $\Phi \subseteq |\mathrm{Sen}(\Sigma)|$ and $K \subseteq |\mathrm{Mod}(\Sigma)|$ we write $K \models \Phi$ with the obvious meaning.

(d) For any signature Σ and $\Phi \subseteq |\mathrm{Sen}(\Sigma)|$, $\mathrm{Mod}(\Phi)$ denotes the collection of all Σ-models M that satisfy Φ (i.e., such that $M \models \Phi$).

However, this very elegant and extremely general framework is too general for our purposes. We restrict the notion of institution and deal only with *abstract algebraic institutions*, which are institutions with factorization systems subject to several technical conditions. Before we give a formal definition we need some more terminology.

For any signature Σ, the morphisms of the category of Σ-models are called *Σ-morphisms*. We identify any class K of Σ-models with the full subcategory of $\mathrm{Mod}(\Sigma)$ with objects K. We say that a class of Σ-models K is *definable* if there is a

[1] Of course, some foundational difficulties are connected with the use of this category, as discussed in [MacL 71]. We do not discuss this point here, and we disregard other such foundational issues in this paper.

set of Σ-sentences $\Phi \subseteq |\mathrm{Sen}(\Sigma)|$ such that K consists of exactly those Σ-models that satisfy Φ, i.e., $K = \mathrm{Mod}(\Phi)$. For any signature morphism $\sigma: \Sigma \to \Sigma'$ by a *σ-expansion* of a Σ-model M we mean any Σ'-model M' such that $M'|_\sigma = M$. Similarly, by a *σ-expansion* of a Σ-morphism f we mean any Σ'-morphism f' such that $f'|_\sigma = f$.

DEFINITION 3.2. [Tar 85]. An *abstract algebraic institution* is an institution INS together with for any signature Σ a factorization system $\langle \mathbf{E}_\Sigma, \mathbf{M}_\Sigma \rangle$ for $\mathrm{Mod}(\Sigma)$ such that the following conditions hold:

(1) The category of signatures is finitely cocomplete and Mod preserves finite colimits (i.e., Mod translates finite colimits in Sign to limits in Cat).

(2) For any signature Σ, the category $\mathrm{Mod}(\Sigma)$ of Σ-models has an initial object and all products (of sets of models). Moreover, it is $\mathbf{E}_\Sigma$-co-well-powered.

(3) For any signature morphism $\sigma: \Sigma \to \Sigma'$ the σ-reduct functor preserves submodels (i.e., for any $m' \in \mathbf{M}_{\Sigma'}$ $m'|_\sigma \in \mathbf{M}_\Sigma$) and products.

(4) (Abstraction condition) For any signature Σ, $A, B \in |\mathrm{Mod}(\Sigma)|$ and $\varphi \in |\mathrm{Sen}(\Sigma)|$, if A and B are isomorphic then $A \models \varphi$ iff $B \models \varphi$.

(5) (Definability of ground varieties) For any signature Σ, any ground variety of Σ-models is definable.

(6) (Existence of diagrams) For any signature Σ and model $M \in |\mathrm{Mod}(\Sigma)|$ there is a signature $\Sigma(M)$ and a signature morphism $\iota: \Sigma \to \Sigma(M)$ such that

(a) M has a reachable ι-expansion $E(M)$.

(b) For any Σ-morphism $f: M \to N$ there is a unique ι-expansion of N, $E_f(N)$, such that f has a (unique, since $E(M)$ is reachable) ι-expansion from $E(M)$ to $E_f(N)$. Moreover, for any Σ-morphisms $f: M \to N1$ and $h: N1 \to N2$, h has a unique ι-expansion, denoted by $E(h)$, from $E_f(N1)$ to $E_{f;h}(N2)$.

(c) For models "containing" $E(M)$ the ι-reduct functor preserves quotients, i.e., for any factorization epimorphism $e \in \mathbf{E}_{\Sigma(M)}$ with domain in $\mathrm{Ext}(E(M))$ $e|_\iota \in \mathbf{E}_\Sigma$ as well.

If this is the case we call $\Sigma(M)$ the *diagram signature* for M with the signature inclusion ι and we call $E(M)$ the *diagram expansion* of M.

By the basis of an abstract algebraic institution we mean the triple $\mathbf{B}_{\mathrm{INS}} = \langle \mathrm{Sign}, \mathrm{Mod}, \{\langle \mathbf{E}_\Sigma, \mathbf{M}_\Sigma \rangle\}_{\Sigma \in |\mathrm{Sign}|} \rangle$.

Discussion. The above requirements may seem to be rather restrictive. We feel, however, that they are quite natural and, moreover, they are satisfied in a number of standard institutions (see below). This was discussed more extensively in [Tar 85], so only the very basic intuition behind these requirements is given here.

(1) is a quite standard requirement which appears whenever the institution is supposed to provide some tools for "putting things together" (cf., e.g., [BG 80, EWT 83, ST 84]). The existence of factorization systems together with (2) and (3)

provide an institution with notions of submodel and quotient model which are necessary to formulate our results. (4) just says that we want to define and consider models only up to isomorphism. (5) guarantees that abstract algebraic institutions have a certain minimal specification power. In the standard algebraic case it reduces to the requirement of expressibility of ground equations. Finally, (6) guarantees that in abstract algebraic institutions we can use the method of diagrams (in the sense of, e.g., [CK 73]). This corresponds to the requirement in [MM 84] than an algebraic specification language must be "rich enough." Notice that (b) and (c) in (6) mean exactly that the ι-reduct functor is an ismorphism of the comma categories (cf. [HS 73, Definition 4.18]) $\langle E(M), \mathrm{Mod}(\Sigma(M))\rangle$ and $\langle M, \mathrm{Mod}(\Sigma)\rangle$ with the factorization systems inherited from the categories of models.

Let INS be an abstract algebraic institution.

By a *specification* in INS we mean a pair $\langle \Sigma, \Phi\rangle$, where Σ is a signature and Φ is a set of Σ-sentences. Note, however, that when dealing with a specification we can use not only the properties explicitly stated in Φ; we can also use all their logical consequences, i.e., sentences that hold in any model of the specification. By a *theory* we mean a specification in which the set of sentences already contains all its logical consequences. A bit more formally: for any signature Σ and $K \subseteq |\mathrm{Mod}(\Sigma)|$ let $\mathrm{Th}(K)$ denote the set of all Σ-sentences that hold in K, i.e., $\mathrm{Th}(K) = \{\varphi \in |\mathrm{Sen}(\Sigma)| \mid K \models \varphi\}$. A theory is a specification $\langle \Sigma, \Phi\rangle$ where $\Phi = \mathrm{Th}(\mathrm{Mod}(\Phi))$. If $T = \langle \Sigma, \Phi\rangle$ is a theory, we use the notation $\mathrm{Mod}(T)$ for the collection of all T-models, i.e., all models that satisfy Φ. For any signature Σ, by the *empty* Σ-theory we mean the theory consisting of all trivial Σ-sentences, i.e., the theory $\langle \Sigma, \mathrm{Th}(|\mathrm{Mod}(\Sigma)|)\rangle$.

For any two theories $T1 = \langle \Sigma 1, \Phi 1\rangle$ and $T2 = \langle \Sigma 2, \Phi 2\rangle$, by a *theory morphism* from $T1$ to $T2$, $\sigma: T1 \to T2$, we mean a signature morphism $\sigma: \Sigma 1 \to \Sigma 2$ such that $\sigma(\varphi) \in \Phi 2$ for any $\varphi \in \Phi 1$.

Note that if $\sigma: T1 \to T2$ is a theory morphism then the σ-reduct functor $\_|_\sigma$ translates $T2$-models to $T1$-models, $\_|_\sigma: \mathrm{Mod}(T2) \to \mathrm{Mod}(T1)$.

Three Examples

Let $\sigma: \Sigma \to \Sigma'$ be an algebraic signature morphism. For any total Σ'-algebra A we define its *σ-reduct* $A|_\sigma \in |\mathrm{TALg}(\Sigma)|$ by $|A|_\sigma|_s = |A|_{\sigma(s)}$ for $s \in S$ and $f_{A|_\sigma} = \sigma(f)_A$ for $f: w \to s$ in Σ. Similarly, for any Σ'-homomorphisms $h: A \to B$ the σ-reduct of h is the Σ-homomorphism $h|_\sigma: A|_\sigma \to B|_\sigma$ defined by $(h|_\sigma)_s = h_{\sigma(s)}$ for $s \in S$. The mappings $A \mapsto A|_\sigma$, $h \mapsto h|_\sigma$ form a functor from $\mathrm{TAlg}(\Sigma')$ to $\mathrm{TAlg}(\Sigma)$; σ-reduct functors for partial and continuous algebras are defined in a similar way. These definitions determine functors $\mathrm{TAlg}: \mathrm{AlgSig}^{\mathrm{op}} \to \mathrm{Cat}$, $\mathrm{PAlg}: \mathrm{AlgSig}^{\mathrm{op}} \to \mathrm{Cat}$, and $\mathrm{CAlg}: \mathrm{AlgSig}^{\mathrm{op}} \to \mathrm{Cat}$.

Now consider

$$\mathbf{B}_{\mathrm{TAlg}} = \langle \mathrm{AlgSig}, \mathrm{TAlg}, \langle \mathbf{TE}_\Sigma, \mathbf{TM}_\Sigma\rangle_{\Sigma \in |\mathrm{AlgSig}|}\rangle,$$

$$\mathbf{B}_{\mathrm{PAlg}} = \langle \mathrm{AlgSig}, \mathrm{PAlg}, \langle \mathbf{PE}_\Sigma, \mathbf{PM}_\Sigma\rangle_{\Sigma \in |\mathrm{AlgSig}|}\rangle,$$

$$\mathbf{B}_{\mathrm{CAlg}} = \langle \mathrm{AlgSig}, \mathrm{CAlg}, \{\mathbf{CE}_\Sigma, \mathbf{CM}_\Sigma\rangle_{\Sigma \in |\mathrm{AlgSig}|}\rangle.$$

$\mathbf{B}_{TAlg}$, $\mathbf{B}_{PAlg}$, and $\mathbf{B}_{CAlg}$ are bases of abstract algebraic institutions. To prove this, it is sufficient to verify requirements (1), (2), (3), and (6) of Definition 3.2 (see the beginning of the next section, where we show how a basis satisfying these requirements may be extended to an abstract algebraic institution). The cocompleteness of AlgSig is stated explicitly in [GB 84a, Proposition 5] and the rest of (1) was proved in [BW 82] for total algebras, but the proof essentially carries over to the two other cases. We mentioned (2) already in the previous section, and (3) may be checked directly. Finally, for (6) note that in each of these three cases, a diagram signature for a total, partial or continuous Σ-algebra A may be defined as $\Sigma(|A|)$, the extension of Σ by a constant of the appropriate sort for each element of the carrier of A.

Abstract algebraic institutions with these bases will be called, respectively, *standard algebraic institutions,* (abstract algebraic) *institutions of partial algebras,* and (abstract algebraic) *institutions of continuous algebras* (examples in the next section).

4. Ground Sentences

Let $\mathbf{B} = \langle \text{Sign}, \text{Mod}, \{\langle \mathbf{E}_\Sigma, \mathbf{M}_\Sigma \rangle\}_{\Sigma \in |\text{Sign}|} \rangle$ be a basis of an abstract algebraic institution.

By the institution of *ground positive elementary sentences* in **B**, GPES(**B**), we mean any abstract algebraic institution with the basis **B** such that all and only ground quasi-varieties are definable in GPES(**B**).

Note that Lemma 2.1 implies that this is equivalent to the following two requirements:

(1) For any signature Σ and ground positive elementary Σ-sentence $\delta \in |\text{Sen}_{\text{GPES}(B)}(\Sigma)|$, δ is preserved under submodels, products and extensions of Σ-models (i.e., the class of models of δ is closed under submodels, products and extensions).

(2) For any signature Σ and reachable Σ-model A there is a set of ground positive elementary Σ-sentences $\Delta \subseteq |\text{Sen}_{\text{GPES}(B)}(\Sigma)|$ such that $A \in \text{Mod}(\Delta) \subseteq \text{Ext}(A)$.

An institution of ground positive elementary sentences exists for any basis of abstract algebraic institutions, although it may be impossible to construct it in a nice "syntactic" form. Perhaps the most "non-syntactic" way to define it is to accept for any signature Σ as ground positive elementary Σ-sentences just all ground varieties of Σ-models with membership as the satisfaction relation. The translation of such "sentences" may be defined as follows: for any signature morphism $\sigma: \Sigma \to \Sigma'$ and ground variety V of Σ-models define $\sigma(V) = V|_\sigma^{-1} = \{M \in |\text{Mod}(\Sigma')| \mid K|_\sigma \in V\}$. To see that this is well-defined, observe that under our assumptions about the reduct functors (Definition 3.2) $V|_\sigma^{-1}$ is closed under products, submodels and extensions whenever V is, which is the case when V is a ground variety, and so

$V|_{\sigma}^{-1}$ is a ground variety by Lemma 2.1. To prove that the above construction yields in fact an institution of ground positive elementary sentences, we have to check the satisfaction and abstractness conditions from Definition 3.2 and the requirements (1) and (2) above, which is trivial.

Let $\Sigma \in |\mathbf{Sign}|$ be a signature.

By an *infinitary conditional ground positive Σ-sentence* we mean a pair $\langle \Delta 1, \Delta 2 \rangle$ of sets of ground positive elementary Σ-sentences, written in the form $\Delta 1 \Rightarrow \Delta 2$.

For any Σ-model $M \in |\mathbf{Mod}(\Sigma)|$, we say that M satisfies an infinitary conditional ground positive Σ-sentence ρ, written $M \models \rho$, where $\rho \equiv \Delta 1 \Rightarrow \Delta 2$, if $M \models \Delta 2$ or $M \not\models \Delta 1$.

The above notions combine in the obvious way to form the institution ICGPS(**B**) of infinitary conditional ground positive sentences in **B**, with the translation of sentences induced by the translation of ground positive elementary sentences: for any signature morphism $\sigma: \Sigma \to \Sigma'$ and infinitary conditional ground positive Σ-sentence $\Delta 1 \Rightarrow \Delta 2$, $\sigma(\Delta 1 \Rightarrow \Delta 2) = \sigma(\Delta 1) \Rightarrow \sigma(\Delta 2)$, where for any set Δ of ground elementary Σ-sentences $\sigma(\Delta) = \{\sigma(\delta) | \delta \in \Delta\}$ is the image of Δ under the σ-translation of sentences in GPES(**B**).

LEMMA 4.1. *Any class of models definable in* ICGPS(**B**) *is a strict quasi-variety.*

Proof. Since any intersection of strict quasi-varieties is a strict quasi-variety, by Theorem 2.1 it is sufficient to prove that for any infinitary conditional ground positive sentence the class of its models is definable by a family of cones of the form $\langle M, \{e\} \rangle$ where e is a factorization epimorphism.

Let $\Sigma \in |\mathbf{Sign}|$ and let $\Delta 1$ and $\Delta 2$ be sets of ground positive elementary Σ-sentences. Then, let $M1$ and $M2$ be reachable Σ-models such that $\mathbf{Mod}(\Delta 1) = \mathrm{Ext}(M1)$ and $\mathbf{Mod}(\Delta 1 \cup \Delta 2) = \mathrm{Ext}(M2)$.

$M2 \models \Delta 1$ and so there is a Σ-morphism $e: M1 \to M2$. Moreover, from the properties of reachable objects it follows that $e \in \mathbf{E}_{\Sigma}$. We prove that a Σ-model A is injective w.r.t. e if and only if $A \models \Delta 1 \Rightarrow \Delta 2$.

Assume that $A \models \Delta 1 \Rightarrow \Delta 2$ and let $f: M1 \to A$. Thus, $A \models \Delta 1$, hence also $A \models \Delta 2$ and so there is $g: M2 \to A$. Now, since $M1$ is reachable, $e; g = f$, which proves that A is injective w.r.t. e.

Then, let A be injective w.r.t. e and assume that $A \models \Delta 1$. Thus, there is a Σ-morphism from $M1$ to A. By injectivity of A w.r.t. e, there is a Σ-morphism from $M2$ to A as well and so $A \models \Delta 2$, which proves that $A \models \Delta 1 \Rightarrow \Delta 2$. ▌

To deal with quasi-varieties (i.e., to drop the assumption of strictness) we extend the institution ICGPS(**B**) to the institution ICGS(**B**) of *infinitary conditional ground sentences.* For any signature $\Sigma \in |\mathbf{Sign}|$, these are either infinitary conditional ground positive Σ-sentences as defined above or sentences written in the form $\Delta \Rightarrow \mathrm{false}$, where $\Delta \subseteq |\mathbf{Sen}_{\mathrm{GPES}}(\Sigma)|$. A Σ-model $M \in |\mathbf{Mod}(\Sigma)|$ satisfies $\Delta \Rightarrow \mathrm{false}$, $M \models \Delta \Rightarrow \mathrm{false}$, if $M \not\models \Delta$.

LEMMA 4.2. *Any class of models definable in* ICGS(**B**) *is a quasi-variety.*

Proof. Let Δ be a set of ground positive elementary Σ-sentences and let M be a reachable Σ-model such that $\text{Mod}(\Delta) = \text{Ext}(M)$.

For any $A \in |\text{Mod}(\Sigma)|$, there is no Σ-morphism from M to A if and only if $A \not\models \Delta$. Thus, $A \models \Delta \Rightarrow \text{false}$ iff A is injective w.r.t. $\langle M, \phi \rangle$, which, by Lemma 4.1 and Theorem 2.1, completes the proof. ∎

Three Examples

In all three of our examples of bases of abstract algebraic institutions, ground elementary positive sentences may be defined in the expected form.

Let $\Sigma \in |\text{AlgSig}|$ be an algebraic signature.

By a ground (finitary) Σ-equation we mean any pair $\langle t1, t2 \rangle$, written in the form $t1 = t2$, of ground Σ-terms of the same sort. A total Σ-algebra A satisfies a ground Σ-equation $t1 = t2$ if $t1^A = t2^A$. A partial Σ-algebra A satisfies a ground Σ-equation $t1 = t2$ if $t1^A$ and $t2^A$ are defined and equal.

By a ground (infinitary) Σ-inequality we mean any pair $\langle t1, t2 \rangle$, written in the form $t1 \sqsubseteq t2$, of ground infinitary Σ-terms of the same sort. A continuous Σ-algebra A satisfies a ground Σ-inequality $t1 \sqsubseteq t2$ if $t1^A$ and $t2^A$ are defined and $t1^A \leqslant^A t2^A$.

It is easy to check that ground Σ-equations are preserved under subalgebras, products and extensions of total and partial algebras and ground Σ-inequalities are preserved under continuous subalgebras, products and extensions of continuous algebras.

Moreover, for any partial (resp. continuous) reachable Σ-algebra A let $\Delta^+(A)$ denote the set of all ground Σ-equations (resp. ground Σ-inequalities) which hold in A. Then, for any partial (resp. continuous) Σ-algebra B, if $B \models \Delta^+(A)$ then there is a (resp. continuous) Σ-homomorphism from A to B (which maps any element $a \in |A|$ to t^B, where t is a ground Σ-term such that $t^A = a$—we leave details of the proof as an exercise.

The above proves that the standard algebraic institution of ground equations (resp. the institution of ground equations in partial algebras, the institution of ground inequalities in continuous algebras) is an institution of ground positive elementary sentences in $\mathbf{B}_{\text{TAlg}}$ (resp. $\mathbf{B}_{\text{PAlg}}$, $\mathbf{B}_{\text{CAlg}}$). (The translation of equations and inequalities along algebraic signature morphisms is induced by the usual translation of ground finitary terms.)

5. Open Formulae in an Arbitrary Institution

In logic, formulae may contain free variables (such formulae are called *open*). To interpret an open formula, we have to provide not only an interpretation for the symbols of the underlying signature (a model) but also an interpretation for the free variables (a valuation of variables into the model). This provides a natural way to deal with quantifiers. Thus, we need institutions in which sentences may contain free variables. Fortunately we do not have to change the notion of institution—we can provide open formulae in the present framework (this idea, first outlined in [ST 84], was influenced by the treatment of variables in [Bar 74]). Note that we use here the term "formula" rather than "sentence," which is reserved for the sentences of the underlying institution.

Let $\Sigma = \langle S, \Omega \rangle$ be an algebraic signature. For any (S-sorted) set X, define $\Sigma(X)$ to be the extension of Σ by the elements of X as new constants of the appropriate sorts.

Now, any sentence over $\Sigma(X)$ may be viewed as an open formula over Σ with free variables X. Given a Σ-algebra A, to determine whether an open Σ-formula with variables X holds in A we have first to fix a valuation of variables X into $|A|$. Such a valuation corresponds exactly to an expansion of A to a $\Sigma(X)$-algebra, which additionally contains an interpretation of the constants X.

Given a translation of sentences along an algebraic signature morphism $\sigma: \Sigma \to \Sigma'$ we can extend it to a translation of open formulae. Roughly, we translate an open Σ-formula with variables X, which is a $\Sigma(X)$-sentence, to the corresponding $\Sigma'(X')$-sentence, which is an open Σ'-formula with variables X'. Here X' results from X by an appropriate renaming of sorts determined by σ (we also have to avoid unintended "clashes" of variables and operation symbols).

The above ideas generalize to an arbitrary institution INS. (INS need not be abstract algebraic, but we have to assume that the category of signatures is finitely cocomplete and that the model functor preserves finite colimits—requirement (1) in Definition 3.2.)

Let Σ be a signature.

Any pair $\langle \varphi, \theta \rangle$, where $\theta: \Sigma \to \Sigma'$ is a signature morphism and $\varphi \in |\mathrm{Sen}(\Sigma')|$, is an open Σ-formula with variables "$\Sigma' - \theta(\Sigma)$." (Note the quotation marks—since $\Sigma' - \theta(\Sigma)$ makes no sense in an arbitrary institution, it is only meaningful as an aid to our intuition.) If M is a Σ-model, $M \in |\mathrm{Mod}(\Sigma)|$, then a valuation of variables "$\Sigma' - \theta(\Sigma)$" into M is a Σ'-model $M' \in |\mathrm{Mod}(\Sigma')|$ which is a θ-expansion of M, i.e., $M'|_\theta = M$.

Note that in the standard logical framework there may be no valuation of a set of variables into a model containing an empty carrier. Similarly, here a valuation need not always exist (although there may be more reasons for that). For example, for an algebraic signature morphism $\theta: \Sigma \to \Sigma'$ which is not injective some total (partial, continuous) Σ-algebras have no θ-expansion.

If $\sigma: \Sigma \to \Sigma1$ is a signature morphism and $\langle \varphi, \theta \rangle$ is an open Σ-formula then we define the translation of $\langle \varphi, \theta \rangle$ along σ as $\sigma(\langle \varphi, \theta \rangle) = \langle \sigma'(\varphi), \theta' \rangle$, where

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\sigma'} & \Sigma1' \\ {\scriptstyle\theta}\uparrow & & \uparrow{\scriptstyle\theta'} \\ \Sigma & \xrightarrow[\sigma]{} & \Sigma1 \end{array}$$

is a pushout in the category of signatures.

Remark 5.1. There is a rather subtle problem we have to point out here: pushouts are defined only up to isomorphism, so strictly speaking the translation of open formulae is not well-defined. Fortunately, from the definition of an institution one may easily prove that whenever $\iota: \Sigma 1' \to \Sigma 1''$ is an isomorphism in Sign with inverse ι^{-1} then $\mathrm{Sen}(\iota): \mathrm{Sen}(\Sigma 1') \to \mathrm{Sen}(\Sigma 1'')$ is a bijection, $\mathrm{Mod}(\iota): \mathrm{Mod}(\Sigma 1'') \to \mathrm{Mod}(\Sigma 1')$ is an isomorphism in Cat and moreover for any $\Sigma 1'$-sentence $\varphi \in |\mathrm{Sen}(\Sigma 1')|$ and any $\Sigma 1'$-model $M1' \in |\mathrm{Mod}(\Sigma 1')|$

$$M1' \models \varphi \qquad \text{iff} \quad M1'|_{\iota^{-1}} \models \iota(\varphi)$$

This shows that (at least for semantic analysis) we can pick out an arbitrary pushout to define the translation of open formulae and so we may safely accept the above definition of translation.

Note that sometimes we want to restrict the class of signature morphisms which may be used to construct open formulae. In fact, above we used only algebraic signature inclusions $\iota: \Sigma \to \Sigma'$, where the only new symbols in Σ' were constants. To guarantee that the translation of open formulae is defined under such a restriction, we consider only restrictions to a collection **I** of signature morphisms which is closed under pushing out along arbitrary signature morphisms, i.e., for any signature morphisms $\sigma: \Sigma \to \Sigma 1$ if $\theta: \Sigma \to \Sigma'$, $\theta \in \mathbf{I}$ then there is a pushout in Sign

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\sigma'} & \Sigma 1' \\ {\scriptstyle\theta}\uparrow & & \uparrow{\scriptstyle\theta'} \\ \Sigma & \xrightarrow[\sigma]{} & \Sigma 1 \end{array}$$

such that $\theta' \in \mathbf{I}$.

Examples of such collections **I** in AlgSig include: the collection of all algebraic signature inclusions, the restriction of this to inclusions $\theta: \Sigma \to \Sigma'$ such that Σ' contains no new sorts, the further restriction of this by the requirement that Σ' contains new constants only (as above), the collection of all algebraic signature morphisms which are onto w.r.t. sorts, the collection of all identities and the collections of all morphisms. Note that most of the above permit variables denoting operations or even sorts.

In the rest of this section we show how to universally close the open formulae introduced above (the construction is based on the notion of a syntactic operation in [Bar 74]).

Let **I** be a collection of signature morphisms which is closed under pushing out along arbitrary morphisms in Sign. Let Σ be a signature and let $\langle \varphi, \theta \rangle$ be an open Σ-formula such that $\theta: \Sigma \to \Sigma'$ and $\theta \in \mathbf{I}$. Consider the universal closure of $\langle \varphi, \theta \rangle$, written $\forall \Sigma' - \theta(\Sigma).\varphi$, as a new Σ-sentence. The satisfaction relation and the translation of sentences $\forall \Sigma' - \theta(\Sigma).\varphi$ along a signature morphism are defined in the expected way:

— A Σ-model satisfies $\forall\Sigma' - \theta(\Sigma).\varphi$ if each of its θ-expansions satisfies φ, i.e., for any $M \in |\mathrm{Mod}(\Sigma)|$ $M \models \forall\Sigma' - \theta(\Sigma).\varphi$ iff for any $M' \in |\mathrm{Mod}(\Sigma')|$ such that $M'|_\theta = M$, $M' \models \varphi$.

— For any signature morphisms $\sigma: \Sigma \to \Sigma 1$, $\sigma(\forall\Sigma' - \theta(\Sigma).\varphi) = \forall\Sigma 1' - \theta'(\Sigma 1).\sigma'(\varphi)$, where

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\sigma'} & \Sigma 1' \\ {\scriptstyle\theta}\uparrow & & \uparrow{\scriptstyle\theta'} \\ \Sigma & \xrightarrow[\sigma]{} & \Sigma 1 \end{array}$$

is a pushout in Sign (with $\theta' \in \mathbf{I}$).

THEOREM 5.1. *For any signature morphism $\sigma: \Sigma \to \Sigma 1$, open Σ-formula $\langle \varphi, \theta \rangle$ and $\Sigma 1$-model $M1 \in |\mathrm{Mod}(\Sigma 1)|$*

$$M1|_\sigma \models \forall\Sigma' - \theta(\Sigma).\varphi \qquad \text{iff} \quad M1 \models \sigma(\forall\Sigma' - \theta(\Sigma).\varphi)$$

Proof. $\Rightarrow$ Assume that $M1|_\sigma \models \forall\Sigma' - \theta(\Sigma).\varphi$ and let $M1'$ be a θ'-expansion of $M1$. Let M' denote $M1'|_{\sigma'}$. Obviously, $M'|_\theta = M1|_\sigma$. Thus, by the assumption, $M' \models \varphi$. Hence, by the satisfaction condition for the underlying institution, $M1' \models \sigma'(\varphi)$, which proves that $M1 \models \forall\Sigma 1' - \theta'(\Sigma 1).\sigma'(\varphi)$.

$\Leftarrow$ Assume that $M1 \models \forall\Sigma 1' - \theta'(\Sigma 1).\sigma'(\varphi)$ and let M' be a θ-expansion of $M1|_\sigma$, i.e., $M'|_\theta = M1|_\sigma$. Now, from the construction of pullbacks in Cat it follows that there is a $\Sigma 1'$-model $M1' \in |\mathrm{Mod}(\Sigma 1')|$ such that $M1'|_{\sigma'} = M'$ and $M1'|_{\theta'} = M1$. Then, by the assumption $M1' \models \sigma'(\varphi)$. Hence, by the satisfaction condition for the underlying institution, $M' \models \varphi$, which proves that $M1|_\sigma \models \forall\Sigma' - \theta(\Sigma).\varphi$. ∎

Note that in the above we have extended our underlying institution INS. Formally, Theorem 5.1 guarantees that the following extension of INS by universal closure w.r.t. $\mathbf{I}$, $\mathrm{INS}^\forall(\mathbf{I})$, is an institution (modulo Remark 5.1.):

(i) $\mathrm{Sign}_{\mathrm{INS}^\forall(\mathbf{I})}$ is $\mathrm{Sign}_{\mathrm{INS}}$.

(ii) For any signature Σ, $\mathrm{Sen}_{\mathrm{INS}^\forall(\mathbf{I})}(\Sigma)$ is the collection of all universal closures $\forall\Sigma' \to \theta(\Sigma).\varphi$ of open Σ-formulae, where $\theta: \Sigma \to \Sigma'$, $\theta \in \mathbf{I}$; for a signature morphism $\sigma: \Sigma \to \Sigma 1$ $\mathrm{Sen}_{\mathrm{INS}^\forall(\mathbf{I})}(\sigma)$ is the translation of universally closed open formulae as defined above.

(iii) $\mathrm{Mod}_{\mathrm{INS}^\forall(\mathbf{I})}$ is $\mathrm{Mod}_{\mathrm{INS}}$.

(iv) The satisfaction relation in $\mathrm{INS}^\forall(\mathbf{I})$ is determined by the notion of satisfaction for universally closed open formulae as defined above.

Obviously, other quantifiers (there exists, there exist infinitely many, there exists a unique, for almost all...) may be introduced to institutions in the same manner as we have just introduced universal quantifiers. It is also worth mentioning that one may similarly introduce logical connectives (cf. [Bar 74]); thus our construction of the institution of infinitary conditional ground sentences out of ground positive elementary sentences may be easily generalized. Note that by iterating this idea we can, for example, derive the institution of first-order logic from the institution of ground atomic formulae.

6. A Characterization of Quasi-varieties

Having defined the institution of infinitary conditional ground sentences and the notion of universal quantification of formulae of an arbitrary institution, the obvious possibility to get the institution in which all and only strict quasi-varieties are definable is to universally quantify the infinitary conditional ground positive sentences, i.e., to consider an institution of the form $\mathrm{ICGPS}(\mathbf{B})^{\forall}(\mathbf{I})$. The only problem is the characterization of the class $\mathbf{I}$ of signature morphisms which we allow to introduce variables.

Obviously, as in the previous section we have to require that $\mathbf{I}$ is closed under pushing out along arbitrary signature morphisms. Moreover, we have to assume that $\mathbf{I}$ contains enough signature morphisms to admit the method of diagrams. Finally, we need some additional restriction to guarantee that the definable classes are in fact strict quasi-varieties.

Thus, throughout the rest of this section let $\mathbf{I}$ be a class of morphisms in Sign such that

(1) $\mathbf{I}$ is closed under pushing out along arbitrary signature morphisms in Sign.

(2) $\mathbf{I}$ contains all identities and *admits diagrams*, i.e., for any signature Σ and Σ-model M there is a diagram signature $\Sigma(M)$ for M with a signature inclusion $\iota: \Sigma \to \Sigma(M)$ such that $\iota \in \mathbf{I}$.

(3) The reduct functors corresponding to signature morphisms in $\mathbf{I}$ *locally create submodels and products*, i.e., for any $\theta \in \mathbf{I}$, $\theta: \Sigma \to \Sigma'$, if A is a Σ-model then any θ-expansion of a submodel of A is a submodel of a θ-expansion of A; if $\langle A_\beta \rangle_{\beta<\alpha}$, $\alpha \geqslant 0$, is a family of Σ-models then any θ-expansion of a product of $\langle A_\beta \rangle_{\beta<\alpha}$ is a product of a family of θ-expansions of $\langle A_\beta \rangle_{\beta<\alpha}$.

Note that under the above assumption about $\mathbf{I}$, for any abstract algebraic institution INS, $\mathrm{INS}^{\forall}(\mathbf{I})$ is abstract algebraic as well (in fact, we need only assumptions (1) and (2) here).

Now, by the institution of *infinitary conditional positive sentences* in $\mathbf{B}$ we mean the institution $\mathrm{ICPS}(\mathbf{B}) = \mathrm{ICGPS}(\mathbf{B})^{\forall}(\mathbf{I})$. We call the classes of models definable in $\mathrm{ICPS}(\mathbf{B})$ *strictly implicational.*

Theorem 6.1. *A class of models is strictly implicational if and only if it is a strict quasi-variety.*

Proof. $\Rightarrow$ Let $\theta: \Sigma \to \Sigma'$, $\theta \in \mathbf{I}$ be a signature morphisms and let ρ be an infinitary conditional ground positive Σ'-sentence. We prove that the class of models of Σ-sentence $\forall \Sigma' - \theta(\Sigma).\rho$ is closed under submodels and products.

Let A be a Σ-model, $A \models \forall \Sigma' - \theta(\Sigma).\rho$, and let B be a submodel of A. Let B' be a θ-expansion of B. Since $\theta \in \mathbf{I}$, there is a θ-expansion of A, say A', such that B' is a submodel of A'. Now, by definition, $A' \models \rho$, and so by Lemma 4.1 $B' \models \rho$, which proves that $B \models \forall \Sigma' - \theta(\Sigma).\rho$.

Let A_β for $\beta < \alpha$, $\alpha \geqslant 0$, be a Σ-model, $A_\beta \models \forall \Sigma' - \theta(\Sigma).\rho$, and let B be a product of $\langle A_\beta \rangle_{\beta < \alpha}$. Let B' be a θ-expansion of B. Since $\theta \in \mathbf{I}$, for $\beta < \alpha$ there is a θ-expansion of A_β, say A'_β, such that B' is a product of $\langle A'_\beta \rangle_{\beta < \alpha}$. Now, by definition, for $\beta < \alpha$ $A'_\beta \models \rho$, and so by Lemma 4.1 $B' \models \rho$, which proves that $B \models \forall \Sigma' - \theta(\Sigma).\rho$.

Since the intersection of strict quasi-varieties is a strict quasi-variety, this completes the prove of the "only if" part.

$\Leftarrow$ By Theorem 2.1 it is enough to prove that for any factorization epimorphism e, the class of models injective w.r.t. e is definable by infinitary conditional positive sentences.

Let Σ be a signature and $e: A \to B$ be a Σ-morphism, $e \in \mathbf{E}_\Sigma$. Let $\Sigma(A)$ be a diagram signature for A with the signature inclusion $\iota: \Sigma \to \Sigma(A)$, $\iota \in \mathbf{I}$, and let $E(A)$ be a diagram expansion of A. Recall that $E_e(B)$ is a ι-expansion of B such that there is a $\Sigma(A)$-morphism $E(e): E(A) \to E_e(B)$ which is a ι-expansion of e (Definition 3.2). Note that under our assumptions about diagram signatures $e \in \mathbf{E}_\Sigma$ implies $E(e) \in \mathbf{E}_{\Sigma(A)}$, and so $E_e(B)$ is reachable. Let $\Delta 1$ and $\Delta 2$ be sets of ground positive elementary $\Sigma(A)$-sentences such that $\mathrm{Mod}(\Delta 1) = \mathrm{Exp}(E(A))$ and $\mathrm{Mod}(\Delta 2) = \mathrm{Exp}(E_e(B))$.

We prove that for any Σ-model M, M is injective w.r.t. e if and only if $M \models \forall \Sigma(A) - \iota(\Sigma).(\Delta 1 \Rightarrow \Delta 2)$.

Assume that M is injective w.r.t. e and let M' be a ι-expansion of M such that $M' \models \Delta 1$. The definition of $\Delta 1$ directly implies that there is a $\Sigma(A)$-morphism $f: E(A) \to M'$. By injectivity of M, there is a Σ-morphism $g: B \to M$ such that $f|_\iota = e; g$. Now, by the definition of a diagram signature there is a $\Sigma(A)$-morphism from $E_e(B)$ to M', which proves that $M' \models \Delta 1 \Rightarrow \Delta 2$, and so that $M \models \forall \Sigma(A) - \iota(\Sigma).(\Delta 1 \Rightarrow \Delta 2)$.

Now, assume that $M \models \forall \Sigma(A) - \iota(\Sigma).(\Delta 1 \Rightarrow \Delta 2)$ and let $f: A \to M$. By the definition of a diagram signature we have $E(f): E(A) \to E_f(M)$ for some ι-expansions $E(f)$ of f and $E_f(M)$ of M. Thus, $E_f(M) \models \Delta 1$ and so $E_f(M) \models \Delta 2$ as well. Hence, there is a $\Sigma(A)$-morphism $g: E_e(B) \to E_f(M)$. Since $E(A)$ is reachable, $E(f) = E(e); g$ and so $f = e; g|_\iota$, which proves that M is injective w.r.t. e. ∎

A similar result holds for quasi-varieties: by the institution of *infinitary conditional sentences* in $\mathbf{B}$ we mean the institution $\mathrm{ICS}(\mathbf{B}) = \mathrm{ICGS}(\mathbf{B})^\forall(\mathbf{I})$. We call the classes of models definable in $\mathrm{ICS}(\mathbf{B})$ *implicational.*

THEOREM 6.2. *A class of models is implicational if and only if it is a quasi-variety.*

Proof. $\Rightarrow$ By Theorem 6.1 it is enough to prove that for any $\theta: \Sigma \to \Sigma'$, $\theta \in \mathbf{I}$, and set Δ of ground positive elementary Σ'-sentences, the class of models of Σ-sentence $\forall\Sigma' - \theta(\Sigma).(\Delta \Rightarrow \text{false})$ is closed under submodels and nonempty products, which follows from Lemma 4.2 and assumptions about $\mathbf{I}$ in the same way as the analogous result in the proof of Theorem 6.1.

$\Leftarrow$ By Theorems 2.1 and 6.1, it is sufficient to prove that for any cone γ of the form $\langle A, \phi \rangle$, the class of models injective w.r.t. γ is definable in ICS(**B**).

Let Σ be a signature and A be Σ-model, Let $\Sigma(A)$ be a diagram signature for A with the signature inclusion $\iota: \Sigma \to \Sigma(A)$, $\iota \in \mathbf{I}$. Let $E(A)$ be a diagram expansion of A and let Δ be a set of ground positive elementary $\Sigma(A)$-sentences such that $\text{Mod}(\Delta) = \text{Exp}(E(A))$ (recall that $E(A)$ is reachable). We prove that for any Σ-model B there is no Σ-morphism from A to B if and only if $B \models \forall\Sigma(A) - \iota(\Sigma).(\Delta \Rightarrow \text{false})$: by definition of a diagram signature, there is no Σ-morphism from A to B iff there is no ι-expansion B' of B with a $\Sigma(A)$-morphism from $E(A)$ to B', i.e., iff there is no ι-expansion B' of B such that $B' \models \Delta$, i.e., iff every ι-expansion of B satisfies $\Delta \Rightarrow \text{false}$, i.e., iff $B \models \forall\Sigma(A) - \iota(\Sigma).(\Delta \Rightarrow \text{false})$. The above proves that the class of Σ-models injective w.r.t. the cone $\langle A, \phi \rangle$ is defined by the infinitary conditional Σ-sentences $\forall\Sigma(A) - \iota(\Sigma).(\Delta \Rightarrow \text{false})$ ∎

Note that the above characterization of quasi-varieties (but not that of strict quasi-varieties) remains true under slightly weaker assumptions about **I**: instead of (3) it is enough to require only that the reduct functors corresponding to signature morphisms in **I** locally create submodels and *nonempty* products.

Three Examples

Let $\mathbf{I}_{\text{Alg}}$ be a class of algebraic signature morphisms $\theta: \Sigma \to \Sigma'$ such that θ is an algebraic signature inclusion (in the usual set-theoretic sense) and, moreover, the only new symbols in Σ' are constants (i.e, no new sorts, no new non-constant operations).

$\mathbf{I}_{\text{Alg}}$ is closed under pushing out along arbitrary signature morphisms in AlgSig. Moreover, since for any algebraic signature Σ and total, partial or continuous Σ-algebra A a diagram signature for A may be given as the enrichment of Σ by a constant of the appropriate sort for each element of $|A|$, $\mathbf{I}_{\text{Alg}}$ admits diagrams. Finally, in each of our three cases the reduct functors corresponding to algebraic signature morphisms from $\mathbf{I}_{\text{Alg}}$ locally create submodels and nonempty products. This is trivially true for standard and continuous algebras. For partial algebras: if $\iota \in \mathbf{I}_{\text{Alg}}$, $\iota: \Sigma \to \Sigma(X)$ and A, B are partial Σ-algebras such that B is a submodel of A, then for any ι-expansion B' of B, B' is a submodel of a ι-expansion of A in which exactly the same constants from X are defined as in B', similar arguments prove that the ι-reduct functor locally creates nonempty products.

Thus, Theorem 6.2 yields a uniform characterization of quasi-varieties of total, partial and continuous algebras as implicational classes.

Moreover, for total and continuous algebras, for any $\iota \in \mathbf{I}_{\text{Alg}}$, the only (up to isomorphism) ι-expansion of a terminal model (over the appropriate signature) is a terminal model. Thus, for these two cases we can directly apply Theorem 6.1: strict quasi-varieties of total and continuous algebras are exactly strictly implicational classes.

Note that for total algebras, infinitary conditional positive sentences are exactly infinitary conditional equations, i.e., formulae of the form $\forall X.(\{t1_i = t2_i\}_{i \in I} \Rightarrow \{t1_j = t2_j\}_{j \in J})$, where $\{t1_i = t2_i\}_{i \in I}$ and $\{t1_j = t2_j\}_{j \in J}$ are sets of equations with variables X; a total algebra A satisfies the above infinitary conditional equation if for any valuation $v: X \to |A|$, $t1_j^A(v) = t2_j^A(v)$ for $j \in J$ provided that $t1_i^A(v) = t2_i^A(v)$ for $i \in I$.

Similarly, infinitary conditional sentences are exactly infinitary conditional equations (as above) or else infinitary conditional inequations, i.e., formulae of the form $\forall X.(\{t1_i = t2_i\}_{i \in I} \Rightarrow \text{false})$, where $\{t1_i = t2_i\}_{i \in I}$ is a set of equations with variables X; a total algebra A satisfies the above infinitary conditional inequation if for no valuation $v: X \to |A|$ $t1_i^A(v) = t2_i^A(v)$ for $i \in I$. Infinitary conditional sentences for continuous algebras may be described in an analogous way (using inequalities rather than equations). In both these cases the conditional sentences are as expected.

This is not quite the case for partial algebras, though. Here, infinitary conditional positive sentences are infinitary conditional equations *with partial valuations*, i.e., formulae of the form $\forall X.(\{t1_i = t2_i\}_{i \in I} \Rightarrow \{t1_j = t2_j\}_{j \in J})$, where $\{t1_i = t2_i\}_{i \in I}$ and $\{t1_j = t2_j\}_{j \in J}$ are sets of equations with variables X; a partial algebra A satisfies the above formula if for any *partial* valuation $v: X \to |A|$, $t1_j^A(v)$ and $t2_j^A(v)$ are defined and equal for $j \in J$ provided that $t1_i^A(v)$ and $t2_i^A(v)$ are defined and equal for $i \in I$. Similarly, infinitary conditional sentences are infinitary conditional equations or inequations with partial valuations.

The partiality of valuations we have to allow here follows from the fact that for any algebraic signature Σ, set X and the inclusion $\iota: \Sigma \to \Sigma(X)$, a ι-expansion of a partial Σ-algebra is a *partial* $\Sigma(X)$-algebra, and so values of variables/constants X may be undefined in it. For this very reason, for partial algebras the reduct functors corresponding to morphisms in $\mathbf{I}_{\mathrm{Alg}}$ do not locally create products of empty sets of partial algebras, and we cannot apply Theorem 6.1 directly.

To restrict consideration to total valuations of variables we can use the following trick (cf. [ST 84], see also [Rei 84] where a similar idea of including definedness requirements into a notion of signature is extensively analysed).

By *definedness formulae* we mean equations of the form $t = t$ (where t is a term) with the satisfaction relation, translation under algebraic signature morphisms, etc., defined as in the institution of ground equations in partial algebras. Let DefTh be the category of *definedness theories*; i.e., the category which has theories of definedness formulae in partial algebras as objects and theory morphisms between them as morphisms. Now, we can consider the basis of *abstract algebraic institutions of partial algebras with definedness axioms*, $\mathbf{B}_{\mathrm{PAlgDef}}$, which has DefTh as the category of signatures and for any $\langle \Sigma, \Delta \rangle \in |\mathrm{DefTh}|$ the category of partial Σ-algebras which satisfy Δ as the category of $\langle \Sigma, \Delta \rangle$-models (with reduct functors and factorization systems inherited from the usual category of partial Σ-algebras). Note that a class of partial algebras is a quasi-variety (resp. strict quasi-variety) "in the old sense" (i.e., without definedness axioms) if and only if it is a quasi-variety (resp. strict quasi-variety) "in the new sense" (i.e., with definedness axioms included in the notion of signature).

Note also that for a partial $\langle\Sigma, \Delta\rangle$-algebra A, a diagram signature for A (in $\mathbf{B}_{\text{PAlgDef}}$) may be given as $\langle\Sigma(|A|), \Delta\cup\{a=a \mid a\in|A|\}\rangle$, i.e., as the extension of $\langle\Sigma, \Delta\rangle$ by a constant of the appropriate sort for each element of $|A|$ and the requirement that these new constants must have defined values.

Let $\mathbf{I}_{\text{PAlgDef}}$ be the class of all signature inclusions of this form (i.e. signature inclusions of the form $\iota\colon \langle\Sigma, \Delta\rangle \to \langle\Sigma(X), \Delta\cup\{x=x \mid x\in X\}\rangle$ for some set X). It is easy to check that $\mathbf{I}_{\text{PAlgDef}}$ is closed under pushing out along arbitrary signature morphisms, contains all identities and admits diagrams, and moreover, the reduct functors corresponding to signature morphisms in $\mathbf{I}_{\text{PAlgDef}}$ locally create submodels and products. Thus, we can apply both our characterization theorems (Theorems 6.1 and 6.2).

Now, infinitary conditional positive sentences for partial algebras with definedness axioms are the usual infinitary conditional equations, i.e., formulae of the form $\forall X.(\{t1_i = t2_i\}_{i\in I} \Rightarrow \{t1_j = t2_j\}_{j\in J})$, where $\{t1_i = t2_i\}_{i\in I}$ and $\{t1_j = t2_j\}_{j\in J}$ are sets of equations with variables X; a partial algebra A satisfies the above infinitary conditional equation if for any (*total*) valuation $v\colon X\to|A|$, $t1_j^A(v)$ and $t2_j^A(v)$ are defined and equal for $j\in J$ provided that $t1_i^A(v)$ and $t2_i^A(v)$ are defined and equal for $i\in I$. Similarly, infinitary conditional sentences are infinitary conditional equations (as above) or inequations, i.e., formulae of the form $\forall X.(\{t1_i = t2_i\}_{i\in I} \Rightarrow \text{false})$, where $\{t1_i = t2_i\}_{i\in I}$ is a set of equations with variables X; a partial algebra A satisfies the above infinitary conditional inequation if for no (*total*) valuation $v\colon X\to|A|$ $t1_i^A(v)$ and $t2_i^A(v)$ are defined and equal for $i\in I$.

All this together proves that a class of partial algebras is a strict quasi-variety (resp. quasi-variety) if and only if it is definable by infinitary conditional equations (resp. infinitary conditional equations and inequations).

7. On the Existence of Free Models

Specifications given in most standard institutions often are *loose*, i.e., admit many nonisomorphic models. The most widely accepted mechanism for imposing additional constraints on the models admitted by a specification is to require initiality (cf. [GM 83] for an extensive treatment of this notion). In this approach, from among all possible models of a set of axioms we choose as an acceptable realization of the specified abstract data type only the unique (up to isomorphism) initial model. Moreover, quite often we want some parts of a data type to be interpreted loosely—and some others to be interpreted in a standard "initial" way given an interpretation of these "loose" parts. In other words we require that some part of a model must be a "free extension" of some other parts. This may be formally expressed using "initially restricting algebraic theories" [Rei 80] or, more generally, data constraints as introduced in [BG 80], cf. also [EWT 83, GB 84].

Unfortunately, initial (or, more generally, free) models need not always exist. Thus, if one wants to avoid proving their existence for each specification separately (see, e.g., [CMPPV 80] and [WPPDB 83] where some results supporting such an approach are given) he has to use an institution that guarantees the existence of initial models of any consistent set of axioms (or, more generally, the existence of free extensions of models along any theory morphism). It is well-known that, for example, equational logic has this property but first-order logic does not.

We say that an abstract algebraic institution *strongly admits initial semantics* if any nonempty class of models definable in it constains a reachable initial model.

We say that an abstract algebraic institution is *strongly liberal* if for any theory morphism $\sigma: T1 \to T2$, the σ-reduct functor $\_|_\sigma: \mathrm{Mod}(T2) \to \mathrm{Mod}(T1)$ has a left adjoint $\mathbf{F}_\sigma: \mathrm{Mod}(T1) \to \mathrm{Mod}(T2)$ and, moreover, for any $A \in \mathrm{Mod}(T1)$ $\mathbf{F}_\sigma(A)$ is σ-reachable, i.e., $\mathbf{F}_\sigma(A)$ has no proper submodel with an isomorphic σ-reduct.

THEOREM 7.1. [Tar 85]. (1) *An abstract algebraic institution strongly admits initial semantics if and only if every class definable in it is a quasi-variety.*

(2) *An abstract algebraic institution is strongly liberal if and only if every class definable in it is a strict quasi-variety.*

These results were obtained by a step-by-step generalization (cf. [Tar 84, 84a, 85]) of a characterization of *standard* algebraic institutions which strongly admit initial semantics essentially due to Mahr and Makowsky [MM 84].

COROLLARY 7.1. *Let* INS *be an abstract algebraic institution in which there exists a class of signature morphisms satisfying the assumptions of Section* 6.

(1) INS *strongly admits initial semantics if and only if every class of models definable in it is implicational.*

(2) INS *is strongly liberal if and only if every class of models definable in* INS *is strictly implicational.*

For any two abstract algebraic institutions INS1 and INS2 with the same basis, we say that INS1 is *reducible* to INS2 if any class of models definable in INS1 is also definable in INS2, or equivalently, if for any signature Σ and Σ-sentence φ in INS1, $\varphi \in |\mathrm{Sen}_{\mathrm{INS1}}(\Sigma)|$, φ is expressible in INS2, i.e., there is a set Φ of Σ-sentences in INS2, $\Phi \subseteq |\mathrm{Sen}_{\mathrm{INS2}}(\Sigma)|$, such that for any $A \in |\mathrm{Mod}(\Sigma)|$ $A \models_{\mathrm{INS1}} \varphi$ iff $A \models_{\mathrm{INS2}} \Phi$.

Corollary 7.1 states that for any basis of abstract algebraic institutions the most general (w.r.t. reducibility) among abstract algebraic institutions with this basis which strongly admit initial semantics (resp. are strongly liberal) is the institution of infinitary conditional sentences (resp. of infinitary conditional positive sentences).

Three Examples

Using the results states in the previous section, we can specialize Corollary 7.1 as follows:

The most general standard algebraic institution which strongly admits initial semantics is the institution of infinitary conditional equations and inequations; the most general standard algebraic institution which is strongly liberal is the institution of infinitary conditional equations.

The most general institution of partial algebras which strongly admits initial semantics is the institution of infinitary conditional equations and inequations in partial algebras; the most general institution of partial algebras which is strongly liberal is the institution of infinitary conditional equations in partial algebras.

The most general institution of continuous algebras which strongly admits initial semantics is the institution of infinitary conditional inequalities and in-inequalities; the most general institution of continuous algebras which is strongly liberal is the institution of infinitary conditional inequalities.

8. Summary of Results

We recalled the notion of institution introduced by Goguen and Burstall [GB 84] to formalize the concept of a logical system for writing specifications. We specialized their extremely general definition and dealt with *abstract algebraic institutions*, i.e., institutions equipped with factorization systems for the categories of models which satisfy a number of additional requirements. Namely, besides some purely technical conditions, we required that abstract algebraic institutions identify isomorphic models, allow the definition of any ground variety of models and guarantee the existence of a diagram expansion for any model (Sect. 3).

In this framework we introduced a notion of ground positive elementary sentence (Sect. 4), which together with a notion of open formula and universal quantification in an arbitrary institution (Sect. 5) led to a notion of infinitary conditional sentence (Sect. 6).

Using these tools we generalized a standard characterization of quasi-varieties and strict quasi-varieties: we proved (Sect. 6) that in abstract algebraic institutions these are, respectively, implicational and strictly implicational classes. This allowed us to present characterizations of the most general abstract algebraic institution which strongly admits initial semantics (resp. is strongly liberal; cf. [Tar 85]) in more standard syntactic terms.

Throughout the paper we specialized the accepted definitions and obtained results for three typical notions of model (over standard algebraic signatures): total, partial and continuous algebras. The general results of Section 8 yield characterizations of quasi-varieties and strict quasi-varieties of, respectively, total, partial and continuous algebras (Sect. 6). Of course, these characterizations are not new (see [Grä 79, Theorem 63.4] for the standard case; [AN 76], also [Tar 85] for the case of partial algebras; [ANR 84] for the case of continuous algebras). Here we obtained them, however, in a uniform way, as special cases of our Theorems 6.1 and 6.2, which may also be applied in the context of other notions of signature and of model, such as for example order-sorted [Goguen 78] and polymorphic

[Mil 78] signatures, error algebras [Goguen 77, GDLE 82] and algebras with subsorts [Gogolla 83].

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