INFORMATION AND CONTROL 30, 38–48 (1976)

Resolution of Composite Fuzzy Relation Equations

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This paper provides a methodology for solution of certain basic fuzzy relational equations, with fuzzy sets defined as mappings from sets into complete Brouwerian lattices, covering a large class of types of fuzzy sets.

1. INTRODUCTION

Zadeh (1965) characterizes a *fuzzy set* (*class*) A in a nonempty set X by a *membership* (*characteristic*) *function* f_A which associates with each point x in X a real number in the interval [0, 1], with the value of $f_A(x)$ representing the *grade of membership* of x in A.

Goguen (1967) generalizes the concept of fuzzy sets, defining them in terms of maps from a nonempty set to a suitable *partially ordered set* (*poset*), the most interesting results arising when posets are *lattices*; with *Boolean lattices*, Brown (1971) shows that Zadeh's basic results carry over to this case.

Using the ordinary symbol $\leqslant$ for the partial order relation on a poset L, let us now recall some useful definitions of lattice theory (Birkhoff, 1967).

DEFINITION 1. A *lattice* is a poset L any two of whose elements x and y have a *greatest lower bound* (g.l.b.) or *meet* denoted by $x \wedge y$, and a *least upper bound* (l.u.b.) or *join* denoted by $x \vee y$.

DEFINITION 2. A lattice L is *complete* when each of its subsets X has a l.u.b., denoted by sup X or $\bigvee X$, and a g.l.b., denoted by inf X or $\bigwedge X$, in L.

DEFINITION 3. By a *greatest element* of a poset L, we mean an element $b \in L$ such that $x \leqslant b$ for all $x \in L$, the *least element* of L *being* defined dually.

DEFINITION 4. A *Brouwerian lattice* is a lattice L in which, for any given elements a and b, the set of all $x \in L$ such that $a \wedge x \leqslant b$ contains a greatest element, denoted $a \alpha b$, the *relative pseudocomplement* of a in b.

In this work, fuzzy sets will be defined as mappings from sets into complete *Brouwerian* lattices covering a large class of types of fuzzy sets as indicated in the following section.

Certain *basic fuzzy relational equations* being next defined, we give a fundamental theorem for existence and determination of solutions.

We then relate the obtained results to similar results involving nonfuzzy relations of which the fundamental theorem is shown to be a generalization.

2. Fuzzy Sets and Fuzzy Relations

Definition 5. If L is a fixed complete Brouwerian lattice, and E is a nonempty set, a fuzzy set A of E is a function $A : E \to L$. The class of all the fuzzy sets of E is denoted by $\mathscr{L}(E)$.

Let us remember some theorems on complete Brouwerian lattices (Birkhoff, 1967).

A complete lattice is Brouwerian iff the meet operation is completely distributive on joins, so that $a \wedge (\vee x_i) = \vee(a \wedge x_i)$ for any set $\{x_i\}$ and for any a.

It is a corollary that any $\bigcup$-ring of sets (i.e., a family of sets closed under finite intersection and arbitrary union) is a complete Brouwerian lattice. Hence, the open sets of any topological space form a complete Brouwerian lattice.

The congruence relations on any lattice form a complete Brouwerian lattice.

The ideals of any distributive lattice form a complete Brouwerian lattice.

Birkhoff's list is enlarged by a theorem presented by De Luca and Termini (1972), to the effect that $\mathscr{L}(E)$ in Definition 5 above is a complete Brouwerian lattice.

If in Definition 5 above, L is taken to be the closed interval [0, 1] of the real line, L is then a complete lattice in which $x \wedge y$ is simply the smaller and $x \vee y$ the larger of x and y.

For any given elements a and b in $L = [0, 1]$, define $c = a \alpha b$ by $c = 1$ if $a \leqslant b$ and $c = b$ if $a > b$, then c is the relative pseudocomplement of a in b, so that L is a Brouwerian lattice.

Fuzzy sets according to Definition 5 are then Zadeh's membership functions, so that the results of this paper apply to Zadeh's fuzzy sets definition.

However, any Boolean lattice is easily verified to be a Brouwerian lattice with $a \alpha b$ defined as $a' \vee b$, where a' denotes the complement of a.

In addition to the Boolean structure, we will need completeness of the lattice in order to be able to define the composition of fuzzy relations.

If L is the Boolean lattice consisting of only the points 0 and 1, then a fuzzy set according to Definition 5 is just the *characteristic function* defining a subset of a set E.

DEFINITION 6. The fuzzy set $A \in \mathscr{L}(E)$ is *contained* in the fuzzy set $B \in \mathscr{L}(E)$ (written $A \subseteq B$) whenever $A(x) \leqslant B(x)$ for all $x \in E$.

DEFINITION 7. The fuzzy sets A and $B \in \mathscr{L}(E)$ are *equal* (written $A = B$) whenever $A \subseteq B$ and $B \subseteq A$, i.e., $A(x) = B(x)$ for all $x \in E$.

DEFINITION 8. A *fuzzy relation* between two nonempty sets X and Y is a fuzzy set R of $X \times Y$, i.e., an element of $\mathscr{L}(X \times Y)$. As usual $R((x, y))$ is written $R(x, y)$ for all $(x, y) \in X \times Y$.

According to Definitions 6 and 7, if R and $S \in \mathscr{L}(X \times Y)$ are two fuzzy relations, we have

$$R \subseteq S, \quad \text{iff} \quad R(x, y) \leqslant S(x, y) \quad \text{for all} \quad (x, y) \in X \times Y. \tag{1}$$

$$R = S, \quad \text{iff} \quad R(x, y) = S(x, y) \quad \text{for all} \quad (x, y) \in X \times Y. \tag{2}$$

DEFINITION 9. Let $R \in \mathscr{L}(X \times Y)$ be a fuzzy relation, the fuzzy relation R^{-1}; the *inverse* or *transpose* of R, is defined by

$$R^{-1} \in \mathscr{L}(Y \times X) \quad \text{and} \quad R^{-1}(y, x) = R(x, y) \quad \text{for all } (y, x) \in Y \times X. \tag{3}$$

DEFINITION 10. Let $Q \in \mathscr{L}(X \times Y)$ and $R \in \mathscr{L}(Y \times Z)$ be two fuzzy relations; we define $T = R \circ Q$, $T \in \mathscr{L}(X \times Z)$, the *$\circ$-composite fuzzy relation* of R and Q, by

$$(R \circ Q)(x, z) = \bigvee_{y} [Q(x, y) \wedge R(y, z)], \quad \text{where} \quad y \in Y,$$
$$\text{for all} \quad (x, z) \in X \times Z. \tag{4}$$

When L is a complete Boolean lattice, (4) stands for a *Boolean matrix product*. It is easy to verify that

$$\text{if } R_1 \text{ and } R_2 \in \mathscr{L}(Y \times Z) \text{ and if } R_1 \subseteq R_2, \text{ then}$$
$$R_1 \circ Q \subseteq R_2 \circ Q, \text{ where } Q \in \mathscr{L}(X \times Y). \tag{5}$$

DEFINITION 11. Let $Q \in \mathscr{L}(X \times Y)$ and $R \in \mathscr{L}(Y \times Z)$ be two fuzzy relations, we define $T = Q ⓐ R$, $T \in \mathscr{L}(X \times Z)$, the *ⓐ-composite fuzzy relation* of Q and R, by

$$(Q ⓐ R)(x, z) = \bigwedge_y [Q(x, y) \alpha R(y, z)] \qquad \text{where} \quad y \in Y,$$

$$\text{for all} \quad (x, z) \in X \times Z. \tag{6}$$

Comment on Definition 11. According to Definition 4, the α operation in L defines the relative pseudocomplement of $Q(x, y)$ in $R(y, z)$, for each $y \in Y$.

Let us now point out some useful properties of the α operation which allow us to derive some theorems in the next section.

With $a, b \in L$, $c = a \alpha b$ is the greatest element in L such that $a \wedge c \leqslant b$. In fact,

$$a \wedge (a \alpha b) \leqslant b. \tag{7}$$

With $a, b, d \in L$, it is easy to verify that

$$a \alpha (b \vee d) \geqslant a \alpha b \qquad (\text{or} \geqslant a \alpha d), \tag{8}$$

$$a \alpha (a \wedge b) \geqslant b. \tag{9}$$

3. RESOLUTION OF COMPOSITE FUZZY RELATION EQUATIONS

THEOREM 1. *For every pair of fuzzy relations $Q \in \mathscr{L}(X \times Y)$ and $R \in \mathscr{L}(Y \times Z)$, we have*

$$R \subseteq Q^{-1} ⓐ (R \circ Q). \tag{10}$$

Proof. Let $U = Q^{-1} ⓐ (R \circ Q) \in \mathscr{L}(Y \times Z)$. From (3), (4), and (6), we have

$$U(y, z) = \bigwedge_x [Q(x, y) \alpha (R \circ Q)(x, z)], \qquad x \in X, y \in Y, z \in Z,$$

$$U(y, z) = \bigwedge_x \left[Q(x, y) \alpha \bigvee_t (Q(x, t) \wedge R(t, z))\right], \qquad t \in Y,$$

$$U(y, z) = \bigwedge_x \left[Q(x, y) \alpha \left[(Q(x, y) \wedge R(y, z)) \vee \bigvee_{\substack{t \\ t \neq y}} (Q(x, t) \wedge R(t, z))\right]\right].$$

From (8) we have

$$U(y, z) \geqslant \bigwedge_{x} [Q(x, y) \alpha (Q(x, y) \wedge R(y, z))].$$

From (9) we have

$$U(y, z) \geqslant R(y, z).$$

THEOREM 2. *For every pair of fuzzy relations* $Q \in \mathscr{L}(X \times Y)$ *and* $R \in \mathscr{L}(Y \times Z)$, *we have*

$$Q \subseteq (R \circledast (R \circ Q)^{-1})^{-1}. \tag{11}$$

The proof is analogous to the proof of Theorem 1.

THEOREM 3. *For every pair of fuzzy relations* $Q \in \mathscr{L}(X \times Y)$ *and* $T \in \mathscr{L}(X \times Z)$, *we have*

$$(Q^{-1} \circledast T) \circ Q \subseteq T. \tag{12}$$

Proof. Let $S = (Q^{-1} \circledast T) \circ Q \in \mathscr{L}(X \times Z)$.

$$S(x, z) = \bigvee_{y} [Q(x, y) \wedge (Q^{-1} \circledast T)(y, z)], \qquad x \in X, y \in Y, z \in Z;$$

$$S(x, z) = \bigvee_{y} \left[Q(x, y) \wedge \left[\bigwedge_{t} (Q(t, y) \alpha T(t, z))\right]\right], \qquad t \in X;$$

$$S(x, z) = \bigvee_{y} \left[Q(x, y) \wedge \left[(Q(x, y) \alpha T(x, z)) \wedge \bigwedge_{\substack{t \\ t \neq x}} (Q(t, y) \alpha T(t, z))\right]\right];$$

$$S(x, z) \leqslant \bigvee_{y} [Q(x, y) \wedge (Q(x, y) \alpha T(x, z))].$$

From (7) we have

$$S(x, z) \leqslant T(x, z).$$

THEOREM 4. *For every pair of fuzzy relations* $R \in \mathscr{L}(Y \times Z)$ *and* $T \in \mathscr{L}(X \times Z)$, *we have*

$$R \circ (R \circledast T^{-1})^{-1} \subseteq T. \tag{13}$$

The proof is analogous to the proof of Theorem 3.

We can now state two *fundamental theorems.*

THEOREM 5. *Let $Q \in \mathscr{L}(X \times Y)$ and $T \in \mathscr{L}(X \times Z)$ be two fuzzy relations, $\mathscr{X}$ be the set of fuzzy relations $R \in \mathscr{L}(Y \times Z)$ such that $R \circ Q = T$; then*

$$\mathscr{X} = \{\text{fuzzy } R \in \mathscr{L}(Y \times Z) \mid R \circ Q = T\} \neq \varnothing, \text{ iff,}$$

$$Q^{-1} \circledast T \in \mathscr{X}; \textit{ then it is the greatest element in } \mathscr{X}. \tag{14}$$

Proof. We prove only the nontrivial implication. $\mathscr{X} \neq \varnothing$, so let $R \in \mathscr{X}$, we have $R \circ Q = T$. From (10) in Theorem 1, we have

$$R \subseteq Q^{-1} \circledast T, \text{ i.e., } R \subseteq \check{R} \text{ denoting } \check{R} = Q^{-1} \circledast T.$$

If we prove that $\check{R} \in \mathscr{X}$, then $\check{R}$ will be the greatest element in $\mathscr{X}$. Since $R \subseteq \check{R}$, from (5) we have $R \circ Q \subseteq \check{R} \circ Q$, i.e., $T \subseteq \check{R} \circ Q$; but from (12) in Theorem 3, we have $\check{R} \circ Q \subseteq T$, hence, $\check{R} \circ Q = T$, i.e., $\check{R} \in \mathscr{X}$.

THEOREM 6. *Let $R \in \mathscr{L}(Y \times Z)$ and $T \in \mathscr{L}(X \times Z)$ be two fuzzy relations, $\aleph$ be the set of fuzzy relations $Q \in \mathscr{L}(X \times Y)$ such that $R \circ Q = T$; then*

$$\aleph = \{\textit{fuzzy } Q \in \mathscr{L}(X \times Y) \mid R \circ Q = T\} \neq \varnothing, \textit{ iff}$$

$$(R \circledast T^{-1})^{-1} \in \aleph; \textit{ then it is the greatest element in } \aleph. \tag{15}$$

The proof is analogous to the proof of Theorem 5, using (11) in Theorem 2 and (13) in Theorem 4.

Comment on the Fundamental Theorems 5 *and* 6. From (3) and (4) it is easy to verify that $(R \circ Q)^{-1} = Q^{-1} \circ R^{-1}$, hence, $R \circ Q = T$, iff $Q^{-1} \circ R^{-1} = T^{-1}$. From (14), $Q^{-1} \circledast T \in \mathscr{X}$, iff $\mathscr{X} \neq \varnothing$, but $(Q^{-1} \circledast T) \circ Q = T$, iff $Q^{-1} \circ (Q^{-1} \circledast T)^{-1} = T^{-1}$. If we now change Q^{-1} into R and T^{-1} into T, we obtain (15).

This comment still holds to get (13) from (12), and (11) from (10). In fact, we can choose either Theorem 5 or Theorem 6 as a unique fundamental theorem and deduce the other one as a corollary.

We mention also the following weaker theorems which are easy to handle.

THEOREM 7. *Let $Q \in \mathscr{L}(X \times Y)$ and $T \in \mathscr{L}(X \times Z)$ be two fuzzy relations, if $\mathscr{X} = \{\text{fuzzy } R \in \mathscr{L}(Y \times Z) \mid R \circ Q = T\} \neq \varnothing$, then $T(x, z) \leqslant \bigvee_y Q(x, y)$ for all $(x, z) \in X \times Z$.*

Proof. Let us assume $\mathscr{X} \neq \varnothing$ and let $R \in \mathscr{X}$.

$$T(x, z) = (R \circ Q)(x, z) = \bigvee_y [Q(x, y) \wedge R(y, z)],$$

where $y \in Y$, for all $(x, z) \in X \times Z$; but for all $y \in Y$, $Q(x, y) \wedge R(y, z) \leqslant Q(x, y)$, then, $T(x, z) \leqslant \bigvee_y Q(x, y)$ for all $(x, z) \in X \times Z$.

THEOREM 8. *Let $R \in \mathscr{L}(Y \times Z)$ and $T \in \mathscr{L}(X \times Z)$ be two fuzzy relations, if $\aleph = \{$fuzzy $Q \in \mathscr{L}(X \times Y) \mid R \circ Q = T\} \neq \varnothing$, then $T(x, z) \leqslant \bigvee_y R(y, z)$ for all $(x, z) \in X \times Z$.*

The proof is analogous to the proof of Theorem 7.

EXAMPLES. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$, $Z = \{z_1, z_2, z_3\}$ and let us consider two fuzzy relations $R \in \mathscr{L}(Y \times Z)$ and $Q \in \mathscr{L}(X \times Y)$ where $L = [0, 1]$.

$Q =$

	y_1	y_2	y_3	y_4
x_1	0.2	0	0.8	1
x_2	0.4	0.3	0	0.7
x_3	0.5	0.9	0.2	0

,

$R =$

	z_1	z_2	z_3
y_1	0.3	0.5	0.2
y_2	0.8	1	0
y_3	0.7	0	0.5
y_4	0.6	0.3	1

.

From the $\circ$-composition (4) we have $T = R \circ Q$, $T \in \mathscr{L}(X \times Z)$.

$T =$

	z_1	z_2	z_3
x_1	0.7	0.3	1
x_2	0.6	0.4	0.7
x_3	0.8	0.9	0.2

.

Let us now assume that Q and T are two given fuzzy relations, $Q \in \mathscr{L}(X \times Y)$ and $T \in \mathscr{L}(X \times Z)$; we can ask if $\mathscr{X} \neq \varnothing$. We already know the answer, but the purpose is to apply (14).

We can point out that the property given in Theorem 7 is easily verified.

Recalling that when $L = [0, 1]$, for a and $b \in L$, $c = a \alpha b = 1$ if $a \leqslant b$ and $c = a \alpha b = b$ if $a > b$, and using (3) and (6), we form $\check{R} = Q^{-1} @ T$.

$Q^{-1} =$

	x_1	x_2	x_3
y_1	0.2	0.4	0.5
y_2	0	0.3	0.9
y_3	0.8	0	0.2
y_4	1	0.7	0

, $\check{R} = Q^{-1} @ T =$

	z_1	z_2	z_3
y_1	1	1	0.2
y_2	0.8	1	0.2
y_3	0.7	0.3	1
y_4	0.6	0.3	1

.

$\check{R} \circ Q = (Q^{-1} @ T) \circ Q =$

	z_1	z_2	z_3
x_1	0.7	0.3	1
x_2	0.6	0.4	0.7
x_3	0.8	0.9	0.2

$= T$ and we have $R \subseteq \check{R}$.

Let us point out a more sophisticated example (Sanchez, 1974). Let X, Y, and Z be the set of positive real numbers, and we define two fuzzy relations, $Q \in \mathscr{L}(X \times Y)$ and $R \in \mathscr{L}(Y \times Z)$, where $L = [0, 1]$, by

$$Q(x, y) = \exp[-k(x - y)^2] \qquad \text{for all } (x, y) \in X \times Y \text{ and}$$

$$R(y, z) = \exp[-k(y - z)^2] \qquad \text{for all } (y, z) \in Y \times Z, \text{ where } k \geqslant 1.$$

Q and R may be interpreted as "is near from."

The $\circ$-composition of R and Q gives $T = R \circ Q$, $T \in \mathscr{L}(X \times Z)$ defined by

$$T(x, z) = \exp[-K(x - z)^2] \qquad \text{for all } (x, z) \in X \times Z, \text{ where } K = k/4.$$

Suppose now that Q and T are given, and apply (14). We find $\check{R} = Q^{-1} @ T$, $\check{R} \in \mathscr{L}(Y \times Z)$ defined by

$$\check{R}(y, z) = \exp[-kz^2/4] \qquad \text{if } y \leqslant z/2$$

and

$$\check{R}(y, z) = \exp[-k(y - z)^2] \qquad \text{if } y \geqslant z/2.$$

We have $\check{R} \circ Q = T$ and $R \subseteq \check{R}$.

4. Remarks on the Resolution of Relational Equations

Resolution of a Dual Composite Fuzzy Relation Equation

In Definition 5 the fixed lattice L is choosen to be Brouwerian in order to solve $\circ$-composite fuzzy relation equations according to Definition 10. To solve a dual composite fuzzy relation equation we need the lattice L to be *dually Brouwerian.* This means that for any given elements a and b, the set of all $x \in L$ such that $a \vee x \geqslant b$ contains a least element, denoted $a \epsilon b$.

In this case we would define a fuzzy set A of a nonempty set E to be a function $A : E \to L$, where L is a fixed *complete dually Brouwerian lattice*, and denote $\mathscr{F}(E)$ the class of all the fuzzy sets of E.

Let $Q \in \mathscr{F}(X \times Y)$ and $R \in \mathscr{F}(Y \times Z)$ be two fuzzy relations; we define $T = R \mathbin{\Delta} Q$, $T \in \mathscr{F}(X \times Z)$, the *$\Delta$-composite fuzzy relation* of R and Q by

$$(R \mathbin{\Delta} Q)(x, z) = \bigwedge_y [Q(x, y) \vee R(y, z)] \qquad \text{where} \quad y \in Y,$$
$$\text{for all} \quad (x, z) \in X \times Z.$$

Denoting $\circledcirc$ the dual composition of the @-composition, if $Q \in \mathscr{F}(X \times Y)$ and $R \in \mathscr{F}(Y \times Z)$ are two fuzzy relations, we define $T = Q \circledcirc R$, $T \in \mathscr{F}(X \times Z)$, by

$$(Q \circledcirc R)(x, z) = \bigvee_y [Q(x, y) \epsilon R(y, z)], \qquad \text{where} \quad y \in Y,$$
$$\text{for all} \quad (x, z) \in X \times Z.$$

With analogous proofs to proofs in the latter section one can verify the following fundamental theorem.

Let $Q \in \mathscr{F}(X \times Y)$ and $T \in \mathscr{F}(X \times Z)$ be two fuzzy relations, $\mathscr{A}$ be the set of fuzzy relations $R \in \mathscr{F}(Y \times Z)$ such that $R \mathbin{\Delta} Q = T$; then,

$$\mathscr{A} = \{\text{fuzzy } R \in \mathscr{F}(Y \times Z) \mid R \mathbin{\Delta} Q = T\} \neq \varnothing, \quad \text{iff}$$
$$\hat{R} = Q^{-1} \circledcirc T \in \mathscr{A}. \text{ It is then the least element in } \mathscr{A}.$$

As a corollary one can deduce the following theorem.

Let $R \in \mathscr{F}(Y \times Z)$ and $T \in \mathscr{F}(X \times Z)$ be two fuzzy relations, $\mathscr{B}$ be the set of fuzzy relations $Q \in \mathscr{F}(X \times Y)$ such that $R \mathbin{\Delta} Q = T$; then,

$$\mathscr{B} = \{\text{fuzzy } Q \in \mathscr{F}(X \times Y) \mid R \mathbin{\Delta} Q = T\} \neq \varnothing, \quad \text{iff}$$
$$(R \circledcirc T^{-1})^{-1} \in \mathscr{B}. \text{ It is then the least element in } \mathscr{B}.$$

When $L = [0, 1]$, with $a, b \in L$, $c = a \epsilon b = b$ if $a < b$ and $c = a \epsilon b = 0$ if $a \geqslant b$.

Results when L is a Fixed Complete Boolean Lattice

With Brown's definition of fuzzy sets, when L is a complete Boolean lattice (therefore, a complete Brouwerian lattice) with $a, b \in L$, denoting the complement of an element a by a', $c = a \alpha b = a' \vee b$.

In a complete Boolean lattice, the de Morgan laws hold; hence, for all $(y, z) \in Y \times Z$,

$$\begin{aligned}(Q^{-1} @ T)(y, z) &= \bigwedge_x [Q(x, y) \alpha T(x, z)] \\ &= \bigwedge_x [Q'(x, y) \vee T(x, z)] \\ &= \left[\bigvee_x [Q(x, y) \wedge T'(x, z)]\right]' \\ &= (Q^{-1} \circ T')' (y, z).\end{aligned}$$

$Q^{-1} @ T = (Q^{-1} \circ T')'$ for (14) in Theorem 5. We can also deduce $(R @ T^{-1})^{-1} = [(R \circ (T^{-1})')']^{-1} = [(R \circ (T')^{-1})^{-1}]' = (T' \circ R^{-1})'$. $(R @ T^{-1})^{-1} = (T' \circ R^{-1})'$ for (15) in Theorem 6.

Remembering that the $\circ$-composition stands for the usual Boolean matrix product, matrix equation solutions hold with $(Q^{-1} \circ T')'$ and $(T' \circ R^{-1})'$ in the two fundamental theorems, as previously indicated by many authors (for example, Sanchez, 1972).

5. Conclusion

Zadeh's introduction and investigation of fuzzy sets since 1965, provided a means of mathematically describing situations which give rise to objects with "grades of membership" in sets, thus opening a large field of research.

We feel that the resolution of composite fuzzy relation equations could give interesting results in transportation problems and in belief systems. We plan to investigate medical aspects of fuzzy relations at some future time.

Acknowledgment

The author is indebted to Professor D. O. Walter, UCLA, Brain Research Institute of Los Angeles, for helpful criticisms.

Received: February 13, 1975

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