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REVERSE-TIME DIFFUSION EQUATION MODELS*

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Reverse-time stochastic diffusion equation models are defined and it is shown how most processes defined via a forward-time or conventional diffusion equation model have an associated reverse-time model.

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1. Introduction

Stochastic differential equations have a built-in direction of time flow since future increments in the driving process are assumed independent of present and past values of the process defined by the solution of the equation. The differential equations are thought of as evolving forward in time, normally from some fixed initial time, and the integral representation of a solution, involving as it does an Ito integral, emphasizes again, via the detailed approximation rule for the integral, the forward time flow. In this paper, we discuss reverse-time stochastic differential equations, and for a wide variety of diffusion processes, we show that each (forward-time) representation of a diffusion process generates a reverse-time representation as well. The only sorts of restrictions needed are those which ensure that the Kolmogorov equations for associated probability densities (not just distribution functions) all have unique smooth solutions; such restrictions, though hard to translate into requirements on the diffusion and drift quantities, seem nevertheless intrinsic.

Results in this vein for diffusion processes described by linear stochastic differential equations have been recently developed, see especially [1] but also [2–9]. Some of these references contain applications of the reverse-time models to problems of

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stochastic realization, signal processing, and electric circuit theory. The ideas of this paper have been developed partly with the hope of such applications fall-out in the nonlinear case.

Some results for nonlinear diffusion equations are also available. For example, Nelson [10] considers diffusion equations with constant diffusion matrix, and fails to relate the driving noise in the reverse-time equations to that in the forward equation. Stratonovich [11] analyzes a scalar diffusion equation, and also fails to relate the driving noise in the reverse-time equations to that in the forward equation.

The paper is structured as follows. In Section 2 we review some key features of the construction of reverse-time models for linear stochastic differential equations. In Section 3 we define (in the obvious way) reverse-time models for nonlinear diffusion equations and state the main result. This is established in Section 4, with further insights and a much simpler proof of part of the main result given in Section 5; it may prove easier for some readers to consider Section 5 prior to Section 4. The method of this section appears to have been first suggested in [11], where it is applied to a scalar diffusion process; it should be noted that the result in [11] is incorrectly stated.

2. The linear problem

The fullest exposition of results on forward and reverse-time stochastic realizations for Gauss–Markov processes can be found in [1]. We sketch some of the ideas here, as motivation for the subsequent results.

Suppose x is a purely nondeterministic, wide-sense stationary n-dimensional process, described as the solution of the stochastic differential equation set

$$dx = Ax\, dt + B\, dw. \tag{2.1}$$

Here, A and B are constant matrices, $\mathrm{Re}[\lambda_i(A)] < 0$ for all i and $w(\cdot)$ is a vector Wiener process such that $x(t)$ is independent of future increments of w, but not of past ones, i.e., $x(t)$ is independent of $w(t_2) - w(t_1)$ for all $t_2 > t_1 \geqslant t$ but not (in general) of $w(t_3) - w(t_4)$ for $t \geqslant t_3 > t_4$. (One could consider such a situation as arising, for example, by considering (2.1) with a finite initial time t_0, and initial state a random variable $x(t_0)$ independent of the $w(\cdot)$ process; then one can let $t_0 \to -\infty$.)

Such a model will be called a forward time model. One thinks of (2.1) as evolving forward in time, and can consider its solution as

$$x(t) = \int_{-\infty}^{t} e^{A(t-s)} B\, dw(s). \tag{2.2}$$

A reverse time model on the other hand is one for which

$$dx = \bar{A}x\, dt + \bar{B}\, d\bar{w} \tag{2.3}$$

where now $\mathrm{Re}[\lambda_i(\bar{A})]>0$ for all i, and $\bar{w}(\cdot)$ is a vector Wiener process with $x(t)$ independent of past increment of $\bar{w}$, but not of future ones.

One might think of (2.3) as evolving backward in time, having a solution

$$x(t) = -\int_t^\infty e^{\bar{A}(t-s)}\bar{B}\, d\bar{w}(s). \tag{2.4}$$

Such models are useful in studying smoothing problems [1, 3], questions of reversibility [5], and electric network synthesis [7].

Now in [1], the problem is considered of obtaining from a given forward-time representation of $x(t)$ a reverse-time representation. This problem is solved in the following way. Let

$$P = E[x(t)x'(t)]. \tag{2.5}$$

(The matrix P is the solution of the linear matrix equation $PA'+AP=-BB'$, and is nonsingular precisely when rank $[B\ AB\ \cdots\ A^{n-1}B]=n$.) Suppose P is nonsingular, and define a vector process $\bar{w}$ by

$$d\bar{w} = dw - B'P^{-1}x\, dt, \qquad \bar{w}(0)=0, \tag{2.6}$$

which in conjunction with (2.1) implies

$$dx = (A+BB'P^{-1})\, dt + B\, d\bar{w}. \tag{2.7}$$

Then it can be proved that $\mathrm{Re}[\lambda_i(A+BB'P^{-1})]>0$ for all i, and that $\bar{w}(\cdot)$ is a vector Wiener process with $x(t)$ independent of past increments of $\bar{w}$, (but not of future ones), i.e., with the definitions (2.5) and (2.6), (2.7) is a reverse time model.

Two further points may be noted. First, the requirement that $P=E[x(t)x'(t)]>0$ can be interpreted as a requirement that (2.1) be a minimal dimension model in a certain sense, and the nonsingularity ensures that the probability density of $x(t)$ exists. Second, stationarity is not an essential ingredient of these results, see for example [2, 3, 7] where some of the ideas are presented free of a stationarity assumption.

3. Construction of reverse time nonlinear models

Let $(\Omega, \mathscr{A}, P)$ be a fixed probability space, let $\{\mathscr{A}_t, -\infty<t<\infty\}$ be an increasing family of sub-σ-algebras on $\mathscr{A}$, and let $\{w_t, -\infty<t<\infty\}$ be an r-vector Brownian motion process such that w_t is $\mathscr{A}_t$-measurable for each t, and w_t-w_s for $t\geq s$ is independent of $\mathscr{A}_s$; we require for $s\geq 0$

$$\mathbf{E}[w_{t+s}\,|\,\mathscr{A}_t] = w_t, \tag{3.1}$$

$$\mathbf{E}[(w_{t+s}-w_t)(w_{t+s}-w_t)'\,|\,\mathscr{A}_t] = sI. \tag{3.2}$$

We study an Ito stochastic differential equation of the form

$$dx_t = f(x_t, t)\, dt + g(x_t, t)\, dw_t. \tag{3.3}$$

Here, x_t is an n-vector process, and $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are $n\times 1$ and $n\times n$ matrix functions with certain smoothness and growth properties which guarantee existence and uniqueness of a solution, see e.g. [9].

Eq. (3.3) is understood to be defined in some region $t\geq t_0$, x_{t_0} is an almost surely bounded random variable independent of $\{w_t, -\infty<t<\infty\}$, and it may be possible for certain equations to allow $t_0\to-\infty$. Commonly $\mathcal{A}_t$ is the minimal σ-algebra with respect to which x_{t_0} and w_s, $s\leq t$ are measurable. The solution has certain properties, depending on the smoothness and growth of f and g, as set out in standard references, e.g. [9, 12]. We shall term (3.3) a forward Ito equation or forward-time model.

We now describe what is meant by a reverse-time model. The idea is *not* simply to make some adjustment to (3.3) to permit use of a backward rather than forward Ito integral [13] for expressing the solution of (3.3).

We require a decreasing family $\{\bar{\mathcal{A}}_t, -\infty<t<\infty\}$ of sub-σ-algebras on $\mathcal{A}$ and an n-vector process $\{\bar{w}_t, -\infty<t<\infty\}$ such that $\bar{w}_t$ is $\bar{\mathcal{A}}_t$-measurable for each t, $\bar{w}_t-\bar{w}_s$ for $t\geq s$ is independent of $\bar{\mathcal{A}}_t$, and for $s\geq 0$

$$\mathbf{E}[\bar{w}_t\,|\,\bar{\mathcal{A}}_{t+s}]=\bar{w}_{t+s}, \tag{3.4}$$

$$\mathbf{E}[(\bar{w}_t-\bar{w}_{t+s})(\bar{w}_t-\bar{w}_{t+s})'\,|\,\bar{\mathcal{A}}_{t+s}]=sI. \tag{3.5}$$

This process drives a reverse-time Ito equation of the form

$$\mathrm{d}x_t=\bar{f}(x_t,t)\,\mathrm{d}t+\bar{g}(x_t,t)\,\mathrm{d}\bar{w}_t, \tag{3.6}$$

which is understood to be defined in some region $t\leq T$, where it may be possible to have $T\to\infty$. One has x_T a random variable independent of $\{\bar{w}_t, -\infty<t<\infty\}$ and (3.6) is shorthand for

$$x_T-x_t=\int_t^T\bar{f}(x_t,t)\,\mathrm{d}t+\int_t^T\bar{g}(x_t,t)\,\mathrm{d}\bar{w}_t, \tag{3.7}$$

in which the second integral is a backward Ito integral. Again, it is understood that $\bar{f}$ and $\bar{g}$ satisfy the growth and smoothness properties sufficient for existence and uniqueness of a solution.

Evidently in the forward model, x_t is independent of future increments of the driving Wiener process, while in the reverse time model, x_t is independent of past increments of the driving process.

The main result explains how to construct a reverse-time realization from a forward-time realization. In order to formulate this result, as further background we recall the fact that associated with (3.3) are forward and backward Kolmogorov equations, see, e.g. [9, 12]. These take various forms; for example, the forward equation for the probability density[1] $p(x_t, t\,|\,x_s, s)$ for $t>s$ is

[1] A lower case p will be reserved henceforth to designate a probability density, with the displayed arguments implicitly defining the random variables of which p is the density.

$$\frac{\partial p(x_t, t\,|\,x_s, s)}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_t^i}[p(x_t, t\,|\,x_s, s) f^i(x_t, t)] + \tfrac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_t^i\,\partial x_t^j}\{p(x_t, t\,|\,x_s, s)[g(x_t, t)g'(x_t, t)]^{ij}\}. \tag{3.8}$$

while the backward equation, again for $t > s$, is

$$\frac{-\partial p(x_t, t\,|\,x_s, s)}{\partial s} = \sum_{i=1}^{n} \frac{\partial}{\partial x_s^i}\{p(x_t, t\,|\,x_s, s) f^i(x_s, s)\} + \tfrac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_s^i\,\partial x_s^j}\{p(x_t, t\,|\,x_s, s)[g(x_s, s)g'(x_s, s)]^{ij}\}. \tag{3.9}$$

Appropriate boundary conditions are usually associated with the equation. Unconditioned forward equations, or partially conditioned forward equations also exist. Sufficient conditions for the transition density to satisfy the Kolmogorov equation are (see, e.g. [9, pp. 297–8]), that $f(\cdot\,,\cdot)$ and $g(\cdot\,,\cdot)$ guarantee existence of a unique strong solution to (3.3) for an almost surely bounded initial condition, that $f(\cdot\,,\cdot)$ $g(\cdot\,,\cdot)$ are twice continuously differentiable in x, that their first order partials in x are bounded, that the second order partials grow no faster than $\|x\|^m$ as $x \to \infty$ for some $m > 0$ and that the transition probability density $p(x_t, t\,|\,x_s, s)$ is twice continuously differentiable in x_t and continuously differentiable in t. Sufficient conditions for there to be no other solution of the Kolmogorov equations satisfying the same boundary conditions, i.e., for uniqueness, are unknown to us. We now have the following theorem.

Theorem. *Let x_t be the process described by* (3.3), *and suppose $f(\cdot\,,\cdot)$ and $g(\cdot\,,\cdot)$ are such as to guarantee the existence of the probability density $p(x_t, t)$ for $t_0 \leq t \leq T$ as a smooth and unique solution of its associated Kolmogorov equation. Suppose further that an r-vector process $\bar{w}_t$ is defined by $\bar{w}_{t_0} = 0$ and*

$$\mathrm{d}\bar{w}_t^k = \mathrm{d}w_t^k + \frac{1}{p(x_t, t)}\sum_j \frac{\partial}{\partial x_t^j}[p(x_t, t)g^{jk}(x_t, t)]\,\mathrm{d}t, \tag{3.10}$$

and that the forward Kolmogorov equation associated with the joint process $(x_t, \bar{w}_t)$ yields a smooth and unique solution in $t > t_0$ for $p(x_t, \bar{w}_t, t)$ and in $t > s \geq t_0$ for $p(x_t, \bar{w}_t, t\,|\,\bar{w}_s, s)$. Then

(i) *x_t and $\bar{w}_t - \bar{w}_s$ are independent for all $t \geq s \geq t_0$.*

(ii) *With $\bar{\mathscr{A}}_t$ the minimal σ-algebra with respect to which x_s for $s \geq t$ and $\bar{w}_s$ for $s \geq t$ are measurable, conditions* (3.4) *and* (3.5) *hold.*

(iii) *A reverse time model for x_t is defined by*

$$\mathrm{d}x_t = \bar{f}(x_t, t)\,\mathrm{d}t + g(x_t, t)\,\mathrm{d}\bar{w}_t \tag{3.11}$$

where

$$\bar{f}^i(x_t, t) = f^i(x_t, t) - \frac{1}{p(x_t, t)}\sum_{j,k} \frac{\partial}{\partial x_t^j}[p(x_t, t)g^{ik}(x_t, t)g^{jk}(x_t, t)]. \tag{3.12}$$

We remark that in the light of the linear results, *it is not surprising that the probability density* $p(x_t, t)$ *should have to exist*; our proof makes essential use however not just of existence, but of unique solvability of the Kolmogorov forward equations for this and a related density.

4. Proof of main theorem

While this section contains a full proof of the main theorem, the next section contains a different, simpler proof of only some of the claims of the theorem.

Let us consider the joint process $(x_t, \bar{w}_t)$ defined by

$$dx_t = f(x_t, t)\,dt + g(x_t, t)\,dw_t, \tag{4.1}$$

$$d\bar{w}_t^k = \frac{1}{p(x_t, t)} \sum_j \frac{\partial}{\partial x_t^j}[p(x_t, t) g^{jk}(x_t, t)]\,dt + dw_t^k, \tag{4.2}$$

with $k = 1, \ldots, r$. The associated forward Kolmogorov equation is

$$\begin{aligned}\frac{\partial p(x_t, \bar{w}_t, t)}{\partial t} = &- \sum_{i=1}^{n} \frac{\partial}{\partial x_t^i}[p(x_t, \bar{w}_t, t) f^i(x_t, t)] \\ &- \sum_{k=1}^{r} \frac{\partial}{\partial \bar{w}_t^k}\left\{\frac{p(x_t, \bar{w}_t, t)}{p(x_t, t)} \sum_j \frac{\partial}{\partial x_t^j}[p(x_t, t) g^{jk}(x_t, t)]\right\} \\ &+ \tfrac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_t^i\, \partial x_t^j}\{p(x_t, \bar{w}_t, t)[g(x_t, t) g'(x_t, t)]^{ij}\} \\ &+ \tfrac{1}{2} \sum_{k,l=1}^{r} \frac{\partial^2}{\partial \bar{w}_t^k\, \partial \bar{w}_t^l}[p(x_t, \bar{w}_t, t)] \\ &+ \sum_{i=1}^{n} \sum_{k=1}^{r} \frac{\partial^2}{\partial x_t^i\, \partial \bar{w}_t^k}[p(x_t, \bar{w}_t, t) g^{ik}(x_t, t)],\end{aligned} \tag{4.3}$$

and the boundary condition we take is the natural one

$$p(x_{t_0}, \bar{w}_{t_0}, t_0) = p(x_{t_0}, t_0)\delta(\bar{w}_{t_0}). \tag{4.4}$$

The bulk of the proof will now be completed via several lemmas. The first relates $p(x_t, \bar{w}_t, t)$ to $p(x_t, t)$.

Lemma 1. *Suppose* $p(x_t, t)$ *is the solution of the forward Kolmogorov equation for* (4.1) *and*

$$\phi(\bar{w}_t, t) = \frac{1}{[2\pi(t - t_0)]^{r/2}} \exp\left[-\frac{\bar{w}_t'\bar{w}_t}{2(t - t_0)}\right]. \tag{4.5}$$

Then the solution of (4.3) *and* (4.4) *is given by*

$$p(x_t, \bar{w}_t, t) = p(x_t, t)\phi(\bar{w}_t, t). \tag{4.6}$$

Note. The notation in (4.5) is chosen to emphasize that, at this point, there is no claim that $\phi(\bar{w}_t, t)$ is actually the probability density of $\bar{w}_t$, though below we shall demonstrate this property.

Proof. When $p(x_t, t)\phi(\bar{w}_t, t)$ replaces $p(x_t, \bar{w}_t, t)$ on the right-hand side of (4.3), straightforward manipulations yield for this right-hand side the expression

$$\phi(\bar{w}_t, t)\left\{-\sum_i \frac{\partial}{\partial x_t^i}[p(x_t, t)f^i(x_t, t)] + \frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial x_t^i\, \partial x_t^j}\{p(x_t, t)[g(x_t, t)g'(x_t, t)]^{ij}\}\right.$$

$$\left. + \frac{1}{2}p(x_t, t)\sum_{k,l} \frac{\partial^2 \phi(\bar{w}_t, t)}{\partial \bar{w}_t^k\, \partial \bar{w}_t^l}\right\}.$$

Taking cognizance of the forward Kolmogorov equation for $p(x_t, t)$ and of (4.5), which implies that

$$\frac{\partial \phi(\bar{w}_t, t)}{\partial t} = \frac{1}{2}\sum_{k,l} \frac{\partial^2 \phi(\bar{w}_t, t)}{\partial \bar{w}_t^k\, \partial \bar{w}_t^l}, \tag{4.7}$$

the expression becomes

$$\frac{\partial}{\partial t}[p(x_t, t)\phi(\bar{w}_t, t)].$$

This agrees with the left-hand side of (4.3) when $p(x_t, \bar{w}_t, t)$ is replaced by $p(x_t, t)\phi(\bar{w}_t, t)$. So (4.6) satisfies (4.3). That (4.6) ensures that (4.4) holds is also trivial. □

As promised, we now identify $\phi(\bar{w}_t, t)$ with the probability density of $\bar{w}_t$.

Lemma 2. *With the same hypotheses as Lemma* 1,

$$p(x_t, \bar{w}_t, t) = p(x_t, t)p(\bar{w}_t, t). \tag{4.8}$$

Proof. An elementary application of Bayes' theorem to (4.6) yields

$$p(\bar{w}_t, t \mid x_t) = \phi(\bar{w}_t, t),$$

and since $\phi(\bar{w}_t, t)$ is independent of x_t we must have

$$p(\bar{w}_t, t) = \phi(\bar{w}_t, t). \qquad \square$$

Eq. (4.8) shows that x_t is independent of any increment $\bar{w}_t - \bar{w}_{t_0} = \bar{w}_t$. We must now extend the independence to include increments $\bar{w}_t - \bar{w}_s$ for arbitrary $s \in (t_0, t)$. This requires several further lemmas which pin down the form of $p(x_t, \bar{w}_t, t \mid \bar{w}_s, s)$ for $t \geq s$. The first is akin to Lemma 1, and has a proof using the result of Lemma 2.

Lemma 3. *Suppose $p(x_t, t)$ is the solution of the forward Kolmogorov equation for* (4.1). *Then the conditional density, $p(x_t, \bar{w}_t, t \mid \bar{w}_s, s)$ associated with* (4.1) *and* (4.2) *for $t \geq s \geq t_0$ is*

$$p(x_t, \bar{w}_t, t \mid \bar{w}_s, s) = p(x_t, t)\psi(\bar{w}_t, \bar{w}_s, t-s) \tag{4.9}$$

where

$$\psi(\bar{w}_t, \bar{w}_s, t-s) = \frac{1}{[2\pi(t-s)]^{r/2}} \exp\left[-\frac{(\bar{w}_t - \bar{w}_s)'(\bar{w}_t - \bar{w}_s)}{2(t-s)}\right]. \tag{4.10}$$

Proof. The conditional density on the left of (4.9) satisfies a certain Kolmogorov equation with certain boundary conditions. We must show the quantity on the right of (4.9) satisfies the same equation and boundary condition.

The equation to be satisfied is (4.3) with $p(x_t, \bar{w}_t, t)$ replaced by $p(x_t, \bar{w}_t, t \mid \bar{w}_s, s)$, and the same argument as used in proving Lemma 1 establishes that the right-hand side of (4.9) also satisfies the equation.

The boundary condition satisfied by $p(x_t, \bar{w}_t, t \mid \bar{w}_s, s)$ is obtained as follows. (The independence of x_s' and $\bar{w}_s$ established in Lemma 2 is critical here.) We know that

$$\lim_{t \downarrow s} p(x_t, \bar{w}_t, t \mid x_s, \bar{w}_s, s) = \lim_{t \downarrow s} \delta(x_t - x_s)\delta(\bar{w}_t - \bar{w}_s)$$

and that

$$\begin{aligned} p(x_t, \bar{w}_s, t \mid \bar{w}_s, s) &= \int p(x_t, \bar{w}_t, t \mid x_s, \bar{w}_s, s) p(x_s, s \mid \bar{w}_s, s)\, \mathrm{d}x_s \\ &= \int p(x_t, \bar{w}_t, t \mid x_s, \bar{w}_s, s) p(x_s, s)\, \mathrm{d}x_s. \end{aligned}$$

Hence

$$\lim_{t \downarrow s} p(x_t, \bar{w}_t, t \mid \bar{w}_s, s) = \lim_{t \downarrow s} p(x_t, t)\delta(\bar{w}_t - \bar{w}_s).$$

The right-hand side of (4.9) has the same limit, in view of the definition of ψ. Accordingly, (4.9) is established. □

Before getting the desired independence result we note without proof the following simple lemma.

Lemma 4. *Let A, B, C be three jointly distributed random variables, and let $p_A(a)$, etc. denote the probability density of A evaluated at $A = a$. Then, if*

$$p_{B|C}(b \mid c) = f(b - c)$$

for some function f, defining $D = B - C$ results in

$$p_D(d) = f(d).$$

If

$$p_{A,B|C}(a, b\,|\,c) = p_A(a) f(b - c)$$

for some function f, defining $D = B - C$ results in

$$p_{A,D} = p_A(a) p_D(d) = p_A(a) f(d).$$

Now we have the following lemma.

Lemma 5. *With x_t, $\bar{w}_t$ defined as above, and for $t \geqslant s \geqslant t_0$,*

$$p(x_t, \bar{w}_t - \bar{w}_s, t, s) = p(x_t, t) p(\bar{w}_t - \bar{w}_s, t, s).$$

Proof. Apply Lemma 4 to (4.9), identifying $a = x_t$, $b = \bar{w}_t$, $c = \bar{w}_s$, $f = \psi$. □

This lemma completes the proof of claim (i) of the theorem. Claim (ii) is immediate using this lemma, the Markovian character of x_t, and the density $p(\bar{w}_t - \bar{w}_s, t, s)$, which is the expression ψ in (4.10). We turn to claim (iii).

From (4.1) and (4.2), we have

$$\mathrm{d}x_t^i = \left\{ f^i(x_t, t) - \frac{1}{p(x_t, t)} \sum_k g^{ik}(x_t, t) \sum_j \frac{\partial}{\partial x_t^j} [p(x_t, t) g^{jk}(x_t, t)] \right\} \mathrm{d}t + \sum_k g^{ik}(x_t, t)\, \mathrm{d}\bar{w}_t^k$$

or

$$\mathrm{d}x_t = \hat{f}(x_t, t)\, \mathrm{d}t + g(x_t, t)\, \mathrm{d}\bar{w}_t,$$

with obvious definition of $\hat{f}$. Since (4.1) and (4.2) have integral forms involving the standard (forward) Ito integral, the integral form of this equation also involves a standard Ito integral, save that $\bar{w}_t$ should be regarded as a semi-martingale, (3.10) defining the decomposition into a martingale and a bounded-variation process. In order to convert this equation to one to be understood as using backward Ito integration, we must make an adjustment if g depends explicitly on x_t—double in fact that required to convert this equation to a Stratonovich equation, [12], or double that required to obtain a symmetrized integral with respect to the usual Wiener martingale, and by extension, with respect to $\bar{w}_t$, regarded as a semi-martingale. The reverse-time model, using a backward Ito integral, [13], is

$$\mathrm{d}x_t = \bar{f}(x_t, t)\, \mathrm{d}t + g(x_t, t)\, \mathrm{d}\bar{w}_t$$

where

$$\begin{aligned} \bar{f}^i(x_t, t) &= \hat{f}^i(x_t, t) - \sum_{j,k} \frac{\partial g^{ik}(x_t, t)}{\partial x_t^j} g^{jk}(x_t, t) \\ &= f^i(x_t, t) - \frac{1}{p(x_t, t)} \sum_{j,k} \frac{\partial}{\partial x_t^j} [p(x_t, t) g^{ik}(x_t, t) g^{jk}(x_t, t)]. \end{aligned}$$

This establishes claim (iii) of the theorem.

5. Another approach to the main result

In this section, we describe an alternative (and historically our first) approach to the main result which yields most, but not all, of it. This alternative approach highlights the existence of reverse-time-versions of the usual Kolmogorov equations and was suggested in [11]. In broad outline, the idea is as follows.

There is a one-to-one, onto correspondence between the stochastic differential equation for x_t and the Kolmogorov equation for $p(x_t, t\,|\,x_s, s)$, $t \geqslant s$. Consequently, there should be a one-to-one onto correspondence between a reverse-time equation for x_t and a Kolmogorov equation for $p(x_t, t\,|\,x_s, s)$, $s \geqslant t$. This suggests that the equation for the latter density be sought. We now indicate how it may be easily obtained.

We consider

$$\mathrm{d}x_t = f(x_t, t)\,\mathrm{d}t + g(x_t, t)\,\mathrm{d}w_t \tag{5.1}$$

where w_t has the usual properties. Then the backward (not reverse-time) Kolmogorov equation for $s \geqslant t$ is

$$-\frac{\partial p(x_s, s\,|\,x_t, t)}{\partial t} = \sum_i f^i(x_t, t)\frac{\partial p(x_s, s\,|\,x_t, t)}{\partial x_t^i} + \tfrac{1}{2}\sum_{i,j,k} g^{ik}(x_t, t) g^{jk}(x_t, t)\frac{\partial^2 p(x_s, s\,|\,x_t, t)}{\partial x_t^i\,\partial x_t^j}, \tag{5.2}$$

and the forward, unconditioned, equation yields

$$-\frac{\partial p(x_t, t)}{\partial t} = \sum_i \frac{\partial}{\partial x_t^i}[p(x_t, t) f^i(x_t, t)] - \tfrac{1}{2}\sum_{i,j,k}\frac{\partial^2[g^{ik}(x_t, t) g^{jk}(x_t, t) p(x_t, t)]}{\partial x_t^i\,\partial x_t^j}. \tag{5.3}$$

Now, because

$$p(x_t, t, x_s, s) = p(x_s, s\,|\,x_t, t)\,p(x_t, t), \tag{5.4}$$

we can attempt to obtain a partial differential equation for $p(x_t, t, x_s, s)$, regarding x_t, t as the independent variables and x_s, s as parameters. We obtain, combining (5.2) through (5.4),

$$\frac{\partial p(x_t, t, x_s, s)}{\partial t} = \text{terms involving } f,\ g,\ p(x_t, t) \text{ and } p(x_s, s\,|\,x_t, t) \text{ and their } x_t\text{-derivatives.}$$

We eliminate every occurrence of $p(x_s, s\,|\,x_t, t)$ on the right-hand side, replacing it in accordance with (5.4) by $p(x_t, t, x_s, s)/p(x_t, t)$. The end result of these manipulations is, for $s \geqslant t$,

$$-\frac{\partial}{\partial t} p(x_t, t, x_s, s) = \sum_i \frac{\partial}{\partial x_t^i}[\bar{f}^i(x_t, t) p(x_t, t, x_s, s)] + \frac{1}{2} \sum_{i,j,k} \frac{\partial^2 [p(x_t, t, x_s, s) g^{ik}(x_t, t) g^{jk}(x_t, t)]}{\partial x_t^i \partial x_t^j} \tag{5.5}$$

where $\bar{f}^i$ is as before, viz,

$$\bar{f}^i(x_t, t) = f^i(x_t, t) - \frac{1}{p(x_t, t)} \sum_{j,k} \frac{\partial}{\partial x_t^j}[p(x_t, t) g^{ik}(x_t, t) g^{jk}(x_t, t)]. \tag{5.6}$$

The same partial differential equation (but with different boundary conditions of course) is satisfied by $p(x_t, t \mid x_s, s)$ [and in fact $p(x_t, t)$]—this is trivial to see. Just as (5.3) corresponds to the forward model (5.1), so then (5.5) has to correspond to the reverse model

$$\mathrm{d}\bar{x}_t = \bar{f}(\bar{x}_t, t)\, \mathrm{d}t + g(\bar{x}_t, t)\, \mathrm{d}\bar{w}_t \tag{5.7}$$

where $\bar{w}_t$ is a vector Wiener process with past increments independent of $\bar{x}_t$. (The apparent non correspondence of the signs in (5.1) and (5.2) on the one hand and (5.6) and (5.7) on the other hand is a result of the reversal of the time flow direction.) Note that in arguing this way we have obtained (more easily in fact) all the results of Section 3 save one: we have not shown here that we can identify trajectories $\{x_t, -\infty < t < \infty\}$ and $\{\bar{x}_t, -\infty < t < \infty\}$ of the forward and reverse time models. The approach of this section however allows a simpler appreciation of the result, and also throws up the Kolmogorov equation (5.5), which we might term the reverse-time (as opposed to backward) Kolmogorov equation. There is of course even a further Kolmogorov equation—the reverse-time parallel of the backward equation associated with the forward-time model.

6. Miscellaneous complements

6.1. *Time-invariant problems*

If the forward-time model has $f(x_t, t)$ and $g(x_t, t)$ independent of t, and if there exists a stationary density $\pi(x_t)$, again independent of t, and satisfying (uniquely) the steady-state Kolmogorov equation, then it is immediately verified that the reverse-time model has $\bar{f}(x_t, t)$ independent of t.

6.2. *Forward stability implies reverse-time stability*

The analog of the stability result described in Section 2 for linear equations is as follows. Suppose that the forward-time model has $f(x_t, t)$ and $g(x_t, t)$ independent of t, and that $\lim_{t\to\infty} p(x_t, t \mid x_s, s) = \pi(x_t)$, independent of x_s, s. Suppose further that

$\pi(\cdot)$ is used to define a reverse-time model. Then we can ask whether this reverse-time model has reverse time stability, in the sense that $\lim_{t\to-\infty} p(x_t, t\,|\,x_s, s) = \pi(x_t)$. We now demonstrate that this is the case.

$$\begin{aligned}\lim_{t\to-\infty} p(x_t, t\,|\,x_s, s) &= \lim_{s\to\infty} p(x_t, t\,|\,x_s, s) \quad \text{(by time-invariance of the model)}\\ &= \frac{\lim_{s\to\infty} p(x_s, s\,|\,x_t, t)\pi(x_t)}{\pi(x_s)}\\ &= \frac{\pi(x_s)\pi(x_t)}{\pi(x_s)} \quad \text{(by stability of forward model)}\\ &= \pi(x_t).\end{aligned}$$

6.3. Simple example

Consider the stationary system with scalar x defined by

$$\mathrm{d}x = f(x)\,\mathrm{d}t + g(x)\,\mathrm{d}w.$$

Here, $f(\cdot)$ and $g(\cdot)$ are smooth and confined to the second and fourth quadrants, each lying in a cone whose boundaries are strictly within the quadrants. It follows easily that

$$\pi(x) = \frac{k}{g^2(x)} \exp\left\{\int_0^x \frac{2f(\sigma)}{g^2(\sigma)}\,\mathrm{d}\sigma\right\}$$

for some constant k. Then

$$\mathrm{d}\bar{w} = \mathrm{d}w + \left[\frac{2f(x)}{g(x)} - g'(x)\right]\mathrm{d}t,$$

and the reverse-time equation becomes

$$\mathrm{d}x = -f(x)\,\mathrm{d}t + g(x)\,\mathrm{d}\bar{w}.$$

6.4. Finding a forward-time model given a reverse-time model

This is straightforward. As one could expect, if a forward time model F_2 is constructed from a reverse time model R_1 which itself was constructed from a forward time model F_1, then we have $F_1 = F_2$.

6.5. Linear systems

We can easily verify the main result in the case of the stationary linear system (2.1), repeated for convenience,

$$\mathrm{d}x = Ax\,\mathrm{d}t + B\,\mathrm{d}w.$$

One has

$$p(x_t)=\frac{1}{(2\pi)^{n/2}|P|^{1/2}}\exp\{-\tfrac{1}{2}x'P^{-1}x\}.$$

Here, P is the solution of $PA'+AP=-BB'$, and is assumed nonsingular. Then

$$\frac{1}{p(x_t)}\sum_j\frac{\partial}{\partial x_t^j}[p(x_t)B^{jk})]=-\sum(P^{-1})^{ji}x_t^iB^{jk}$$

$$=-k\text{th entry of } B'P^{-1}x.$$

Thus, following (3.10),

$$\mathrm{d}\bar{w}_t=\mathrm{d}w-B'P^{-1}x\,\mathrm{d}t$$

as stated in (2.6).

6.6. Future extensions

There are at least two directions in which we believe this work can be extended. First, we expect to examine smoothing or interpolation problems, where $p(x_t|z_s, s\in[0,T])$ is sought given the pair

$$\begin{aligned}\mathrm{d}x_t&=f(x_t,t)\,\mathrm{d}t+g(x_t,t)\,\mathrm{d}w_t,\\ \mathrm{d}z_t&=h(x_t,t)\,\mathrm{d}t+j(x_t,t)\,\mathrm{d}w_t.\end{aligned}\tag{6.1}$$

Such problems are studied in, e.g. [14, Chapter 9] and [15], and hitherto have used solution methods which reflect a preference for direction of time flow that the quantity sought appears to lack; linear results along the lines desired are however available [1, 3, 16]. Secondly, we aim to consider problems of reversibility and dynamic reversibility of processes, see e.g. [5, 7]. Such processes arise in many physical situations, e.g. in linear electric networks comprising resistors, capacitors and inductors. We are developing extensions of the known results on reversibility and dynamic reversibility for linear networks to nonlinear, but still passive, networks, see [17] for some results.

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