Intermediation via Credit Chains*

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Abstract

The modern financial system features complicated financial intermediation chains, with each layer performing a certain degree of credit/maturity transformation. We develop a dynamic model in which an entrepreneur borrows from overlapping-generation households via layers of funds, forming a credit chain. Each intermediary fund in the chain faces rollover risks from its lenders. The model delivers new insights regarding the benefits of intermediation via layers: by shortening the maturity of liquidated assets, the chain structure insulates interim negative fundamental shocks and protects the underlying cash flows from being discounted heavily during bad times. We show that the equilibrium chain length minimizes the run risk and that it is constrained efficient.

Keywords: Financial intermediation, Debt runs, Shadow banking, Dynamic economy, Money.

JEL codes: D85, G21, G23, G33, E44, E51

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1 Introduction

Since the mid-1980, the nature of financial intermediation has been changed in a dramatic way by the emergence of securitization and secured lending techniques, giving rise to a more market-based financial system. Shadow banking can be viewed as the product of this market-based financial system; to take one of the most salient examples, it is widely acknowledged that maturity and credit transformation in the shadow banking system contributed to the U.S. real estate market boom prior to the 2007–09 financial crisis.

Although the underlying economic mechanism of shadow banking has been well studied by many leading scholars since the onset of the 2007–09 financial crisis (Adrian and Shin, 2009, 2013; Gennaioli et al., 2013; Duffie, 2019), our paper focuses on one missing piece in the literature on shadow banking. Adrian et al. (2012) explain it vividly:

> Like the traditional banking system, the shadow banking system conducts credit intermediation. However, unlike the traditional banking system, where credit intermediation is performed "under one roof"—that of a bank—in the shadow banking system, it is performed through a daisy-chain of non-bank financial intermediaries in a multi step process. . . . The shadow banking system performs these steps of shadow credit intermediation in a strict, sequential order with each step performed by a specific type of shadow bank and through a specific funding technique. . . . The intermediation chain always starts with origination and ends with wholesale funding, and each shadow bank appears only once in the process.

The thrust of the above description is the concept of a "chain." The common theme in the various shadow banking businesses anatomized by Adrian et al. (2012) is the step-by-step maturity/liquidity and credit transformation, often initiated by loan origination. This is then followed by so-called "loan warehousing," which refers to the act of collecting a significant volume of eligible loans in a special purpose vehicle (SPV), which then issues asset-backed commercial papers (ABCP) to the public, as well as issues loans to the next layer of asset-backed securities (ABS) warehousing. This process might further involve an ABS collateralized-debt-obligation (CDO), but eventually reaches the wholesale funding markets that are populated by money market investors as well as long-term fixed income investors (say pension funds and insurance companies) (Adrian et al., 2012).

The concept of intermediation credit chain is more general than the stark example of the shadow banking system prior-to the 2007–09 financial crisis. In most modern financial systems, money market mutual funds (MMMFs) issue daily "debt" to households, but hold commercial papers with maturity of one to six months; and these commercial papers are issued by banks and other nonbank financial institutions to fund even longer-term and riskier projects. Regulators have increasingly expressed concerns over these nonbank financial intermediaries, which have grown significantly since the global financial crisis (Aramonte et al., 2021). As a result, opaque layers of leverage have piled up among both banks and nonbanks; for example, banks today support funding to private debt funds, who then lend to companies;[1] loan mutual funds hold tranches of CLOs, who then hold baskets of leveraged loans.[2]

Figure 1 plots the credit intermediation index over time, which is the ratio of total liabilities of all sectors in the economy over the total end-user liability. Similar to the "money multiplier" idea, Greenwood and Scharfstein (2013) argue that the credit intermediation index approximates the average credit chain length in the economy, where they measure the total end-user liability by domestic nonfinancial sector liabilities and the total liabilities of all sectors by the sum of financial and nonfinancial sector liabilities. This ratio grew significantly during the 1990s when structured finance and securitization became popular, consistent with the view in Adrian et al. (2012) mentioned above. It decreased slightly after the global financial crisis, but remains at a high level from a historical perspective. During the last decade, each dollar from investors flows through about 2.2 layers of financial intermediaries on average before reaching the final borrower.[3]

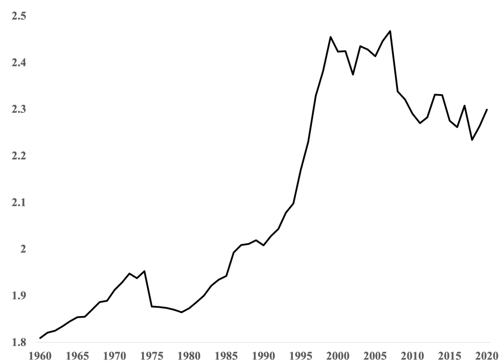
Despite the extensive literature on shadow banking and its policy implications, it still remains an open question why market participants rely on *layers of intermediaries* instead of just one (layer of) intermediary to take funding from households and lend it out directly to firms, as envisioned by the classsic Diamond (1984). A long credit chain could lure unsophisticated household investors in for potentially wrongly perceived "safety;" but professional money market funds often invest on behalf of these households. Another often-mentioned explanation is regulatory arbitrage; under this view, a long financing chain is intentionally created to obscure certain financial activities conducted by financial institutions. The great body of empirical studies (Acharya et al., 2013; Karolyi and Taboada, 2015; Demyanyk and Loutskina, 2016) on regulatory arbitrage certainly lends support to this view, but it does not explain the rapid growth of the securitization market in the first place around the mid-1980's. In fact, there is evidence that securitization is best explained as contracting

[1] Neil Callanan and Silas Brown, "Banking Crisis Raises Concerns About Hidden Leverage in the System," *Bloomberg*, March 27, 2023, https://www.bloomberg.com/news/articles/2023-03-27/banking-crisis-raises-concerns-about-hidden-leverage-in-the-system?leadSource=uverify%20wall.

[2] Laurie DeMarco, Emily Liu, and Tim Schmidt-Eisenlohr, "Who Owns U.S. CLO Securities? An Update by Tranche," *Feds Notes*, June 25, 2020, https://www.federalreserve.gov/econres/notes/feds-notes/who-owns-us-clo-securities-an-update-by-tranche-20200625.html

[3] The increase in the credit intermediation index could also be due to intermediaries increasing their gross exposure to each other, which is unfortunately masked in the Flow of Funds data.

Figure 1: Credit Intermediation Index, 1960–2020



This figure plots the credit intermediation index, following the definition in Greenwood and Scharfstein (2013). It is calculated as the ratio of the total liability of all domestic sectors to the total liability of domestic nonfinancial sectors. Both series are obtained from the Flow of Funds at the annual level.

innovation instead of pure regulatory arbitrage (Calomiris and Mason, 2004).

We study the economics of credit chains by considering a dynamic model, in which a long-lived entrepreneur borrows from overlapping generations (OLG) of households. The impatient entrepreneur is endowed with a project that matures with certain probability each period and only produces cash flows upon maturity. Households, on the other hand, are born with endowments and live for two dates; and different from the impatient entrepreneur, they do not discount their consumption in the second date.

The relative impatience wedge implies that the (impatient) entrepreneur would like to pledge out future cash flows and borrows from (patient) households. Suppose the entrepreneur borrows from households directly. Because households are OLG, their trading in the secondary market needs to be facilitated by financial intermediaries. We call these intermediaries, who are impatient, "experts;" and we assign the same discount rate to experts as the entrepreneur. These experts facilitate liquidation and trading in the secondary market, a process that is costly due to the experts' impatience relative to households.

These experts are also managing funds, via which the entrepreneur can borrow (indirectly) from OLG households. These layers of funds are linked with each other with debt contracts, forming a credit chain with an endogenous chain length. To facilitate analysis, we focus on debt contracts that are with an exogenous contract maturity rate and debt face value; but each layer can adjust the interest rates to rollover its debt, taking as given other layers' contracts and households' strategies. When contracts mature, the borrower—whether the entrepreneur or an intermediary fund—needs to rollover its debt. Rollover fails when the cash-flow realization falls below an endogenous threshold which triggers defaults. Creditors liquidate this borrower's assets in the secondary market, where experts serve as buyers who then resell to the next cohort of households. In addition, households pay an exogenous (dead-weight) bankruptcy cost per layer.

We abstract away from modeling information frictions explicitly. This allows us to focus on the credit chain's role in facilitating maturity transformation, and underscores how our mechanism is distinct from the established literature on asset pooling and tranching (DeMarzo, 2004). Central to our mechanism are the transaction costs associated with secondary market trading and liquidation, facilitated by experts as financial intermediaries (He and Krishnamurthy, 2013; Brunnermeier and Sannikov, 2014). This stems from the fact that experts are impatient, but can also be viewed in the context of intermediaries' inventory costs.

Given the transaction cost in the secondary market on trading long-term securities, the entrepreneur could potentially borrow more from OLG households by using short-term debt (compared with long-term debt), if the entrepreneur can always rollover the debt. Short-term contracts are therefore preferred because they minimize the maturity mismatch between the OLG households and the financial assets they hold. Essentially, our model captures the growing appetite for money-like assets in recent decades, as well documented in Greenwood et al. (2015) and Carlson et al. (2016). However, when cash flows are uncertain, short-term debt exposes the borrowers to too much rollover risks, limiting their debt capacity.

We demonstrate that a credit chain can further increase the entrepreneur's borrowing capacity by reducing liquidation losses when rollover fails. Section 2 presents a three-period example to illustrate the key mechanism. When the entrepreneur borrows directly from households using one-period short-term debt, a negative date-1 interim shock forces the entrepreneur's *project*, which matures in two periods, to be liquidated. Now suppose the entrepreneur borrows (using two period long-term debt) from a fund who then borrows (using one-period short-term debt) from households, i.e., forming a credit chain. Following a date-1 interim negative fundamental shock, it is the fund's asset—which is the *long-term debt* issued by the entrepreneur maturing in one period—that is being liquidated. Because in our model long-term assets (the project) are more costly to liquidate than short-term assets (debt issued by entrepreneur), which arises endogenously due to OLG households and secondary market trading frictions, credit chains increase the entrepreneur's borrowing capacity. Fundamentally, it is because the layered credit chain structure helps preserve the subsequent short-term debt claims over the entrepreneur's project, potentially avoiding secondary market trading frictions in the continuation game if rollover in the future is successful.

In sum, our model features a stylized trade-off: the impatient entrepreneur would like to pledge out future cash flows, but the associated secondary market trading and liquidation loss will be high. By comparing the borrowing capacities induced by direct-borrowing and the layered chain structure, our example highlights that the layered credit chain structure helps shorten the maturity of assets if liquidated, thereby supplying more money-like securities. In this way, the credit chain structure reduces the tension between maximizing the cash flow pledged out and minimizing liquidation losses, just like what special purpose vehicles (SPVs) achieve in practice.

One of the key assumptions of our model is that debt issuance costs in the primary market are lower than i) liquidation costs and ii) secondary market transaction costs. Two points are noteworthy. First, the assumption of frictional secondary market trading and liquidation is common in the money and banking literature (say, Bryant (1980)), leading to demand for money (which corresponds to short-term debt in our model). Second, although ii) does not apply universally for all markets, it does hold for many instruments observed in the shadow banking system. For example, Friewald et al. (2017) document that the average secondary market transaction costs for asset-backed securities (ABS) and mortgage-backed securities (MBS) are 43 bps and 58 bps respectively. They are much higher compared with ABCP issuance costs, which are around 10 bps (Kacperczyk and Schnabl, 2010). As we later explain in Section 2.3, it is the correct comparison between the issuance cost of ABCP, which is the liability side of SPVs, and the transaction cost of ABS or MBS, which sits on the asset side of SPVs.[4] The SPVs that are issuing ABCPs are set up in order to streamline the process of debt rollovers, which minimizes issuance costs.[5]

Section 3 generalizes the three-period example (with two layers) to infinite periods and multiple layers, and Section 4 characterizes the equilibrium credit chain in our model. We show that the equilibrium contracts are both time-invariant and layer-independent. The time-invariance comes from the assumption that the fundamental is i.i.d. over time, while the competitiveness of intermediary funds delivers layer-independence. Both contract features are important for tractability, which allows us to study the equilibrium chain length analytically.

We show that the equilibrium chain length minimizes the interest payments given the borrowing amount. The benefit of borrowing via layers is best illustrated by considering the extreme case without exogenous dead-weight bankruptcy cost on layers; in such a case, the equilibrium chain length is infinity. Similar to the intuition in Section 2's example, a longer chain formed by more layers of financial intermediaries delivers shorter maturity assets during liquidation. Intermediating via credit chains meets the liquidity needs of the households and simultaneously reduces liquidation losses. Finally, Section 4.5 shows that

[4] ABS and MBS are with longer maturity than ABCP. But the correct comparison should be the per-time transaction cost and per-time issuance cost, regardless of the maturity of securities. By issuing short-term debt, investors and issuer save on the secondary market transaction cost but incur additional issuance cost.

[5] There is no role of monitoring performed by creditors in our model, which makes the model more applicable to cases such as MBS.

the equilibrium chain length emerged in a decentralized market, despite of various trading frictions in the network structure, is constrained efficient from the social perspective. This is because the fund in the last layer, which determines the equilibrium chain length, internalizes the trade-offs of longer chains through the interest rate it pays to the households.

As detailed in the literature review, our paper differs fundamentally from the literature of asset trading chains. Oftentimes, these papers focus on certain market frictions that prevent the asset seller (with a relatively low valuation) from directly selling to the first-best buyer (with the highest valuation), and an intermediary who holds the asset temporarily ensues. Our focus, instead, is on intermediation credit chains where one agent's liability is another agent's asset, which is missing in the literature of asset trading chains. Compared to the financial network literature, we study a simple network structure, i.e. chains, but focus on the endogenous formation of chains and the strategic interaction of agents in the chain.

Literature Review

Our paper belongs to a recent literature that studies the role and frictions of credit chains, motivated by the growing intermediation chain in the modern financial system, particularly in the shadow banking sector (Adrian and Shin, 2010; Adrian et al., 2012). Di Maggio and Tahbaz-Salehi (2017) study how the distribution of collateral along the credit chain matters for the intermediation capacity and systemic stability. In Donaldson and Micheler (2018), credit chains arise when banks rely more on non-resaleable debt, i.e., repos, which is similar to borrowing via layers in our setting. In both papers, liquidation losses are smaller in defaults when the borrowing is done via layers; the difference is that, instead of assuming exemption of automatic stay, we start with a common type of frictions and show that having a layer in the middle—by shortening the maturity of liquidated assets—endogenously results in smaller default losses. More recently, Glode and Opp (2021) and Mayer and Gryglewicz (Forthcoming) study the externality of economic agents' decisions in an exogenously given intermediation chain.[6]

There is a long literature on the theory of financial intermediation; we focus on the benefit of having multiple layers of intermediaries instead of just one. One-layer intermediation is the robust prediction in leading models in this field; for instance, Diamond (1984) shows that banks reduce monitoring cost through diversifying projects' idiosyncratic risks, which are absent in our model.

[6]Focusing on strategic debt renegotiation when agents are connected through liabilities in an exogenously given debt chain, Glode and Opp (2021) analyze externalities in renegotiation because even though bargaining is bilateral, it affects renegotiation outcomes in other parts of the chain. Relatedly, Mayer and Gryglewicz (Forthcoming) study the contracting problem where intermediaries are tasked with both designing contracts for the subsequent layer and exerting effort to monitor it; the incentive provision to one layer affects other layers as well.

Conceptually our paper is closer to Diamond and Rajan (2001). There, an intermediary is necessary—but a single layer is enough—as it has specific skill in collecting repayments from firms and can also commit to repaying its creditors by offering demand deposits. Like our paper, Diamond and Rajan (2001) micro-founds the continuation game after asset liquidation, and show that intermediaries increase recovery value if default happens. But inalienable human capital (of entrepreneurs/bankers), which is the backbone of Hart and Moore (1998) and Diamond and Rajan (2001), plays no role in this paper; instead, our mechanism rely on the households' short-term liquidity needs combined with secondary market trading frictions.

We built upon the literature on bank runs and instability of short-term debt (Diamond and Dybvig, 1983; Calomiris and Kahn, 1991; Goldstein and Pauzner, 2005; Acharya et al., 2011), by adopting a dynamic debt run setting akin to He and Xiong (2012).[7] We differ from the existing literature by studying runs in an endogenous multi-layer structure.

On the literature of network and contagion,[8] we focus on a simple form of network, i.e. chains, and endogenize both the contracts among layers as well as the length of the credit chain. In other words, we endogenize network formation within the simple chain network structure. Recently, Donaldson et al. (2022) show the usage of long-term debt in financial networks can be stabilizing, as banks hit by liquidity shocks can raise additional funding using interbank long-term debt as collateral and dilute existing long-term creditors. We rule out debt dilution and focus on maturity transformation along the credit chain.

In addition to credit chains, a recent literature has also investigated asset trading chains, where an asset is bought and re-sold by a sequence of dealers before it reaches the final buyer. Glode and Opp (2016) show trading via a sequence of moderately informed intermediaries can reduce allocation inefficiency caused by asymmetric information.[9] The literature has also examined the length and price dispersion of intermediation chains in an over-the-counter (OTC) market with search frictions (Atkeson et al., 2015; Hugonnier et al., 2019; Sambalaibat, 2021; Shen et al., 2021). Our focus is on credit chains where one agent's liability is

[7]Similar to Qi (1994), we consider an OLG setup where intergenerational transfers through financial institutions improves welfare but could lead to runs. The runs between layers in our model capture the repo market and commercial paper runs by institutional investors during the global financial crisis, which has been well documented (Gorton and Metrick, 2012; Copeland et al., 2014; Krishnamurthy et al., 2014; He and Manela, 2016; Schmidt et al., 2016).

[8]To name a few, Allen and Gale (2000) and Elliott et al. (2014) show how financial networks provide diversification and insurance against liquidity shocks, but on the other hand, lead to fragility and cascades of failures. A similar point is delivered by Acemoglu et al. (2015). Allen et al. (2012) also consider rollover risks of short-term debt in clustered structures, where banks share common assets.

[9]In a follow-up paper, Glode et al. (2019) show that A sufficient long intermediation chain can also eliminate trading inefficiencies caused by agents with monopoly power screening counterparties. In a general equilibrium context, a recent paper by He et al. (2023) studies the role of information technology and intermediation in an economy with asset origination and distribution.

another agent's asset, which is the key for "credit chains."[10]

2 An Example: Model Mechanism and Intuition

This section provides a simplified example to illustrate the key intuition of our paper.

2.1 Set-up

Consider a four-date-three-period setting $t = 0, 1, 2, 3$, with timeline given in Figure 2a. (This is the simplest setting to illustrate our mechanism). All agents are risk neutral.

Project and entrepreneur. There is a long-term project that produces cash flows $\tilde{y} \geq 0$ only at the end of period $t = 3$. Good news could arrive with probability $p \in (0, 1)$ in period $t = 1, 2$. If good news arrives in either period, then $\tilde{y} = 1$; otherwise, $\tilde{y} = 0$. The arrival of good news is independent across periods.

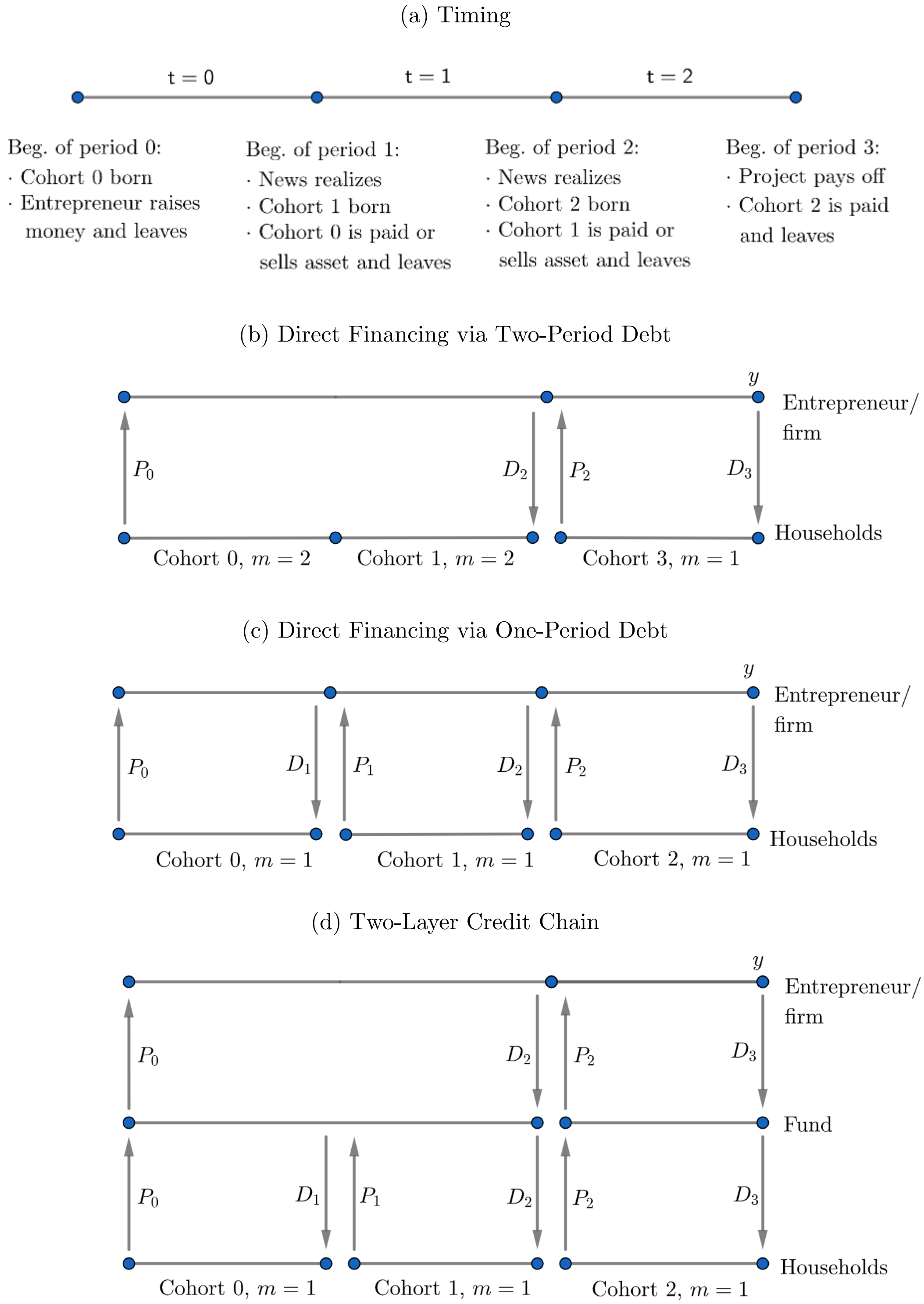
The project is owned by an impatient entrepreneur (he); entrepreneur and firm are used interchangeably in this section. In the example, for illustration purpose we take the extreme assumption that he leaves the economy at the end of period 0, implying that he maximizes the payment of cohort-0 households by pledging out his entire cash flows to households. We will relax this assumption in the main model.

OLG households. Cohort t households are born at the beginning of period t, endowed with 1 unit of consumption goods, and have access to a storage technology with zero net return. This cohort can consume $c_t^t > 0$ or invest in financial markets (storage technology or securities issued by the firm or funds, as explained shortly), and leave the economy in period $t + 1$ after consuming $c_{t+1}^t > 0$, with a utility of $c_t^t + c_{t+1}^t$. Importantly, there is no discount between periods.

Debt refinance/rollover and secondary market. We consider debt contracts with different maturities. In period t, if the contract has matured, then the entrepreneur/firm will refinance the debt payment to cohort-$(t - 1)$ households from cohort-t households. We call this event "rollover the debt," and throughout the paper we use the word "refinance" and "rollover" interchangeably. If refinance/rollover succeeds, there is no cost involved. Otherwise, the firm has to liquidate its asset at a discount, which is α_l fraction of next

[10] With a slightly broader interpretation, our model also sheds light on "rehypothecation," i.e., the reuse of collateral in secured financing transactions, which is also called "collateral chains" and is a widespread practice to enhance market functioning between banks and nonbanks (Infante and Saravay, 2020). As most repo transactions in the U.S. are conducted on an "outright" basis with complete ownership transfer at each leg, rehypothecation in a collateral chain is closer to asset trading chains.

Figure 2: Illustration of the Example



This figure illustrates the timing and different financing structures in Section 2. Panel (a) illustrates timing. Panel (b) illustrates the flow of money in Case 1: direct financing using a two-period debt followed by a one-period debt. Panel (c) illustrates the flow of money in Case 2: direct financing using only one-period debt. Panel (d) illustrates the flow of money in Case 3: two-layer financing.

cohort's valuation of the asset. The micro-foundation is that the firm has to sell the asset first to experts (with specialty in dealing with distressed assets), who then sell it to the cohort-t households. Suppose that experts have discount rate $\alpha_l \in (0,1)$, implying that the proceeds received by the departing cohort is α_l fraction of cohort-t's endogenous valuation in equilibrium. We rule out debt renegotiation, as in many cases, the debt is held by a dispersed set of households and the renegotiation cost is too high.

If instead the contract has not matured yet, the existing households (the $t-1$ cohort) can sell it to a specialized intermediary sector, who then sells the securities to cohort-t at the end of period t. The intermediary has a discount rate of $\alpha_s \in (0,1)$, implying that if cohort-t is willing to pay 1 unit for the security, cohort-$(t-1)$ can only receive α_s units. If short term investor holds long term asset, then this transaction cost α_s will be incurred repeatedly (Amihud and Mendelson, 1986).

Conceptually, assets being liquidated are eventually sold on the secondary market, implying a tight connection between these two discount factors. However, empirically, they often differ; we allow $\alpha_s \neq \alpha_l$ in the example to show that our mechanism does not rely on the relative magnitudes of the two. In the main model (Section 3), we set $\alpha_l = \alpha_s = \alpha \in (0,1)$.

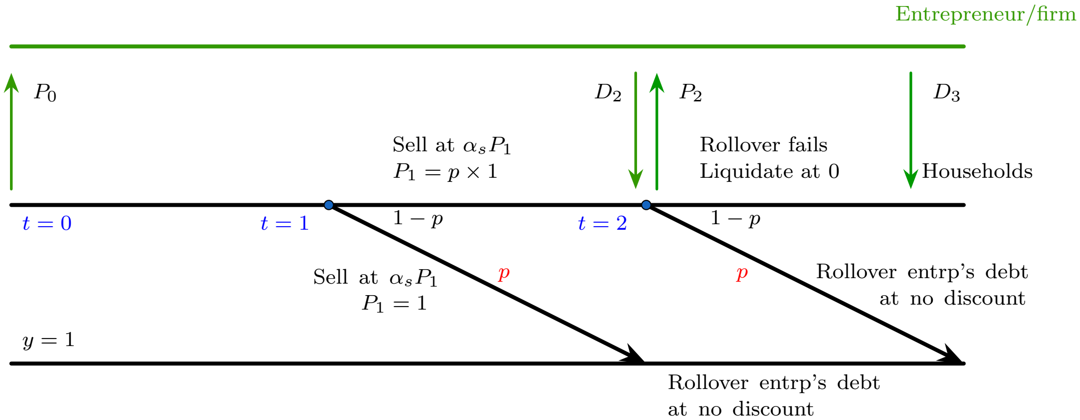
Two-layer financing structure with intermediary fund. We depart from the existing literature by studying a two-layer financing structure. Other than issuing debts directly to households, the firm can also adopt a two-layer financing structure where the entrepreneur issues long-term debt to an intermediary fund, who then finances itself by issuing one-period debt to OLG households. (The intuition of how one layer adds value is new to the literature, and importantly carries through to multiple layers so that two layers are better than one layer.) When rollover fails, either at the fund layer or the firm layer, the corresponding creditors liquidates their debt holdings issued by the layer above.

2.2 Comparison of Financing Structures

To illustrate the model mechanism, we provide a numeric example with $\alpha_l = 0.5$, $\alpha_s = 0.8$ and $p = 0.6$. We assume that all debt contracts are with zero coupon and a face value (denoted by D_t if the debt matures in period t) of 1; Appendix A verifies that they indeed are optimal in our numerical example, thanks to the binary distribution of cash-flows and that entrepreneur maximizes period 0 proceeds. We denote the market price of the debt at time t by P_t.

As a benchmark case, the entrepreneur issues a three-period debt to cohort-0 households, who then sell it to future cohorts later. Due to repeated transaction costs, the entrepreneur here is only able to raise $P_0 = 0.538$. (We leave the detailed calculation to Appendix A.) In

Figure 3: Illustration of Case 1: Long-Term Two-Period Debt



This figure illustrates the cash flow exchanges between the households and the entrepreneur in Case 1.

the rest of this section, we compare three financing structures: direct financing using two-period debt, one-period debt, and a two-layer credit chain. Our discussion focuses on why the two-layer financial intermediation can increase the entrepreneur's borrowing capacity, though the comparison between long-term contracts and short-term debt is also useful in delivering the intuition.

Case 1: Long-Term Two-Period Debt The entrepreneur first issues to households a two-period debt that matures in $t = 2$ with face value D_2. Given a three-period project, the entrepreneur then issues another one-period debt from $t = 2$ to 3, with face value D_3. Figure 2b illustrates this direct financing structure, with cash flows shown in Figure 3.

We work backwards. At $t = 2$, the entrepreneur can raise $P_2 = 1$ if good news has arrived, otherwise $P_2 = 0$. Given period 2 debt has face value 1,[11] following good news a successful rollover delivers 1 to cohort-1 households. (There is no discount applied in the good state, which contributes to Case 1's advantage over the case with three-period debt.) If no good news has arrived, the entrepreneur is forced into liquidation with liquidation value equal to 0.

At the beginning of period 1, following good news cohort-1 households know that they receive $D_2 = 1$ for sure at $t = 2$; otherwise, they receive $D_2 = 1$ with probability $p = 0.6$ at $t = 2$. Hence cohort-1's valuation for debt is $0.6 \times 1 + 0.4 \times 0.6 \times 1 = 0.84$. As a result, in period 1, departing cohort-0 households sell the debt contract to cohort-1 (via

[11] As explained in Appendix A this maximizes the payout to cohort-1 households (and in turn, that to cohort-0 households). In general, setting a lower face value could help avoid liquidation; in our binary cash-flow case, however, liquidation is not costly because the expected output is zero anyway in the liquidation state in period 2.

intermediaries) with a discount rate of $\alpha_s = 0.8$, at price of

$$P_0(\text{two-period debt}) = 0.8 \times 0.84 = 0.672. \tag{1}$$

This dominates 0.538 the case when issuing three-period debt. Unlike the three-period debt case, not all cash flows here are discounted by α_s: when rollover is successful in period 1, the debt payment flowing to cohort-2 does not involve any discount.

Case 2: Short-Term One-Period Debt This direct financing structure is illustrated in Figure 2c, with cash flows shown in Figure 4. The calculation of $t = 2$ is the same as before: we have $P_2 = D_2 = 1$ if good news has arrived. At $t = 1$, P_1 can be calculated as the expected payment at $t = 2$. If good news arrives in period 1, then $P_1 = 1$, otherwise the expected payment is 0.6. Again, rollover is successful only in the good state, where the departing cohort-0 receives the full face value $D_1 = 1$. In the bad state, the firm is forced into liquidation, with a liquidation value of α_l times the $t = 1$ market value of the project P_1.[12] But this market value P_1 equals the secondary market discount α_s multiplied by the expected cash flow 0.6; therefore the liquidation value at the bad state of $t = 1$ is $0.5 \times 0.8 \times 0.6 = 0.24$. (Note, when liquidation happens in period 1, the subsequent short-term debt claims in period 2 are destroyed.) The project gets traded repeatedly in the secondary market as a long-term asset, which is why we multiply by 0.8 to obtain 0.24. We then calculate the $t = 0$ price of D_1 to be:

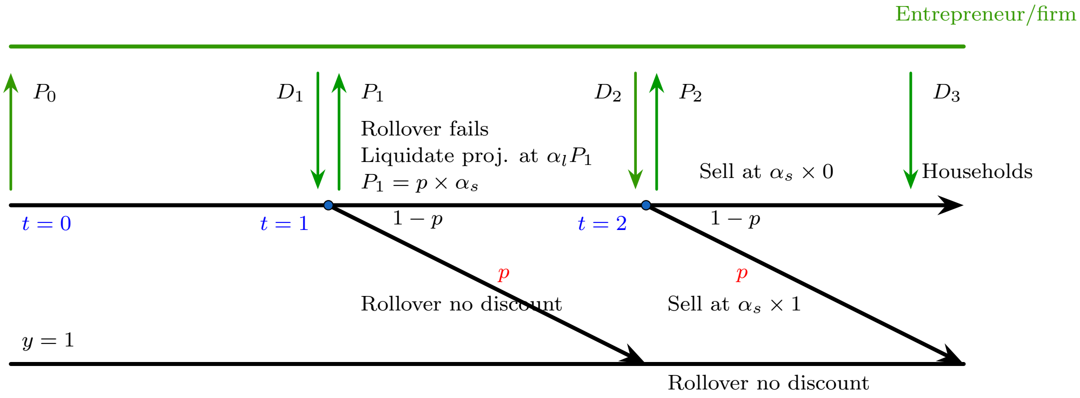
$$P_0\,(\text{one-period debt}) = 0.6 \times 1 + 0.4 \times 0.24 = 0.696. \tag{2}$$

By issuing short-term debt, the entrepreneur can raise 0.696 in period 0. This is larger than the 0.672 in (1) raised by two-period debt (and then another one-period debt).

In this example, if good news arrives in period 1, short-term debts transfer wealth across cohorts in the most efficient way. But given bad news in period 1, short-term debt results in liquidating the project, destroying the subsequent short-term claims. Here we implicitly assume that households cannot issue short-term debt in period 2, since it is generally difficult for households to issue debt against real assets. Households may not have the expertise or resources to issue financial claims against the project. In the dynamic model in Section 3, we relax this assumption and allow issuances of short-term debt even after liquidation, by considering chain restoration. However, it is important that there is some friction such that chain restoration gets delayed by one period with positive probability.

[12] With face value $D_1 = 1$, the short-term debt holders have the full claim over the entire project. Setting a significantly lower face value (D_1) could help avoid liquidation in the bad state, but the payout of this riskless debt is too small.

Figure 4: Illustration of Case 2: Short-Term Debt



This figure illustrates the cash flow exchanges between the households and the firm in Case 2.

Case 3: Two-Layer Credit Chain We now show that the two-layer structure can reduce the liquidation losses in the absence of good news in period 1. In this case, the entrepreneur issues a two-period debt—at a $t = 0$ price P_0—to the fund from period 0 to period 2, with a face value $D_2 = 1$. The fund then issues one period debt to households, with face value $D_1 = 1$ in period 1 and $D_2 = 1$ in period 2. This structure, which combines Case 1 and Case 2, is illustrated in Figure 2d, with cash flows shown in Figure 5. In this example, the intermediate layer also features a maturity transformation, i.e., the debt contract between the entrepreneur and the fund is longer than the one between the fund and households.

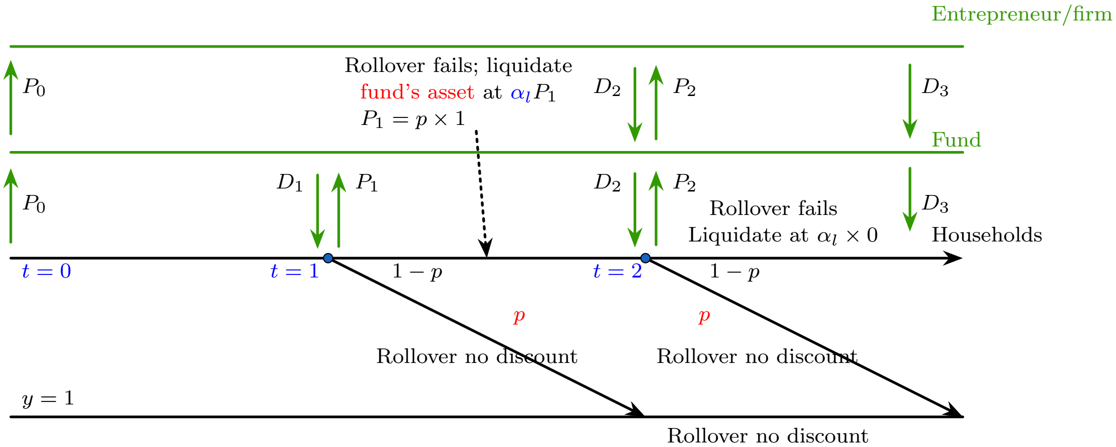
The calculation of $t = 2$ is the same as before: $P_2 = D_2 = 1$ if good news has arrived, otherwise $P_2 = 0$. Similarly, $D_1 = 1$ and rollover is only successful if good news arrives in period 1. When rollover fails in period 1, the fund's asset, which is a debt claim with face value D_2 over the project, is liquidated at the secondary market.[13] The value of that claim is $0.6 \times 1 = 0.6$, hence the liquidation proceeds is $0.5 \times 0.6 = 0.3$. Therefore, P_0 which equals the expected payment to be received in period 1 is

$$P_0\,(\text{two-layer}) = 0.6 \times 1 + 0.4 \times 0.3 = 0.72. \tag{3}$$

This is larger than 0.696 in (2) raised using one-period debt in Case 2.

[13]In Case 2, it is as if the entire project gets liquidated in this scenario; there we rule out the possibility that the entrepreneur sells its liquidated asset to funds, which we allow in the formal model.

Figure 5: Illustration of Case 3: Two-Layer Credit Chain



This figure illustrates the cash flow exchanges between the households, the funds and the firm in Case 3.

2.3 Intuition

Compared with long-term contracts (two-period debt in Case 1), the benefit of issuing short-term debt (in Case 2) comes from the fact that successfully rolling over debt avoids transaction costs in the secondary market. Appendix A shows the difference between Case 2 and Case 1 as:

$$P_0\,(\text{one-period}) - P_0\,(\text{two-period}) = \underbrace{p}_{g}(1-\alpha_s) - \underbrace{(1-p)p}_{bg}\alpha_s(1-\alpha_l). \tag{4}$$

In Eq. (4), the first term captures the benefits of short-term debt: if rollover is successful in period 1 (good news), then there is no discount applied to the final cash flow for one-period debts. The second term captures the cost of short-term debt: if rollover fails in period 1 (bad news), then the entrepreneur's asset has to be liquidated, even if good news eventually arrives (with probability p in period 2); this corresponds to the event "bg," which occurs with probability $(1-p)p$. In this sample path, for one-period debt, α_l (α_s) is applied in the first (second) period; while for two-period debt, if rollover is successful in period 2, then only α_s is applied in $t = 1$ (i.e., α_l at $t = 2$ can be avoided). The term $1 - \alpha_l$ in the second term in Eq. (4) hence captures this difference. Overall, the benefit of short-term debt dominates the cost under our parameterization. The benefit of short-debt over three-period debt is even larger, with a similar mechanism.

We move on to the main result of the paper. The difference between the short-debt case and the two-layer case comes from the fact that liquidating the entrepreneur's asset, which is the project with a two-period remaining maturity, is more costly than liquidating the fund's asset, which is the one-period debt backed by the firm. One can show that

$$P_0\,(\text{two-layer}) - P_0\,(\text{one-period debt}) = \underbrace{(1-p)p}_{bg}\,\alpha_l(1-\alpha_s). \tag{5}$$

As shown in Eq. (5), the difference lies in the "bg" event, which happens with probability $(1-p)p$. In this event, rollover fails in $t=1$ for both cases. So there is an liquidation discount α_l common to both cases. In the short-debt case, the entrepreneur fails to rollover, and the project is liquidated; this implies that the project's final cash flows will be discounted in both period 1 and 2 no matter what happens in the subsequent period $t=2$. In contrast, in the two-layer case, after rollover failure, it is the fund's asset—a one-period debt backed by the firm—that is being liquidated. There, if good news arrive at $t=2$, because the two-layer structure preserves subsequent short-term debt claims, the firm can still successfully rollover the short-term debt without secondary market trading, saving the $t=2$ trading cost which is captured by the term $(1-\alpha_s)$ in Eq. (5). Compared with the previous cases, two-layer financing structure provides the benefit of short-term debt yet avoids the additional liquidation losses in the one-period debt, delivering an endogenously smaller default cost.[14] This benefit of adding layers, which is to reduce liquidation costs, extends to multiple layers.

This numerical example also helps deliver a slightly more general intuition that this paper aims to deliver. Thanks to OLG households and secondary market trading frictions, our model generates a key result that default costs are endogenously increasing in the maturity of liquidating assets. Given a higher liquidating cost of longer-term assets, it is better to issue short-term debt against two-period asset, as in Case 3, than to issue short-term debt against three-period asset, as in Case 2. The relation between default cost and asset maturity can be micro-founded in many ways; but as long as longer-term assets have larger liquidation costs, which is often empirically the case, then there is a benefit for the layered structure with credit chains. Finally, debt with state-contingent maturity could achieve similar outcomes as the layered structure. We explain the comparison in details in Appendix A.

As evident from Eq. (5), our key mechanism works as long as $\alpha_s \in (0,1)$. What we really need is that both secondary market transactions and liquidation processes are more frictional than debt issuance/rollover, which we have assumed to be costless in this paper. This is reasonable in the context of SPVs. The secondary market transaction costs for the

[14]We do not focus on the comparison between the two-period debt and the two-layer case. This is because relative to the two-period debt, the two-layer credit chain in this example (which is formed with one-period debt) shortens the maturity that households face, in addition to adding a layer. The comparison will be confounded by the effect of shorter maturity.

securities that SPVs hold, such as MBS and ABS, are around 50 bps, whereas the issuance cost for shorter term debt is around 10 bps (Kacperczyk and Schnabl, 2010). These vehicles are purposefully set up to minimize the debt rollover costs. Note, the relevant comparison is indeed the per-time issuance cost, which has been normalized to zero in our model, and the per-time transaction cost, which is α_s in our model.[15]

To summarize, our paper highlights a key trade-off that is new to the literature. The impatient entrepreneur would like to pledge out as many future cash flows as possible at $t = 0$, but the associated secondary market liquidation losses will be high. The credit chain structure, just like special purpose vehicles (SPVs) that we observe in the practice, supplies more money-like securities by helping insulate interim negative fundamental shocks and protect the underlying real firms from heavy discounts.

2.4 Connection to the Main Model

Since our mechanism does not rely on the relative magnitudes of secondary market friction α_s versus liquidation friction α_l (see Eq. (5)), we assume both are equal to α in our full model. Furthermore, the final cash-flow at the final period $t = 3$ is similar to the cash-flow structure in the full model, except that there we assume a Poisson arrival of cash-flow to keep the environment stationary.

More importantly, the intuition revealed in the example carries to the main dynamic model featuring general credit chains with L layers. We generalize the numerical example in this section to a project that matures in L periods in Appendix B. The benefit of credit chains illustrated in Section 2.3 suggests that the optimal financing structure features an $(L-1)$-layer credit chain, where layer-ℓ holds debt with maturity $L-\ell$ and issue debt with maturity $L-\ell-1$. In other words, every layer bears some maturity mismatch and the maturity of debt held by layer-ℓ decreases as ℓ increases. This conjecture is formally proven in Appendix B. Clearly, collapsing the $(L-1)$ layers to one-layer does not yield the same result, and this is a key difference from the classic literature on financial intermediation (Diamond, 1984).

In the remaining parts of the paper, we consider a dynamic model to study the connection between financial stability and credit chain length. Both the run probability and credit chain length are endogenously determined. Since deterministic debt maturities lead to intractability in a dynamic setting with Poisson cash-flows, we instead assume that each layer's debt contract matures at some Poisson event, and also matures if above-layers' debts ma-

[15] If a short-term debt is issued instead of a long-term debt, then in the period when the short-term debt matures, one saves on the secondary market transaction cost but incurs additional issuance cost in order to rollover debt.

ture. As we will show, this random maturity setup is much more tractable, while generating similar maturity structure (e.g., the layer-ℓ' debt maturity decreases in ℓ) and same economic mechanisms as in the example.[16] In addition, this setup with Poisson maturity probability and continuous cash-flow distribution allows us to study endogenous run probabilities and related comparative statics in a relatively tractable way.

3 The Model

We consider a discrete-time dynamic model with three types of risk-neutral agents: OLG households, a long-lived entrepreneur, and a group of long-lived experts. After presenting each ingredient in our model, we write down the optimization problem for each fund in different layers in the credit chain. We then define the equilibrium formally in this economy.

3.1 The Setting

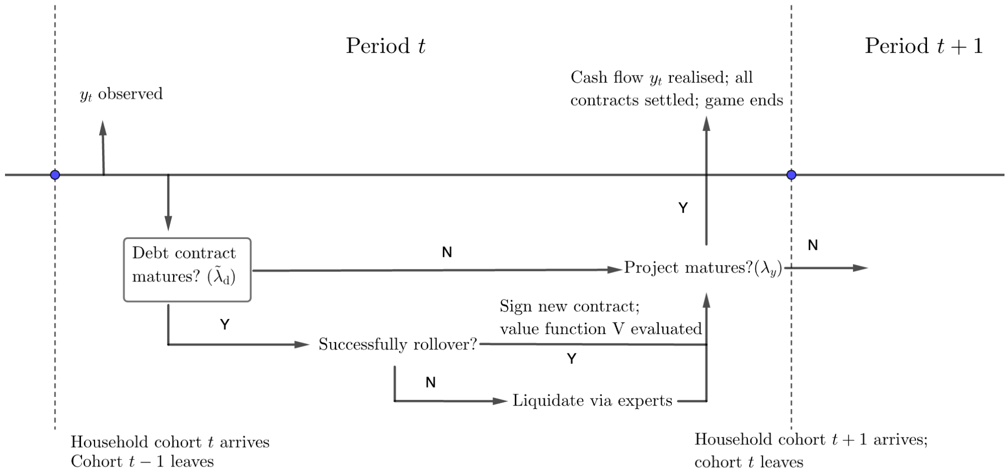
Endowment and agents. A long-lived entrepreneur with a discount rate $\alpha \in (0, 1)$ (hereafter he) has a long term project that produces nothing before maturity. The timeline within each period $t > 0$ is as follows. At the beginning of the period, the public "news" on the (potential) cash-flow $y_t \geq 0$ arrives; we assume y_t is i.i.d. across periods, with $H(\cdot)$ denoting the cumulative distribution function (CDF) and $h(\cdot)$ the corresponding probability density function (PDF). During the period, the project matures with probability $\lambda_y \in (0, 1)$, in which event the project delivers y_t units of consumption good at the end of the period and the game ends. (We will explain the timing in more detail shortly.)

There are OLG households in this economy. Cohort-t is born at the beginning of period t and leaves the economy at the beginning of period $t+1$. Each cohort consists of a measure 1 of representative households, who are endowed with $e > 0$ units of consumption good when born. They can choose to consume c_t^t in period t or invest in the securities issued by the firm or funds, and consume c_{t+1}^t in period $t+1$ (and then leave the economy). Household's utility is $c_t^t + c_{t+1}^t$.

There is a financial intermediary sector which consists of a group of "experts." In contrast to OLG households, each expert (hereafter she) is long lived with a discount rate $\alpha \in (0, 1)$. For simplicity we take the experts' discount rate to be the same as that of the entrepreneur's. In our model, experts can serve different roles in the financial market. They

[16] Suppose that each debt will mature in each period with an exogenous probability λ_d, for all layers. Then effectively, layer-ℓ holds debt that matures with probability $1 - (1 - \lambda_d)^\ell$ and issues debt that matures with probability $1 - (1 - \lambda_d)^{\ell+1}$. Similar to the deterministic debt maturity case, each layer's asset has longer maturity than its liability side, and layer-ℓ's debt matures with higher probability as ℓ increases.

Figure 6: Timing



This figure illustrates the timing of events in each period for Section 3.

can operate some funds who raise financing from households and in turn provide credit to the entrepreneur; serve as market makers to facilitate the secondary market trading between OLG households; or run distress funds who purchase liquidated assets and resell in the secondary market. There are many interpretations for their discount rate α besides their opportunity costs of time. For instance, following He and Krishnamurthy (2012, 2013), experts need to commit certain equity capital to operate the distressed funds, which is costly.

The fact that households are more patient (with a discount rate 1) than the entrepreneur and experts (with discount rate $\alpha < 1$) implies that in our model the gain of trade comes from financing from households. Just as illustrated in Section 2, the key is how to sell the project's cash flows from the hands of relatively impatient entrepreneur to the patient but OLG households.

Timing. As shown in Figure 6, at the beginning of each period, everyone learns the value of y_t first; then whether debt contracts mature or not. Cohort-t households are then born, and after that, cohort-$(t-1)$ households (who receive the debt payment or liquidation value) leave the economy. At the end of each period, whether the project matures or not is realized. We denote by $\mathcal{F}_t$ the information set at the end of period t.

Debt contracts. Financing contracts in our model are restricted to the class of "debt"-like contracts. Let T be the contract termination time (either project or debt matures, which is a stopping time measurable to $\mathcal{F}_t$). Any "debt"-like contract needs to specify i) debt maturity; ii) promised payment upon debt maturity; and iii) promised payment upon project maturity. In our main analysis, we take the first two as exogenous—that is, debt matures with some maturity parameter(s) $\tilde{\lambda}_d$, and the promised payment equals the households endowment e—while focusing on the third to analyze endogenous rollover decisions.

Specifically, our debt contract takes the form of $\{F_{y,s}\}_{s=t}^{T}$, which specifies the following promised future payments from the debtor to the creditor (w.p. stands for with probability):

$$\min(F_{y,s}, y_s) \cdot \mathbf{1}_{\text{project matures at period } s, \text{ w.p. } \lambda_y} + e \cdot \mathbf{1}_{\text{debt matures at period } s+1, \text{ w.p. } \tilde{\lambda}_d}. \tag{6}$$

where $\{F_{y,s}\}$ is $\mathcal{F}_{s-1}$-measurable for any $s \geq t$; in words, the payment upon project maturity takes the form of a debt contract $\min(F_{y,s}, y_s)$. In Eq. (6), one can interpret $F_{y,s}$ as interest payment each period (upon project maturity) and e as the face value to be paid (upon debt maturity); and we will soon explain that it is "debtness" of face value e—rather than the "debtness" of interest payment $F_{y,s}$—that drives our result.

Denote by π_t the sequence of interest payments $\{F_{y,s}\}_{s=t}^{T}$, which lies in the space of $\Pi \equiv \mathbb{R}_+^{T-t+1}$. Each period t, all funds (and the entrepreneur) can choose $\pi_t \in \Pi$ if their existing debt contracts mature. A new debt contract is signed after the existing debt matures with y's information in hand, but before knowing whether project matures; see Figure 6.

It is worth pausing to discuss the restrictiveness of our contracting space. First, we focus on credit chain length and therefore leave endogenous debt maturity choice to future research.[17] We assume the maturity rate of contracts issued to intermediary funds (from the entrepreneur to funds or between funds) is $\tilde{\lambda}_d = \lambda_d \in (0, 1)$, and the maturity rate of contracts issued to households is $\tilde{\lambda}_d = 1$, where "maturity rate" refers to the probability of debt matures within each period. When rollover is successful, the OLG households are always holding the shortest-term debt (which is one, as their debt always matures within a period).[18] Importantly, this maturity structure fixes the total maturity transformation in the system, regardless of the number of layers. In the NBER working paper version (w29632) of this paper (He and Li, 2022), we consider $\tilde{\lambda}_d = \lambda_d$ for all layers, including the households. The main mechanism is the same.

Second, to keep the contract space simple, in the main text we exogenously fix the debt face value at the household endowment e. Appendix D endogenizes the sequence of face values and derive condition under which they indeed binds at e. Intuitively, the benefit of a

[17]For models with endogenous debt maturity structure, see He and Milbradt (2016) and Hu et al. (2021).

[18]However, it is possible that on equilibrium path, conditional on rollover failure households are holding liquidated assets (i.e., debt issued by some funds) that do not mature every period. For details, see the discussion on liquidation value toward end of this section.

larger face value is due to the discount rate wedge between the entrepreneur and households, whereas the cost comes from the possibility of rollover failures in the future. Because the face value cannot exceed the households endowment e due to resource constraint, when e is sufficiently low, the benefit of having a larger face value outweighs the cost, in which case the optimal face value binds at e.

Third, the "debt" form of interest payment upon project maturity is not essential, and one can show that this is indeed the optimal contract under a weaker assumption of limited liability. To see this, consider a more general $\mathcal{F}_s$-measurable interest payment $\hat{F}_{y,s}$, which satisfies limited liability $\hat{F}_{y,s} \leq y_s$. Suppose that debt matures in period s, which means the debtor needs to repay e. If y_s is sufficiently low, then $\hat{F}_{y,s} \leq y_s$ is constrained to be low and the debtor may not be able to raise enough funding from the market to rollover its debt. As we will see, inefficient liquidation caused by rigid debt payment only occurs after a debt contract matures (and when the y_t is sufficiently low), while the game ends without inefficient liquidation after the project matures. Put it differently, it is the "debtness" of the promised payment e upon maturity, not the "debtness" of $F_{y,s}(y_s)$, that drives our result. Nevertheless, the simple debt form on interest payments allows us to make a sharper claim, as we show that in equilibrium $\{F_{y,s}\} = F_y^*$ is stationary (for all layers). Or equivalently, in equilibrium, the optimal $\mathcal{F}_s$-measurable interest payment is $\hat{F}_{y,s} = \min(F_y^*, y_s)$.

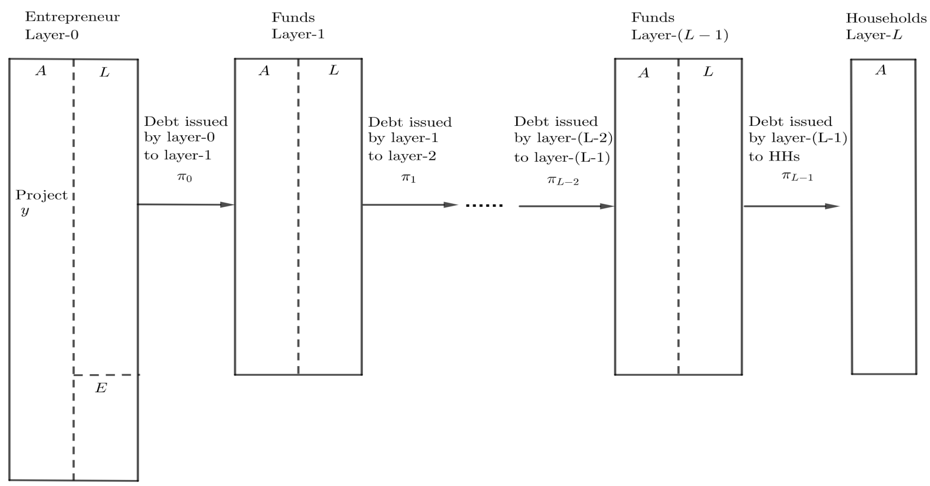
Without loss of generality we focus on the class of issue-at-par debt contracts, i.e., their market values at issuance equal their face value e. We further impose Assumption 1.

Assumption 1 *Issuers cannot raise new debt before existing debt matures. However, issuers can prepay their existing debt anytime.*

First, we rule out dilution by preventing issuers (the entrepreneur or funds) from raise new debt before their existing debt is repaid. Second, we allow debtors, after knowing the realization of y_t, to renegotiate by "prepaying" the debt contract. Effectively, in our model creditors have the option of unilaterally triggering the debt to "mature," so that they pay the lender e and eliminate all future obligations. Under this assumption, without loss of generality we can focus on renegotiation proof contracts; as we show, this implies "stationarity" so that the optimal debt contract chosen at any period along the equilibrium path is independent of history.[19] We suppress the time t index from now on, unless necessary.

[19]Because of the stationary structure of the fundamental (i.e., y_t's are i.i.d.), the optimal debt contracts would have been stationary if we assume debt contracts to be short-term. Essentially, the prepayment option (of the lenders) is the minimum element to guarantee the stationarity of optimal contracting in our model. It is also worth emphasizing that this prepayment option, which is about the debt itself, differs from "the prepayment clauses" introduced shortly, which are regarding prepayments triggered by events along the credit chain.

Figure 7: Credit Chains



This figure illustrates the structure of the credit chain. Layer-0 is the entrepreneur, holding the project on the asset side and issuing debt contract π_0 to layer-1 funds. Funds in layer-1 hold the debt issued by layer-0 on the asset side, and issue debt contract π_1 to layer-2. The households hold debt contract π_{L-1} issued by the last layer of funds, layer-$(L-1)$.

Credit chain and prepayment clauses along the chain. The model starts with the entrepreneur who owns the project issues debt at period 0 to household creditors via a credit chain. See Figure 7 for an illustration.

Consider a credit chain with length L, and a fund in the chain is indexed by its position l, where $0 < l < L$. A fund in layer l borrows from layer $l+1$ using a debt contract $\pi_l = \{F_{y,l}\}$, which prespecifies promised payment when the project matures at layer l. We refer to 0-layer fund of a credit chain as the entrepreneur with real project—the ultimate borrower, and L-layer as the households—the ultimate lenders. And, we call funds that sit at layer $i < l$ ($i > l$) to be the upper (lower) layers of fund l.

Credit chain features contracting externality. To facilitate analysis, we impose Assumption 2, i.e., "prepayment" clauses regarding other players in the chain.

Assumption 2 *When the project matures, all the debt contracts mature; when debt claim issued by layer l matures, all the debt claim issued by layer l' ($\forall l' \geq l$) matures.*

First, when the real project matures, the creditors of layer l get paid by $F_{y,l}$ and the game ends. Due to limited liability, we have

$$F_{y,l} \leq F_{y,l-1} \qquad \text{for} \quad \forall 1 \leq l \leq L, \tag{7}$$

and hence this payment trickles down to households. (In equilibrium $F_{y,l} = F_{y,l-1}$.) Second, when $l+1$'s debt claim issued by l matures, all debts issued by lower layers $i \geq l+1$ mature; therefore the payment from $l+1$—whether l makes it full or gets liquidated—will trickle down to the ultimate departing household creditors.[20]

Finally, it follows from the prepayment clauses that if multiple contracts mature, only the one with the highest layer (the smallest layer number) matters. To simplify expression, we refer to the scenario that "either the project matures, or any debt contract issued by any fund $i \in \{1, \cdots, l-1\}$ matures" simply as that "layers above l mature."

Credit chain, debt rollover, and secondary market. We have explained the payment flow along the credit chain following a debt maturing event in a fund l. Now consider a borrower fund l who needs to refinance/rollover its debt contract (so that contractual payments can ensue as described above).

We call rollover successful when the fund l is able to raise enough money in the market to pay back e to fund $l+1$. In equilibrium, successful rollover occurs when y exceeds above certain endogenous threshold $\hat{y}_l$. Due to prepayment clauses, all debt between layer l and the households matures. When rollover is successful, fund l can use the proceeds raised from new-born households to pay back e, so that all funds below layer l and the departing households are paid in full. Since the optimal chain length does not change, they can renegotiate and form a new credit chain with the optimal length of L.[21]

Otherwise, when $y < \hat{y}$, rollover fails. Creditors take over and liquidate the asset held by fund l, which could be the real project, or the debt issued by fund $l-1$. The liquidation occurs on the secondary market where the (direct) buyers are experts (who run distressed funds), who then sell this asset to the next cohort of households at a price $B_l(y, L)$.

Similar to He and Xiong (2012), there is strategic complementarity among different cohorts of households' rollover decisions. If future cohorts are more likely to rollover their debt, the current cohort of households are less likely to face liquidation, and they will be more willing to rollover their debt as well. In other words, there are runs in equilibrium, and we refer to the probability of rollover failures as the run probability.

In the case of liquidation, we assume that with probability $\beta \in [0, 1]$, the chain is restored immediately, in which case the next cohort values the debt at $V_L(L)$. With probability $1-\beta$, the households need to hold the debt issued by layer-l for one period, and the chain length

[20]Although our analysis takes this "prepayment" clause as given, we conjecture that this will be the outcome of optimal contracting, as it facilitates the payment directly to departing households as soon as possible, avoiding secondary market transaction costs (to be introduced shortly).

[21]There are many different ways to implement the same outcome, as essentially in this arrangement departing households receive the payment e financed by new-born households. For instance, all funds can simply ask their corresponding lender funds for rollover. In the final layer, the new-born households simply replace departing households. The credit chain stays exactly the same going forward.

is restored to its optimal level in the following period absent another run. We essentially need some bankruptcy cost, and a probabilistic delay of chain length restoration is perhaps the simplest way to capture this inefficiency.[22]

We further assume that there is a restructuring/legal cost $c \geq 0$ for each layer during bankruptcy. To summarize, the direct creditor fund $l+1$ recovers $\min(\alpha B_l(y, L), e)$ from the liquidation of fund l's asset (intermediated by experts), where the liquidation value $B_l(y, L)$ is endogenously determined in equilibrium. This payment then trickles down to departing households who receive

$$\min(\alpha B_l(y, L), e) - c \cdot (L - l).$$

Finally, in the case when layer-0 (the firm operated by the entrepreneur) fails to rollover its debt, bankruptcy occurs, but the expert can locate the original entrepreneur who have the most project-specific human capital to continue running the project (so the original chain is restored and the economy is stationary), just like in Diamond and Rajan (2000). This ensures that the private loss in a bankruptcy is the same as the social loss.

3.2 Value Functions and Bellman Equation

Recall that each period, the layer-l fund sets its contract (denoted by π_l), take the project fundamental (y), the total chain length (L), and the contract from the layer above (π_{l-1}) as given. Denote the value function of layer l fund by $V_l(y, \pi_l; \pi_{l-1}, L)$; this is evaluated after debt maturity but before the project maturity (in Figure 6). For layer-0, the entrepreneur's value function only depends on y, π_0 and L. Denote the market price of the debt issued by layer-l ($1 \leq l \leq L-1$) under contract π_l by $P_l(\pi_l, y; \pi_{l-1}, L)$. We may write the price of the debt and the value function simply as $P_l(y)$ and $V_l(y)$ whenever there is no risk of confusion.

Finally, for notational convenience, we denote $\tilde{F}_{y,l} = \min(F_{y,l}, y)$. We also denote by m_l the probability that layer l's asset does not mature:

$$m_l \equiv (1 - \lambda_d)^l \quad \text{for} \quad 0 \leq l \leq L - 1 \tag{8}$$

which satisfies $1 - m_{l+1} = 1 - m_l + m_l \lambda_d$. Since debt held by households always matures (recall $\tilde{\lambda}_d = 1$ for households), we can define $m_L \equiv 0$.

[22] Two points are worth making. First, recall in our example in Section 2, no restoration is allowed (i.e., $\beta = 0$; once an asset is liquidated, it is traded repeatedly among the households and households cannot issue short-term financial claims against the project). Second, our mechanism goes through in another stationary setting where restoration occurs with a constant probability each period, instead of restoration after one period.

3.2.1 Fund managers

For $0 < l < L$, we calculate layer-l's payoff in period 0 to be

$$P_l(\pi_l, y; \pi_{l-1}, L) - P_{l-1}(y) + V_l(y, \pi_l; \pi_{l-1}, L). \tag{9}$$

Here, layer-l issues its debt π_l for a proceed of P_l, and then purchases the debt from layer-$(l-1)$ at a price of P_{l-1}, where P_l and P_{l-1} are the market prices of the underlying debt. The last term captures its continuation payoff.

In subsequent periods, if the debt issued by layer-l ($l < L$) matures, then it needs to refinance its debt to repay e. If $P_l - e \geq 0$, successful rollover implies a layer-l's value to be

$$P_l(\pi_l, y; \pi_{l-1}, L) - e + V_l(y, \pi_l; \pi_{l-1}, L). \tag{10}$$

If rollover fails, the layer-l fund asset gets liquidated and the manager recovers nothing.

Following the convention of using prime to indicate variables in the next period, we can write $V(y, \pi_l; \pi_{l-1}, l, L)$ for $0 < l < L$ recursively as,

$$V_l(y, \pi_l; \pi_{l-1}, L) = \lambda_y \underbrace{(\tilde{F}_{y,l-1} - \tilde{F}_{y,l})}_{\text{Project matures}} \tag{11}$$

$$+ (1-\lambda_y)\alpha \left\{ m_{l+1}\mathbb{E}\Big[\underbrace{V_l(y', \pi_l; \pi_{l-1}, L)}_{\text{Neither debt issued by nor held by layer } l \text{ matures}} \Big] \right. \tag{12}$$

$$+ \sum_{i=0}^{l-1}(m_i - m_{i+1})\mathbb{E}\Big[\underbrace{\mathbf{1}^i_{rollover}(-P'_{l-1} + \max_{\pi'_l}(P'_l + V_l(y', \pi'_l; \pi'_{l-1}, L)))}_{\text{Debt held by layer } l \text{ matures}} \Big] \tag{13}$$

$$\left. + (m_l - m_{l+1})\mathbb{E}\Big[\underbrace{\mathbf{1}^l_{rollover}(-e + \max_{\pi'_l}(P'_l + V_l(y', \pi'_l; \pi_{l-1}, L)))}_{\text{Debt held by layer } l \text{ does not mature but debt issued by layer } l \text{ matures}} \Big] \right\}. \tag{14}$$

In the above expression, (11) captures the payoff to layer-l when the project matures with probability λ_y; otherwise with probability $1 - \lambda_y$, we have the next three terms.

First, (12) in the curly bracket captures the continuation value of layer-l when neither its asset nor liability side matures, which occurs with probability m_{l+1}. Here the fund manager as an expert discount her future by α, and y' is the next period project cash flow realization. For the last layer of fund ($l = L - 1$) who borrows from households, its liability side always matures (recall $m_L = 0$). Hence this scenario never occurs.

Second, (13) captures the payoff if layer-l's asset side matures; this happens whenever debt issued by any layer-i ($i < l$) matures. In this case, if rollover fails, layer-l's payoff is 0. When rollover is successful ($\mathbf{1}^i_{rollover} = 1$), then layer-$l$ receives e from its debtors, and pays e to its creditors — the two terms cancel out. In the refinancing stage, it receives P'_l from its new creditors and gives P'_{l-1} to its debtors. Going forward, layer l's valuation is $V(y', \pi'_l; \pi'_{l-1}, l, L)$, where π'_l is the new contract issued by layer-l and π'_{l-1} is a new contract given to layer-l. We highlight that fund l is optimally choosing a new contract π'_l to maximize the sum of new debt proceeds and its continuation payoff $P'_l + V_l(y', \pi'_l; \pi_{l-1}, L)$.

Finally, (14) considers the expected payoff to layer-l if debt issued by layer-l matures but layer-l's asset has not matured yet. This event occurs with probability $m_l - m_{l+1} = \lambda_d m_l$ for $1 \leq l < L-1$. For layer-$(L-1)$, this event occurs with probability m_{L-1}, as its liability side always matures (layer-L households hold one-period debt). There, if rollover is successful, layer-l raises P'_l, pays off e to existing creditors and chooses a new contract π'_l. Otherwise layer-l's payoff is 0.

3.2.2 Entrepreneur

Recall that the entrepreneur is labeled as layer 0. Like fund managers, his value is:

$$V_0(y, \pi_0; L) = \underbrace{\lambda_y(y - \tilde{F}_{y,0})}_{\text{Project matures}} + (1-\lambda_y)\alpha \Big\{ (1-\lambda_d)(1-\mathbf{1}_{L=1})\mathbb{E}\big[\underbrace{V_0(y', \pi_0; L)}_{\text{Debt issued by layer-0 does not mature}} \big] \tag{15}$$

$$+ (\lambda_d(1-\mathbf{1}_{L=1}) + \mathbf{1}_{L=1})\mathbb{E}\big[\underbrace{\mathbf{1}^0_{rollover}(-e + \max_{\pi'_0}(P'_0 + V_0(y', \pi'_0; L)))}_{\text{Debt issued by layer-0 matures and rollover succeeds}} + \tag{16}$$

$$\underbrace{(1-\mathbf{1}^0_{rollover})[(\beta + (1-\beta)(1-\lambda_y)\alpha)(-P'_{-1} + \max_{\pi'_0}(P'_0 + V_0(y', \pi'_0; L)))]\big]}_{\text{Debt issued by layer-0 matures and rollover fails}} \Big\}. \tag{17}$$

The term $\mathbb{E}\big[V_0(y', \pi_0; L)\big]$ in the second part of (15) captures the continuation value when debt does not mature;[23] (16) captures the value if debt matures and rollover is successful.

The main difference between the entrepreneur's payoff and intermediary funds' payoffs is reflected in the last term in (17), when debt matures but rollover fails. Because of the entrepreneur's unique human capital in the project, he is re-hired back after the bankruptcy if the chain is restored.[24] Essentially, the expert in the distress fund sells the project back to the entrepreneur at price P'_{-1} (one can view the distress fund as layer -1). The entrepreneur takes price P'_{-1} as given, chooses a new contract π'_0 (and hence initializes a new chain) to

[23] In the special one-layer case where the entrepreneur is directly issuing to households one-period debt (which always matures), this term equals zero since $1 - \mathbf{1}_{L=1} = 0$.

[24] The chain is restored with probability β this period, and $(1-\beta)(1-\lambda_y)$ in the next period. Discount rate α is applied to the continuation value if restoration happens in the next period.

maximize the sum of proceeds from issuing debt (P'_0) and his continuation value (V_0).

We allow entrepreneurs to be rehired for keeping the contract stationary over time. Since the entrepreneur has no savings when he is rehired, the price charged by the distress fund P'_{-1} cannot be larger than the debt proceeds that the entrepreneur can raise P'_0. We assume the distress fund has all the bargaining power so that $P'_{-1} = P'_0$.[25]

3.2.3 Households

Now consider the value function of households. When debt matures, the new-born households are paying P_{L-1} for the debt. So their payoff is

$$e - P_{L-1}(y) + V_L(y; \pi_{L-1}, L). \tag{18}$$

In equilibrium, households are paying the competitive price, $P_{L-1}(y) = V_L(y; \pi_{L-1}, L)$, which is defined recursively as below:

$$\begin{aligned} V_L(y; \pi_{L-1}, L) = \quad & \lambda_y \underbrace{\tilde{F}_{y,L-1}}_{\text{Project matures}} \\ & + (1-\lambda_y)\left\{ \sum_{l=0}^{L-1} (m_l - m_{l+1}) \mathbb{E}[\underbrace{\mathbf{1}^l_{rollover} e + (1 - \mathbf{1}^l_{rollover})(\alpha B_l(y, L) - c(L-l))]}_{\text{Rollover happens at layer } l \le L-1} \right\}. \end{aligned} \tag{19}$$

Similar as before, $\tilde{F}_{y,L-1} = \min(F_{y,L-1}, y)$ is the payoff to households who hold debt issued by layer $L-1$ if the project matures. With probability $1 - \lambda_y$, the project does not mature, though households' debt matures with probability 1. When debt issued by layer-l matures (with proability $m_l - m_{l+1}$), the repayment trickles down to households due to the prepayment clauses, as explained in Section 3.1. The departing households get paid by e if rollover is successful (the first part of (19) inside the expectation), or they receive the liquidation proceeds $\alpha B_l(y, L) - c(L - l)$ if rollover fails (the second part of (19)).

The valuation equation from the perspective of buyers' determines $B_l(y, L)$:

$$B_l(y, L) = \beta \underbrace{V_L(L)}_{\text{If chain is restored}} + (1-\beta)\Big\{ \underbrace{\lambda_y \tilde{F}_{y,l-1}}_{\text{Project matures}} + (1-\lambda_y)\Big[m_l \underbrace{\mathbb{E}[\alpha V_L(y'; L)]}_{\text{Debt does not mature}} \tag{20}$$

$$+ \sum_{i=0}^{l-1} (m_i - m_{i+1}) \mathbb{E}\underbrace{\left[\mathbf{1}^i_{rollover} e + (1 - \mathbf{1}^i_{rollover})(\alpha B_i(y, L) - c(l-i))]\right]}_{\text{Debt matures}} \Big\}. \tag{21}$$

With probability β, the chain is restored to length L immediately, in which case the house-

[25]The impatient entrepreneur with discount rate $\alpha < 1$ indeed has no savings, as he would prefer consume his previous debt proceeds immediately. Also recall that we have always assumed that the distress fund has all the bargaining power, so that the liquidation value equals the fair value of the debt when other layers are broken.

holds' valuation for the debt is V_L. With probability $1-\beta$, households hold the liquidated asset (debt issued by layer $l-1$) directly for one period, and the chain is restored to L in the following period. In this case, if the project matures with probability λ_y during this period, then households get paid $\tilde{F}_{y,l-1}$. If neither the project nor the debt matures, which occurs with probability $(1-\lambda_y)m_l$, then it is sold in the secondary market to the next cohort of households at discount α. Since the next cohort of households will hold debt issued by the restored chain, their valuation of debt is V_L; this gives the last term in Eq. (20).

Lastly, if the project does not mature but debt matures (which could occur if any debt issued by layers above l matures), then households either get paid by e given successful rollover or get the liquidation proceeds $\alpha B_i(y, L) - c(l-i)$ if rollover fails. This is captured by Eq. (21).

3.3 Liquidation Value

As illustrated by the simple example in Section 2, the goal for financial intermediaries to form credit chains is to increase the liquidation value $B_l(y, L)$ toward departing households. Denote the equilibrium credit chain length by L^*. The next proposition formally gives the key property of $B_l(y, L)$ that drives the benefit of a long-chain.

Proposition 1 *Liquidation value $B_{L-j}(y, L)$ is increasing in L for $L \leq L^*$ and any $j \leq L$.*

We show this formally in Appendix C.1. By fixing the distance j between the bankruptcy layer $L-j$ and households while varying the chain length L, Proposition 1 shows that the further away from the entrepreneur the higher the liquidation value. To see the intuition, consider the asset that is being liquidated at the breaking point $L-j$. This asset in liquidation can be considered as a collection of debt contracts issued by all layers above; and consistent with the intuition of maturity transformation, the further away the breaking point from the entrepreneur, the shorter-term the liquidated asset. These shorter-term claims are desirable in that if favorable fundamental y realizes later then debt payments can flow toward departing households in a frictionless way (i.e., without the discount factor α), leading to a higher liquidation value.

We highlight that the above intuition is exactly the same as in our example in Section 2, which shows that two-layer structure dominates that of short-term debt. Essentially, during liquidation, instead of liquidating the long-term project as what would happen in the short-term debt structure, the two-layer structure liquidates a short-term asset and protects the underlying long term cash flows from being discounted repeatedly. As explained toward the end of Section 2.3, because of secondary market discount and short-lived households, liquidating short-term asset is less costly than liquidating long-term asset. Proposition 1 formally states this property in our dynamic model: The more the layers between the point of bankruptcy and the underlying project, the shorter-term the liquidation asset is, the higher the liquidation value, and the greater the ex-ante debt value.

3.4 Equilibrium Definition

Define $\hat{\Pi}$ as the set of feasible contracts that are renegotiation proof and subject to the resource constraint (imposed by limited endowment from OLG households):

$$\hat{\Pi} \equiv \{\pi \in \Pi : V_L(\{F_{y,s}\}_{s=t}^{T}, L) \leq e \quad \text{for} \quad \forall t\}. \tag{22}$$

Definition 1 *The equilibrium credit chain is a set of contracts $\{\pi_{l,t}\}_{0\leq l\leq L-1}$ and credit chain length L^* such that*

1. *For $1 \leq l \leq L-1$, when layer-l's liability matures in period t,*[26]

$$\pi_{l,t} = \arg\max_{\pi\in\hat{\Pi}} \quad \mathbf{1}^{l}_{rollover}(P_l(y_t, \pi; \pi_{l-1,t}, L^*) + V_l(y_t, \pi; \pi_{l-1,t}, L^*)), \tag{23}$$

$$s.t. \quad F_{y,l,t} \leq F_{y,l-1,t} \text{ in } (7). \tag{24}$$

When layer-0's liability matures,

$$\pi_0 = \arg\max_{\pi\in\hat{\Pi}} \quad \mathbf{1}^{0}_{rollover}(P_0(y_t, \pi; L^*) + V_0(y_t, \pi; L^*)). \tag{25}$$

2. *The equilibrium L^* is such that the last layer of fund manager (L^*-1) prefers to borrow directly from households than to borrow via other fund managers:*

$$P_{L^*-1}(L^*) + V_{L^*-1}(L^*) \geq P_{L^*-1}(L^*+l) + V_{L^*-1}(L^*+l) \quad \text{for} \quad l \geq 1. \tag{26}$$

Furthermore, for all other funds $0 < l < L^*-1$,

$$P_l(L^*) + V_l(L^*) \geq P_l(l+1) + V_l(l+1). \tag{27}$$

In other words, the funds in intermediary layers prefer to borrow via other funds than to borrow from households.

[26] When $t = 0$, $\mathbf{1}^{l}_{rollover} = 1$ for all l.

3. Due to perfect competition,

$$P_l - P_{l-1} + V_l = 0. \tag{28}$$

4 Equilibrium Credit Chain

In this section, we first show that the equilibrium contract features stationarity and layer independence, i.e., the interest payment F_y is the same for all layers and is stationary over time. We then characterize and analyze the equilibrium credit chain length L^*, and provide comparative statics analysis on equilibrium chain length. Finally, we show that, somewhat surprisingly, the resulting equilibrium chain length in the decentralized market is constrained efficient.

4.1 Equilibrium Contract

Now we show that the equilibrium contract features stationarity and layer-independence; and for this section we put back time subscript t. Each period t, after y_t is observed but before the project matures, layer-l chooses a new contract for its creditors when either the debt issued by himself or the debt held by himself matures, i.e., the event $\mathbf{1}_{rollover}$ in Eq. (13) and (14) occurs. There, we can see layer-l's $(0 < l < L)$ problem is equivalent to:

$$\max_{\pi_{l,t}} \quad P_{l,t} + V_l(y_t, \pi_{l,t}; \pi_{l-1,t}, L) \tag{29}$$

$$s.t. \quad P_{l+1,t} + V_{l+1}(y_t, \pi_{l+1,t}; \pi_{l,t}, L) - P_{l,t} = 0 \tag{30}$$

$$F_{y,l,s} \leq F_{y,l-1,s} \quad \forall s \geq t. \tag{31}$$

Eq. (30) indicates that the payoff of layer-$(l+1)$ is 0 in equilibrium — $P_{l+1,t}+V_{l+1}(y_t, \pi_{l+1,t}; \pi_{l,t}, L)$ is layer-$(l+1)$'s payoff from issuing debt, and $P_{l,t}$ is how much he pays to layer-l. In perfect competition, layer-$(l+1)$'s payoff is 0. We present the first result on the equilibrium contract in Lemma 1.

Lemma 1 *The interest rate payment in the optimal debt contract is stationary and independent of fund position l, so that $\tilde{F}_{y,l,t} = \min(y_t, F_y^*)$.*

Start with stationarity. Recall that successful rollover occurs when y exceeds certain endogenous threshold $\hat{y}_{l,t}$ which is measurable to $\mathcal{F}_{t-1}$. By definition, $\hat{y}_{l,t}$ is the payment to the creditor in period t, such that the present value of the debt contract—i.e., all future promised payments at $t+s$ with $s \geq 1$—equals e. But Assumption 1 in Section 3.1 says that the issuer can always unilaterally prepay his debt. As a result, in a renegotiation proof contract, the funds set $F_{y,l,t+s} = \hat{y}_{l,t+s}$ for $s \geq 0$, i.e., the interest payment equals the run threshold for all periods. Since the amount that needs to be refinanced—i.e., e—is constant over time, it immediately implies that the endogenous rollover threshold $\hat{y}_{l,t}$ is also constant over time, yielding the stationarity of $F_{y,l}$.[27]

Next, given that the face value e is the same across layers, we argue that $F_{y,l}$ has to be the same for all l as well due to perfect competition. In light of limited liability constraint (31), we only need to rule out the case $F_{y,l-1} > F_{y,l}$. But if this is the case, then layer-l earns positive spread when the project matures, implying strictly positive profit in expectation—but this is against perfect competition.

4.2 Special Case: $c = 0$

The special case of no restructuring cost, i.e. $c = 0$, helps illustrate the benefit of setting up long chains. We have the following Corollary.

Corollary 1 *When $c = 0$, the equilibrium length of credit chain is infinity, i.e., $L^* = \infty$.*

To see the benefit of long chains, consider the difference in households value when the chain length is L versus $L+1$. We can simplify the households' value function by taking advantage of the fact that $\mathbb{E}[\mathbf{1}^l_{rollover}] = 1 - H(F_y)$ (recall $H(\cdot)$ is the cumulative distribution function of y); in words, $H(F_y)$ is the probability of rollover failure. In Appendix C.3, we show that

$$V_{L+1}(L+1) - V_L(L) = \frac{(1-\lambda_y)\alpha H(F_y) m_L \overbrace{[B_L(y, L+1) - B_{L-1}(y, L)]}^{>0 \text{ due to Proposition 1}}}{1 - (1-\lambda_y)\alpha H(F_y) K_L} \geq 0, \tag{32}$$

with an endogenous constant $K_l \in (0, 1 - m_l]$ for any $l \geq 0$. Proposition 1 says that a longer credit chain increases liquidation value, which implies that the last fund layer always prefers to keep extending the credit chain. Here we see exactly the same intuition revealed by the comparison between two-layer structure and short-term debt in Section 2: Having multiple layers increases the liquidation value (reduce liquidation loss). Again, longer chains increase liquidation value because the liquidated asset is of shorter maturity in expectation. In other words, an ex-ante longer chain delivers a shorter maturity asset when liquidation happens.

[27] $\hat{y}_{l,t}$ is defined by $V_L(\hat{y}_{l,t}, \{F_{y,l,t+j}\}_{j=1}^{\infty}, L) = e$. Since the intermediate layers all have zero payoffs, a successful rollover is determined by promising the households a value equal to the face value e. In renegotiation-proof contracts, $F_{y,l,t+j} = \hat{y}_{l,t+j}$. With stationarity, $\hat{y}_l$ is pinned down by $V_L(\hat{y}_l, \{\hat{y}_l\}, L) = e$.

Since OLG households value short-term assets, this raises the liquidation value and increases debt value ex-ante.

Furthermore, similar to the example, the benefit of long chains can also be achieved by debt contracts where the maturity is state-contingent. In Appendix C.3.1, we show that the value of some state-contingent maturity debt is exactly the same as the value of debt with $L+1$ layers. The state-contingent contract preserves the subsequent short-term contracts in the case of low output and rollover failures, yielding the liquidated asset effectively shorter-term and achieving similar outcome as longer chains.

Finally, in the special case of $c = 0$, the difference in liquidation value $B_L(L+1) - B_{L-1}(L)$ is proportional to $(1-\beta)$, as shown in Eq. (70) in Appendix C.3. If $\beta = 1$, i.e., chain restoration happens immediately after liquidation (so future short-term debt claims are never destroyed by liquidation), then adding layers does not improve liquidation value.

4.3 Characterizing Equilibrium

Given the contract is stationary and layer independent, the run thresholds for all layers are the same and constant over time. We can simplify the households' value function to the following:

$$V_L(F_y, L; y) = \lambda_y \min(F_y, y) \tag{33}$$

$$+ \underbrace{(1-\lambda_y)\Big\{(1-H(F_y))e + H(F_y)\big[\sum_{l=0}^{L-2} m_l \lambda_d(\alpha\mathbb{E}[B_l(y,L)|y<F_y] - c(L-l)) + m_{L-1}(\alpha\mathbb{E}[B_{L-1}(y,L)|y<F_y] - c)\big]\Big\}}_{v_L(F_y,L)}. \tag{34}$$

Define $v_L(F_y, L) \equiv V_L(L) - \lambda_y \min(F_y, y)$ to be the continuation value if the project does not mature in this period, which is independent of the current realization of y. Because $v_L(F_y, L)$ only depends on total chain length L and interest payment F_y, it is constant over time as both inputs are constant along the equilibrium path.

Conditional on rollover being successful ($y \geq F_y$), the households' valuation of the debt $V_L(L)$ should equal e, representing households' binding participation constraint in equilibrium. Therefore the following equation pins down F_y as a function of L,

$$V_L = e \quad \Rightarrow \quad \lambda_y F_y + v_L(F_y, L) = e. \tag{35}$$

Assumption 3 guarantees that the solution to Eq. (35) is unique.

Assumption 3 *The following inequality holds for all F_y,*

$$\lambda_y - (0,0,...,0,1)\Psi^{-1}\frac{\partial \Psi}{\partial F_y}\Psi^{-1}\eta + (0,0,...,0,1)\Psi^{-1}\frac{\partial \eta}{\partial F_y} \geq 0 \tag{36}$$

where the exact expressions for $\Psi(F_y)$ *and* $\frac{\partial\eta(F_y)}{\partial F_y}$ *are in Appendix C.4.*

Now we are ready to determine L^*. Given the competitive fund sector, the problem faced by the last layer of funds is equivalent to maximizing the sum of the funds' payoff and the households' payoff. Given that part of V_L (specifically, Eq. (33)) is a transfer between the households and the funds, the funds' problem is equivalent to maximizing v_L. But Eq. (35) immediately implies that maximizing v_L amounts to minimizing F_y, by choosing the optimal credit chain length L^*. Intuitively, layer-l will only borrow via another layer of funds if extending the credit chain reduces interest F_y; otherwise, layer-l should borrow directly from households. Proposition 2 characterizes the equilibrium conditions.

Proposition 2 *Under Assumption 3, the equilibrium (promised) interest payment* F_y^* *and equilibrium chain length* L^* *is the unique solution to the following equations*

$$e = \lambda_y F_y^* + \underbrace{\underbrace{\begin{bmatrix}0 & 0 & ... & 0 & 1\end{bmatrix}}_{1\times(L^*+1)}(\Psi(F_y^*)^{-1}\eta(F_y^*))}_{=v_{L^*}(L^*)} \tag{37}$$

$$0 = \frac{\lambda_d m_{L-1} m_L(1-\beta)(1-\lambda_y)\alpha}{(1+\lambda_d m_{L-1}\Omega)(1+m_L\Omega)} e(1-H)(1-\alpha) - \left\{(1-m_L)\frac{1+m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega} - \frac{m_L\Omega}{1+m_L\Omega}\frac{\lambda_d m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega}\right\} c. \tag{38}$$

where Ψ *is a* $(L^*+1)\times(L^*+1)$ *matrix and* η *is a* $(L^*+1)\times 1$ *vector, with both being functions of* F_y^**. The exact expressions for* Ψ *and* η *are in Appendix C.4. Finally,* $\Omega \equiv (1-\beta)(1-\lambda_y)\alpha H(F_y^*)$*.*[28]

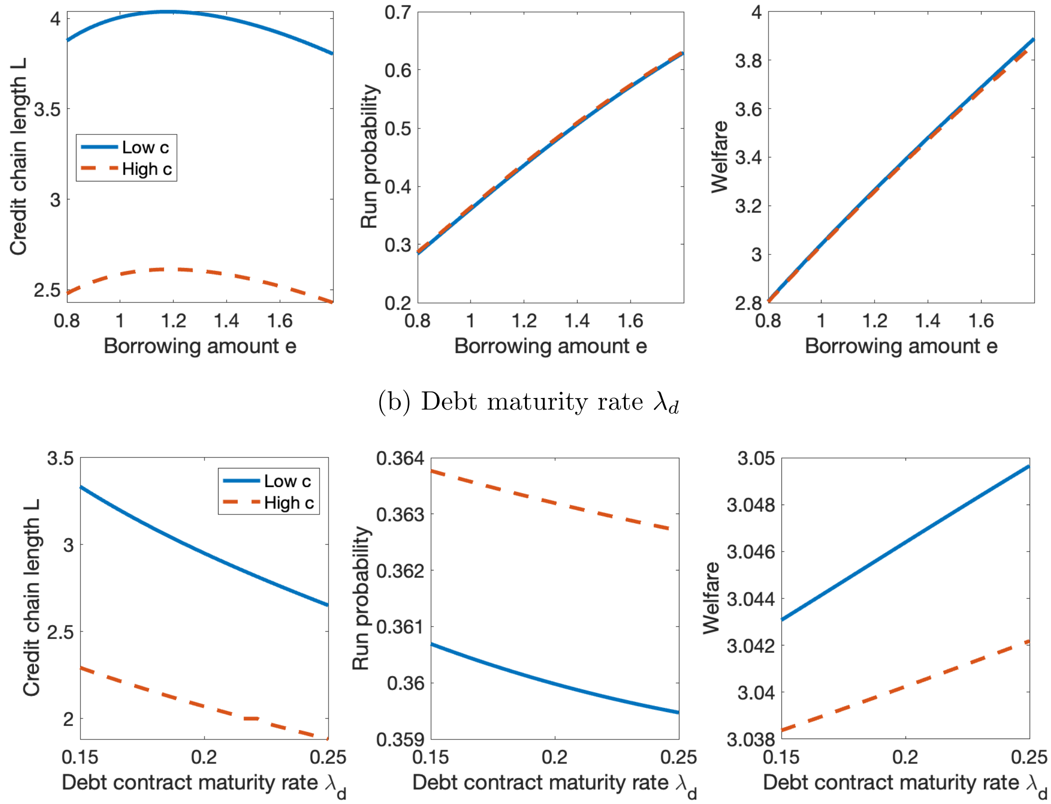
The formal proof is in Appendix C.4. Household's valuation for the debt $V_L(L)$, together with all the liquidation values $B_l(L)$ $(0 \le l \le L-1)$, forms a system of linear equations with dimension $L+1$. We solve this system of linear equations and take the last entry which is the value of $v_L(L)$, to be the second term in the right-hand-side of (37).

As explained, the equilibrium chain length L^* is effectively characterized by maximizing households' continuation payoff v_L, with (38) as the first order condition. The first term gives the marginal benefit of longer chains, as explained in Section 4.2. On the cost side which is the second term in (38), the total bankruptcy cost given rollover failure is increasing in the number of layers disrupted as well as the bankruptcy cost c. Combining this with Eq. (37) (the households' participation constraint (35)) yields the equilibrium F_y^* and L^*.

Figure 8: Comparative Statics with respect to e and λ_d

(a) Borrowing amount e

(b) Debt maturity rate λ_d



Numerical illustration of comparative statics related to chain length L. Parameter values (unless specified in the x-axis): $\beta = 0.1$, $\lambda_d = 0.1$, $\alpha = 0.5$, $\lambda_y = 0.35$, $g(y) = \gamma \exp(-\gamma y)$, $\gamma = 0.3$, $e = 1$. The blue solid line plots equilibrium chain length when $c = 0.01$ and the red dotted line plots chain length when $c = 0.02$.

4.4 Comparative Statics

The previous subsection illustrates that intermediaries in the market would like to lengthen the credit chain. When $c > 0$, additional cost in the case of rollover failure increases with L, leading the equilibrium chain length to be finite.

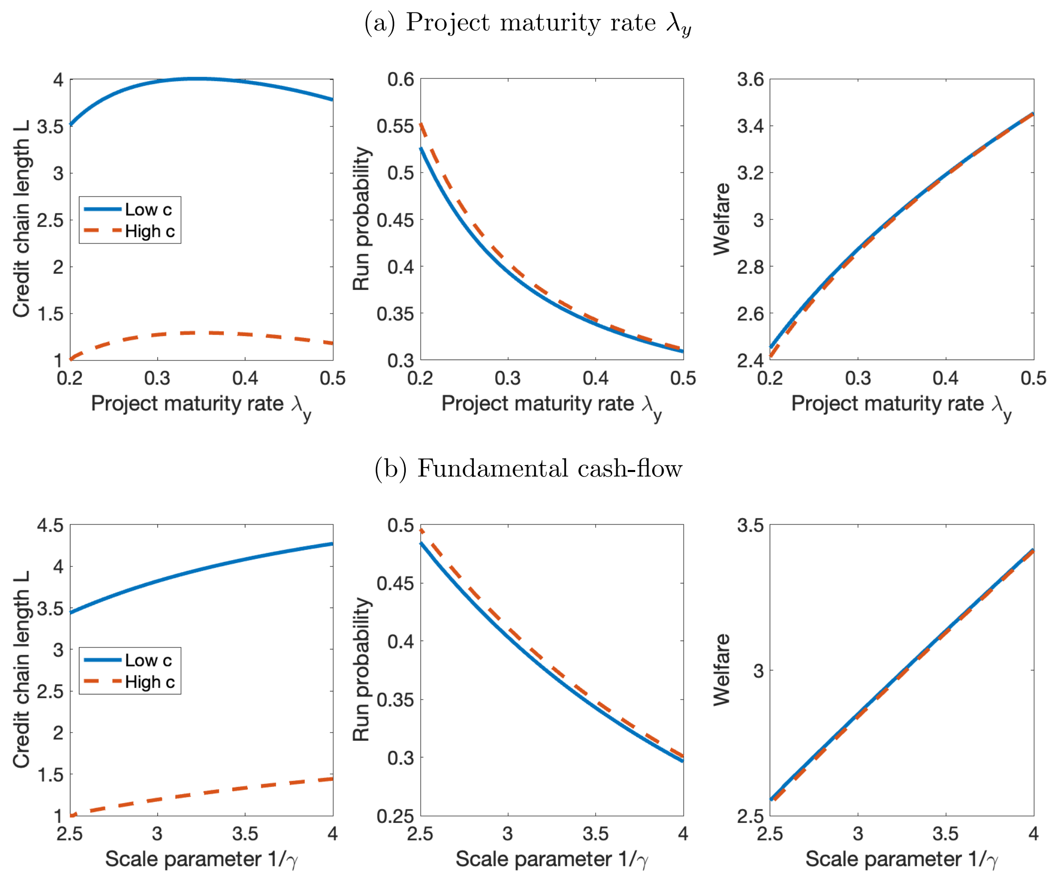
In this section, we analyze how credit chain length varies with certain parameter values.

Proposition 3 *The equilibrium credit chain length is decreasing in bankruptcy cost c, i.e.* $\frac{\partial L^*}{\partial c} \leq 0$.

When the liquidation cost c is higher, it is more costly to add layers, hence the equi-

[28] Although credit chain length should take an integer value, for exposition convenience we do not impose this requirement for L in this article. It is straightforward though to impose this restriction; specifically, if the solution to Eq. (38) lies between two integers (L_1, L_2), then $L^* = \arg\max_{L \in \{L_1, L_2\}} v_L(L)$.

Figure 9: Comparative Statics with respect to λ_y and Distribution of y



Numerical illustration of comparative statics related to chain length L. Parameter values (unless specified in the x-axis): $\beta = 0.1$, $\lambda_d = 0.1$, $\alpha = 0.5$, $\lambda_y = 0.35$, $g(y) = \gamma \exp(-\gamma y)$, $\gamma = 0.3$, $e = 1$. The blue solid line plots equilibrium chain length when $c = 0.01$ and the red dotted line plots chain length when $c = 0.05$.

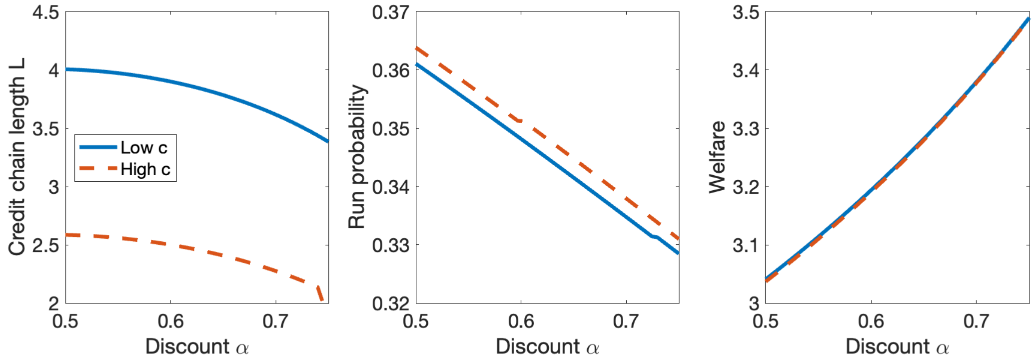
librium chain length is shorter. However, for the other parameter values, the effects are generally mixed. Figure 8-10 plot several numerical illustrations of how equilibrium chain length, run probability and welfare vary with parameter values. In all cases, higher c leads to lower equilibrium chain length, higher run probability and lower welfare.[29]

To understand the opposing forces, consider the marginal benefit of extending the chain length when the borrowing amount e is larger. Recall that the benefit of longer chains comes from a higher liquidated value of the debt, which is proportional to $e(1 - H(F_y^*))(1 - \alpha)$ (as shown in the first part of Eq. (38)). The direct effect of higher leverage e increases the marginal benefit of longer chains. However, higher leverage also increases probability of rollover failures $H(F_y^*)$, as shown in Panel (a) of Figure 8. This indirect effect through equilibrium run threshold reduces the benefit of long chains. We therefore cannot sign the

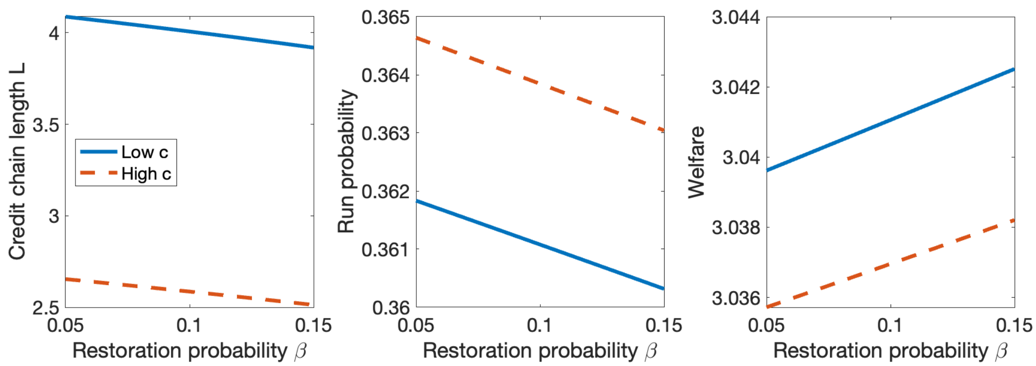
[29]The total welfare is equal to the sum of all agents' payoff. See Section 4.5 for details.

Figure 10: Comparative Statics with respect to α and β

(a) Discount rate α



(b) Chain restoration probability β



Numerical illustration of comparative statics related to chain length L. Parameter values (unless specified in the x-axis): $\beta = 0.1$, $\lambda_d = 0.1$, $\alpha = 0.5$, $\lambda_y = 0.35$, $g(y) = \gamma \exp(-\gamma y)$, $\gamma = 0.3$, $e = 1$. The blue solid line plots equilibrium chain length when $c = 0.01$ and the red dotted line plots chain length when $c = 0.02$.

net effect of larger e in general. Figure 8 Panel (a) presents a case when the effect of e on the chain length is non-monotone. Welfare naturally increases with e because e is households' endowment.

Similar logic applies to all the other parameters: the direct effect on chain length and the indirect effect through the equilibrium rollover threshold operate in opposite directions. In the case of λ_d, when λ_d is small, the asset side of any given layer has long maturity in expectation and it is more costly to liquidate those assets. This force pushes more layers in the chain to "shorten" the effective maturity of liquidated assets. However, the indirect effect via run probability goes in the opposite direction. As Figure 8 Panel (b) shows, run probability becomes larger when λ_d is smaller. In Figure 8 Panel (b), the direct effect dominates and chain length is longer when λ_d is smaller.

Next, we consider comparative statics with respect to project characteristics. First, consider the project maturity rate λ_y, fixing the rollover threshold, a larger λ_y means less maturity mismatch in the system, which reduces the benefit of extending chains (in the limit, when $\lambda_y = 1$, there is no maturity mismatch and hence households lend directly to the entrepreneur). However, larger λ_y also reduces the equilibrium rollover threshold, as shown in Panel (a) of Figure 9. This indirect force reduces the marginal cost of extending chains. As a result, the relationship between equilibrium chain length and λ_y is non-monotone in Figure 9 Panel (a). Welfare increases in λ_y because larger λ_y means the project produces cash-flow sooner, which raises the payoff of the impatient entrepreneur. Furthermore, we find that higher fundamental cash-flow leads to longer credit chains in equilibrium in Figure 9 Panel (b). When the cash-flow is on average higher, the run probability is lower and welfare is higher. Lower run probability reduces the marginal cost of extending the credit chains. Hence, we see longer chains in equilibrium when the fundamental project has higher cash-flow.

Figure 10 illustrates the effect of the discount rate α and chain restoration probability β. The direct effect of higher α reduces the marginal benefit but the indirect effect via probability of rollover failure counteracts it. When $\alpha = 1$, i.e., without any transaction/liquidation cost, we have $L^* = 1$ as there is no liquidation loss to start with, implying no benefit of using long chains. This implies that L^* decreases in α for α being close to 1. But for general α values, the comparative statics is undetermined: a higher α also reduces the rollover threshold and hence the probability of rollover failure, an indirect force that may increase the marginal value of longer chains. Under the parameterization in Figure 10 Panel (a) shows that assets with worse secondary market liquidity are supported by longest credit chains. This pattern is consistent with the case of MBS where the underlying assets (real estate properties) are with illiquid secondary markets and the intermediation chain is long.[30] Higher α also raises total welfare through smaller liquidation costs.

Finally, higher probability of chain restoration right after rollover failures reduces frictions in the system and raises liquidation values. As a result, run probability is lower and total welfare is higher, as shown in Figure 10 Panel (b). However, the benefit of long chains is to preserve the short-term claims in the event of rollover failures. Hence being able to immediately re-issue short-term debt after liquidation diminishes the benefit of having long chains in the first place. In general, we find that the equilibrium chain length decreases with the immediate restoration probability β.

[30] In addition, there is no monitoring from creditors in our model. This is perhaps another reason why our model applies well to market based financing such as the case of MBS.

4.5 Welfare Analysis

We now study whether the decentralized equilibrium is constrained-efficient from the social planner's perspective. Specifically, we ask the following question: Can the social planner improve welfare by restricting the credit chain length, say via a regulation that caps L? The answer is negative.

Consider a constrained planner who can choose L to maximize the sum of all agents' utilities $W(\{\pi_{l,t}\}_{0\leq l\leq L-1}, L)$, subject to that the contracts are determined by the decentralized equilibrium as characterized in Section 4.3. Since equilibrium contracts are layer-independent and time-invariant (Section 4.1), we denote the total welfare by $W(F_y(L), L)$, which is defined as

$$W(F_y(L), L) = e + \lambda_y y + v_0(F_y(L)) + v_1(F_y(L)) + \cdots + v_L(F_y(L).L), \tag{39}$$

Here, we are given y, $F_y(L)$ is the equilibrium interest rates given chain length L, and v_l being layer-l's continuation payoff if the project does not mature this period. Note, only households (layer L) will be directly affected by the chain length L; all other layers, including the entrepreneur only care about the equilibrium interest rates F_y (which, of course, will be determined by L).

Since in equilibrium all intermediary funds earn 0 profit, we can write Eq. (39) as

$$W(F_y(L), L) = e + \lambda_y y + v_0(F_y(L)) + v_L(F_y(L), L). \tag{40}$$

Denote the continuation value of the social welfare by w_L in the case when the project does not mature in this period, i.e., $w_L = v_0 + v_L$. w_L can then be expressed recursively as

$$w_L = (1-\lambda_y)\Big\{\alpha\lambda_y\mathbb{E}[y] + \alpha w_L + \underbrace{(1-H(F_y))(1-\alpha)e - H(F_y)\sum_{l=0}^{L-1}(m_l - m_{l+1})\left[\alpha(v_L(L) - b_l) + c(L-l)\right]}_{=\Delta}\Big\}. \tag{41}$$

To better understand (41), the term Δ represents the gain from trade (via endogenous credit chains) in our economy. To see this, consider the case of autarky without lending. There, the households' payoff is e, while the entrepreneur's payoff $\hat{V}_0 = \lambda_y y + \hat{v}_0$, with $\hat{v}_0$ denoting his continuation value if the project does not mature. Similar to before, we can express $\hat{v}_0$ recursively as

$$\hat{v}_0 = (1-\lambda_y)\alpha\left(\lambda_y\mathbb{E}[y] + \hat{v}_0\right), \tag{42}$$

which makes it clear that the difference between (42) and (41) comes from Δ. The first part of Δ is $(1-H(F_y))(1-\alpha)e$, capturing the impatience wedge between households and the entrepreneur; and the second part captures the cost in the event of rollover failure. When $c=0$, we can show that $\Delta > 0$, representing a gain from trade. See Appendix C.6 for details.

We now show that the equilibrium chain length emerging from the decentralized market is constrained efficient. To formally prove this claim, we first state the formal definition of a planner's problem in Definition 2. Same as before, $\hat{\Pi}$ denotes the set of feasible contracts that are renegotiation proof and subject to the resource constraint.

Definition 2 *The planner's solution L^{**} solves $\max_L W(\{\pi_{l,t}(L)\}_{0\le l\le L-1}, L)$ such that*

1. *For $1 \le l \le L-1$, when layer-l's liability matures,*

$$\pi_l(L) = \arg\max_{\pi\in\hat{\Pi}} \quad \mathbf{1}^l_{rollover}(P_l(y,\pi;\pi_{l-1},L) + V_l(y,\pi;\pi_{l-1},L)), \tag{43}$$

$$s.t. \quad F_{y,l} \le F_{y,l-1} \text{ in } (7). \tag{44}$$

When layer-0's liability matures,

$$\pi_0(L) = \arg\max_{\pi\in\hat{\Pi}} \quad \mathbf{1}^0_{rollover}(P_0(y,\pi;L) + V_0(y,\pi;L)). \tag{45}$$

2. *Due to perfect competition,*

$$P_l - P_{l-1} + V_l = 0. \tag{46}$$

Consider the impact of varying credit chain length on the total welfare, evaluated at the decentralized equilibrium $L = L^*$. From Eq. (40), both the entrepreneur's payoff (v_0) and the households' payoff (v_L) are affected:

$$\frac{dW}{dL} = \frac{dv_0}{dF_y}\underbrace{\frac{dF_y}{dL}}_{=0}\Big|_{L=L^*} + \underbrace{\frac{dv_L}{dL}}_{=0}\Big|_{L=L^*}. \tag{47}$$

As argued in Section 4.3, in the decentralized equilibrium, the privately optimal chain length L^* is chosen to maximize households' payoff v_L. Hence the second part of Eq. (47) is equal to 0 at the decentralized equilibrium L^*. Furthermore, we have also discussed that the optimization amounts to choosing the chain length L^* that minimizes the interest rate F_y, as suggested by $v_L = e - \lambda_y F_y$ in (35). Hence the first part of Eq. (47) is also 0. The following proposition summarizes our key result in this section.

Proposition 4 *The constrained social planner's solution coincides with the decentralized equilibrium. In other words,* $L^* = L^{**}$*.*

Why does the first order condition of L from the social perspective coincides with that in the private case in our model? At a high level, this is because the trade-offs of extending chains is reflected in the interest rate paid by the last layer of funds to their lenders —that is, OLG households. Since households are the ones paying the full restructuring cost, the interest rate that the households are willing to lend at takes this cost into account. If the cost of extending the chain outweighs the benefit, fund managers will directly borrow from households for a lower interest rate—in other words, households are willing to accept a lower interest rate to rollover the debt. Thanks to this force, in our model funds internalize the cost and benefit of longer chains through the interest rate they pay. We expect this force to be general in other settings, though we leave it to future research for a more thorough analysis on this issue.

We emphasize that our constrained-efficiency result is conditional on liquidation discounts and bankruptcy costs being fixed. Incorporating bankruptcy externalities, in which the bankruptcy cost c and/or restructuring probability β are endogenous to how many layers are being liquidated, could lead to too long chain in the decentralized equilibrium. This could be micro-founded by limited resources to handle distressed intermediary funds. Individual agents in the decentralized economy take these equilibrium variables as given, while the planner internalizes the effect of chain length on number of layers being liquidated and eventually the bankruptcy costs. Then, the wedge between the social planner's optimal choice of L^{**} and the equilibrium L^* boils down to the effect of credit chain length on the fraction of funds that go through rollover failures. Finally, if the liquidation cost c is born by each layer, instead of the households, the total liquidation cost will not be internalized by the last layer fund, which determines the equilibrium chain length. This will also lead to the decentralized equilibrium chain length being longer than the socially optimal one.

More broadly, if the overall degree of maturity transformation generated by the system varies with the chain length as in the NBER working paper version (w29632), then there is a wedge between systemic risk (the probability that the credit chain experiences a run) and the rollover risk of a given layer.[31] Standard fire-sale externalities (e.g. Shleifer and Vishny, 1992; Lorenzoni, 2008; He and Kondor, 2016 and many more), in which fire-sale discount α gets more severe when there are more assets being liquidated, also lead to chains that are

[31] As explained in "Debt contracts" in Section 3.1, in the NBER version of this paper (He and Li, 2022) we assume that OLG households are holding debt with that matures with probability λ_d (instead of 1). This implies that for the system as a whole the debt rollover occurs with probability $1 - m_L = 1 - (1 - \lambda_d)^L$, which is increasing in the credit chain length L. The decentralized market equilibrium minimizes the rollover risk of a given layer, but may generate too much systemic risk overall.

too long. In this setting, the planner limits the chain length to reduce systemic risk.

5 Conclusion

By highlighting a feature that we often see in the modern market-based financial system, we study a new dimension of the credit intermediation where one agent's liability is another agent's asset in the credit chain. We develop a new framework to illustrate the novel economic benefit of credit chains, characterize the equilibrium credit chain, and then study welfare implication of the equilibrium credit chain.

Different from existing research that only looks at systemic risk for each part of the financial system one at a time, our paper tries to provide a holistic view of the financial system when analyzing risks and welfare. This is important because regulations that impact one sector of the financial system will induce changes in the whole sector, affecting other institutions that interact with that sector. Without a model that includes the linkages among different institutions, researchers cannot properly assess the impact of any individual institution or policy. We hope future studies can use our model to answer these questions by further incorporating other empirically relevant features.

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Online Appendix

A Equilibrium in the Example

As a benchmark, the entrepreneur can issue three-period debt to cohort 0, who will then sell the debt to cohort 1 and 2 later. Figure 11 illustrates the cash flow exchanges. In $t = 0$, households purchase the debt from the entrepreneur at price P_0; in $t = 3$, the firm pays the households D_3. In $t = 1$ and 2, households trade debt on the secondary market. Since each sale on the market incurs a discount $\alpha_s = 0.8$, at $t = 0$ the entrepreneur is able to raise

$$P_0\,(\text{three-period debt}) = 0.6 \times 0.8^2 + 0.4 \times 0.6 \times 0.8^2 = 0.538. \tag{48}$$

More generally, the price of the equity contract is given by

$$P_0(\text{three-period debt}) = [p + (1-p)p]\alpha_s^2 \tag{49}$$

A.1 Direct Financing Using Two-Period Contract

Here we consider the general case where the debt face value is D_2. We show that in equilibrium D_2 is such that rollover is only successful in the good state.

We solve the problem backward. In period 2, the entrepreneur/firm can at most raise $P_2 = 1$ from cohort 2. This happens when good news has arrived, otherwise, $P_2 = 0$. If $D_2 = 0$, then the firm is never in liquidation; if $1 \leq D_2 > 0$, the firm is only liquidated if no good news has arrived; finally, if $D_2 > 1$, the firm will always be liquidated.

$$D_2 \begin{cases} \leq 0 & \text{never liquidate} \\ \in (0,1] & \text{only liquidate when no good news arrives} \\ > 1 & \text{always liquidate.} \end{cases}$$

The amount of money that can be raised in period 0 is,

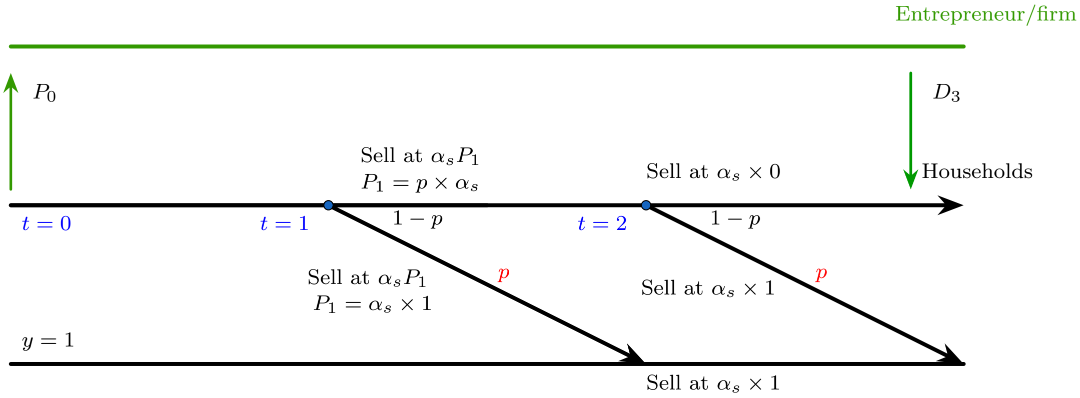
$$P_0(D_2) = \mathbf{1}_{D_2 \leq 0}\alpha_s D_2 + \mathbf{1}_{0<D_2\leq 1}[p + (1-p)p]\alpha_s D_2 + \mathbf{1}_{D_2>1}[p + (1-p)p]\alpha_l\alpha_s$$

The entrepreneur chooses D_2 to maximize P_0, we get $D_2 = 1$, and

$$P_0\,(\text{two-period debt}) = [p + (1-p)p]\alpha_s \tag{50}$$

Liquidation only happens in period 2 and when no good news arrives. Comparing Eq. (50)

Figure 11: Illustration of Case 0: Long-Term Three-Period Debt



This figure illustrates the cash flow exchanges between the households and the entrepreneur in Case 0.

with that in the three-period debt case Eq. (49), we get

$$P_0\,(\text{two-period debt}) - P_0(\text{three-period debt}) = [p + (1-p)p]\alpha_s(1-\alpha_s) \tag{51}$$

In the three-period debt case, discount is always applied twice on the final cash flow. In the two-period debt case, α_s is always applied once due to the trading in period 1. However, if debt is rolled over successful in period 2, then final cash flow need not be discounted again. That situation happens with probability $p + (1-p)p$ and the saving is $1 - \alpha_s$.

A.2 Direct Financing Using One-Period Contract

The problem in period 2 is the same as in the two-period contract case

$$D_2 \begin{cases} \leq 0 & \text{never liquidate} \\ \in (0,1] & \quad \text{only liquidate when no good news arrives} \\ > 1 & \text{always liquidate.} \end{cases}$$

So the proceeds of issuing debt in period 1 if good news arrives is,

$$P_1(g; D_{2,g}) = \mathbf{1}_{D_{2,g} \leq 0} D_{2,g} + \mathbf{1}_{0 < D_{2,g} \leq 1} D_{2,g} + \mathbf{1}_{D_{2,g} > 1} \alpha_l$$

if good news has not arrived in period 1, then

$$P_1(b; D_{2,b}) = \mathbf{1}_{D_{2,b}\leq 0} D_{2,b} + \mathbf{1}_{0<D_{2,b}\leq 1} p D_{2,b} + \mathbf{1}_{D_{2,b}>1} p\alpha_l$$

Given the amount of money that can be raised in period 1 (P_1),

$$D_1 \begin{cases} \leq P_{1,b} & \text{never liquidate} \\ \in (P_{1,b}, P_{1,g}] & \text{liquidate if no good news} > P_{1,g} \qquad \text{always liquidate} \end{cases}$$

The amount of money that can be raised in period 0 is then

$$P_0(D_1, D_2) = \mathbf{1}_{D_1 \leq P_{1,b}} D_1 + \mathbf{1}_{P_{1,b}<D_1\leq P_{1,g}} [pD_1 + (1-p)p\alpha_s\alpha_l] + \mathbf{1}_{D_1>P_{1,g}} \alpha_s\alpha_l [p + (1-p)p] \tag{52}$$

The entrepreneur's problem is the following,

$$\max_{D_1, D_2} \quad P_0(D_1, D_2)$$

Solution is then $D_{2,g} = D_{2,b} = 1$, and $D_1 = 1$. Plug this in Eq. (52) to get,

$$P_0(\text{one-period debt}) = p + (1-p)p\alpha_s\alpha_l \tag{53}$$

So liquidation in period 2 happens when no good news arrives and liquidation in period 1 happens when no good news arrives in period 1.

Comparing the funds raised via one-period debt (Eq. (50)) with the funds raised via two-period debt (Eq. (53)),

$$P_0(\text{one-period debt}) - P_0(\text{two-period debt}) = \underbrace{p(1-\alpha_s)}_{\text{Rollover succeeds}} \underbrace{-(1-p)\alpha_s p(1-\alpha_l)}_{\text{Rollover fails}} \tag{54}$$

The benefit comes from avoiding transaction cost in the secondary market when short-term debt can be successfully rolled-over.

A.3 Financing via Intermediary Funds

Similar to before, we solve the problem backward. The problem at period 2 is exactly the same as the previous subsection,

$$D_2 \begin{cases} \leq 0 & \text{never liquidate} \\ \in (0,1] & \text{only liquidate when no good news arrives} \\ > 1 & \text{always liquidate.} \end{cases}$$

Same as before, the proceeds of issuing debt in period 1 if good news arrives is,

$$P_1(g; D_{2,g}) = \mathbf{1}_{D_{2,g} \leq 0} D_{2,g} + \mathbf{1}_{0 < D_{2,g} \leq 1} D_{2,g} + \mathbf{1}_{D_{2,g} > 1} \alpha_l$$

if good news has not arrived in period 1, then

$$P_1(b; D_{2,b}) = \mathbf{1}_{D_{2,b} \leq 0} D_{2,b} + \mathbf{1}_{0 < D_{2,b} \leq 1} p D_{2,b} + \mathbf{1}_{D_{2,b} > 1} p \alpha_L$$

Next, we consider the issuance of debt in period 0. Given the amount of money that can be raised in period 1 (P_1),

$$D_1 \begin{cases} \leq P_{1,b} & \text{fund never liquidates} \\ \in (P_{1,b}, P_{1,g}] & \text{fund only liquidates when no good news} \\ > P_{1,g} & \text{fund always liquidates} \end{cases}$$

Notice the liquidation in period 1 is at the fund level, i.e. the asset being sold on the market is the debt contract between the entrepreneur and the fund.

The amount of money that can be raised in period 0 is then

$$\begin{aligned} P_0(D_1, D_2) = & \mathbf{1}_{D_1 \leq P_{1,b}} D_1 + \mathbf{1}_{P_{1,b} < D_1 \leq P_{1,g}} [p D_1 + (1-p) \alpha_l p] & (55) \\ & + \mathbf{1}_{D_1 > P_{1,g}} (p + (1-p)p) \alpha_l & (56) \end{aligned}$$

The entrepreneur's problem is $\max_{D_1, D_{2,g}, D_{2,b}} P_0$, which gives us $D_{2,g} = D_{2,b} = 1$, and $D_1 = 1$. Plug the debt face values into Eq. (56) to get,

$$P_0(\text{two-layer}) = p + (1-p) p \alpha_l \tag{57}$$

Comparing this with the one-period direct financing in Eq. (53),

$$P_0(\text{two-layer}) - P_0(\text{one-period debt}) = (1-p)p\alpha_l(1-\alpha_s) \tag{58}$$

The difference occurs in the case when no good news arrive in period 1, so rollover fails in the first period. In the one-period debt financing case, the entrepreneur's asset is being liquidated, where as in the two-period financing case, only the fund's asset is being liquidated. Since short-term asset incurs lower discount on the secondary market, the indirect financing method is able to raise more funds.

Suppose the entrepreneur issues the following debt contract whose maturity depends on period 1 news. If good news arrives in period 1, then the cash flow structure is the same as that in one-period debt (Case 2); otherwise the cash flow structure is the same as that in two-period debt (Case 1). This state-contingent maturity contract provides liquidity in period 1 if good news arrives, but at the same time avoids liquidating the firm's asset in the bad state. The difference in debt value between this case and a credit chain (Case 3) is $(1-p)p(\alpha_s - \alpha_l)$: when there is no good news arriving in period 1, the cohort-0 households incur liquidation discount α_l in a credit chain whereas they incur secondary market trading discount α_s in the state-contingent debt case. (When $\alpha_l = \alpha_s$, the two structures deliver the same outcome.) The state-contingent maturity contract, however, requires more flexibility on the contract space; and from this perspective the layered structure can be viewed as an institutional implementation of contracts whose maturity is state-contingent.

B Generalizing the Example

We generalize the example in Section 2 to multiple layers. The environment is the same as in Section 2, except that the project matures in $T > 3$ periods. We show that the structure that generates the highest debt value in period 0 is the following: there are $T-2$ layers of intermediaries. We label the firm as layer 0, and the households as layer $T-1$. The initial entrepreneur issues debt with maturity T. Funds in layer l ($0 < l \leq T-2$) hold debt with maturity $T-l$. They first issue debt with maturity $T-l-1$. When such debt matures, they try to rollover and issue debt with maturity that has only one period. In other words, they only bear rollover risk for one period (the last period) before their asset side matures. Once the fund's asset side matures, it leaves the economy, and the existing households hold debt directly issued by the layer above. The structure is illustrated in Figure 12, where the arrows indicate cash-flow exchanges.

Here we assume the intermediary funds leave the economy as their asset side matures,

Figure 12: Illustration of the Layer-Structure in the General Case

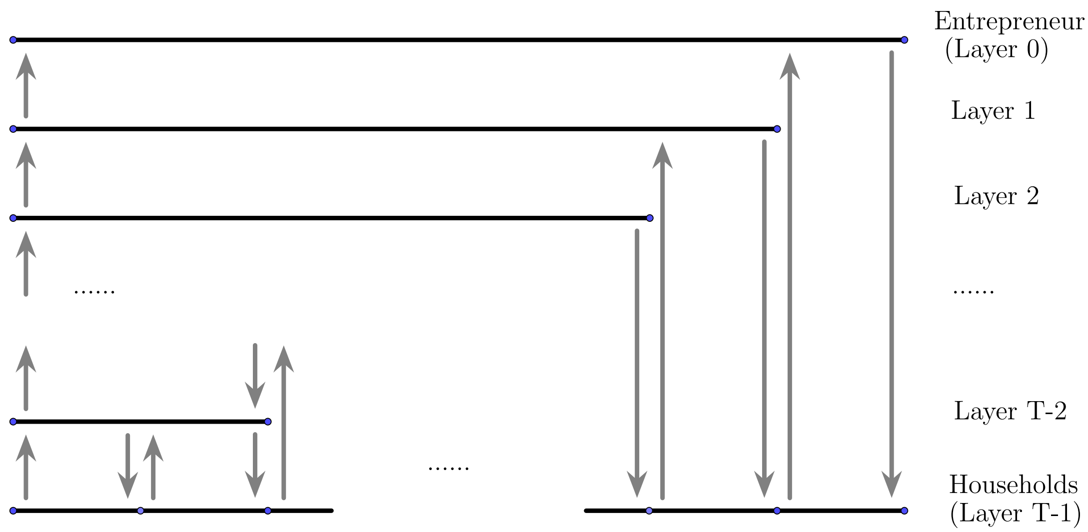
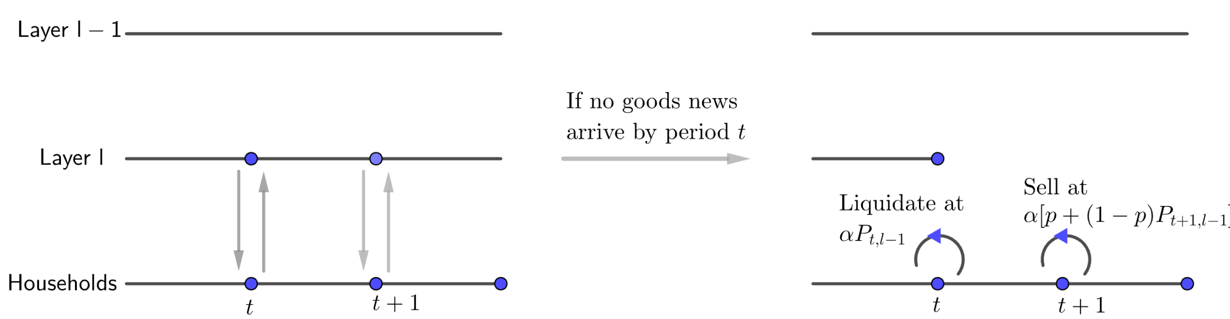


Figure 13: Alternative Substructure

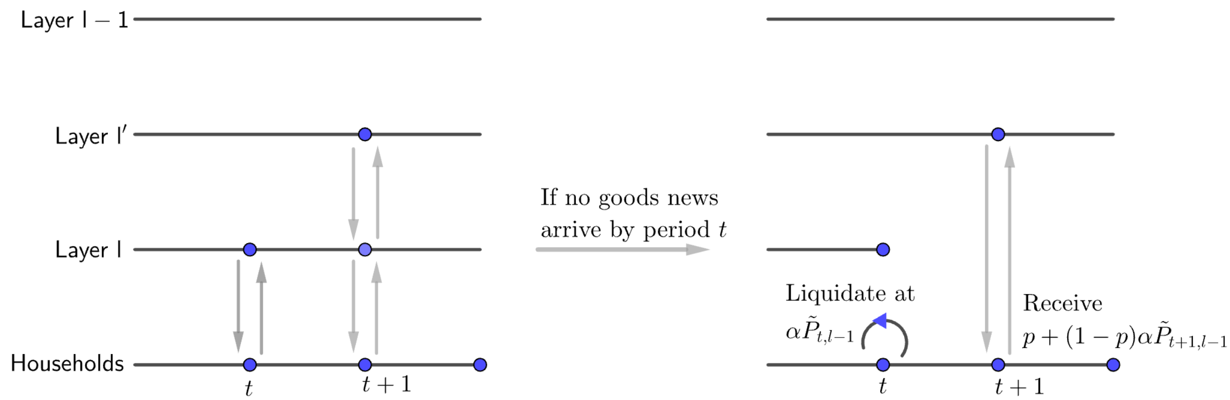


whereas in the example, we assume they issue one period debt until the economy ends. The two structures are equivalent, because the fund is not bearing any maturity mismatch and acts purely as a pass-through. Notice that from the households perspective, they are always holding one-period debt, same as in the 3-period example.

To simplify the analysis, we assume $\alpha_s = \alpha_l = \alpha$, as in the main model. Consider an alternative structure in which layer l bears more than one period maturity mismatch risk, i.e. layer l's asset matures in T_l period, but it's liability matures in $T_l - \Delta$, where $\Delta \geq 2$. See the illustration in Figure 13. We show that debt value can be improved by adding a layer (moving to the structure in Figure 14) and shortening layer l's maturity mismatch.

Suppose layer l's liability matures in period t. We focus the analysis on the amount of money that households receive at time t. Given the iterative structure, the initial debt value P_0 is increasing in the expected payment to the households in any given period. If good

Figure 14: Improvement on the Alternative Substructure



news has arrived by time t, then layer l can rollover its debt successfully and so will all the subsequent debt. Hence households receive 1. However, when good news has not arrived by time t, the rollover fails and layer l has to liquidate its asset at $\alpha P_{t,l-1}$, where $P_{t,l-1}$ is the new cohort of households' valuation of debt issued by layer $l-1$ at time t conditional on no good news has arrived by time t. In the structure in Figure 13,

$$P_{t,l-1} = p \underbrace{\alpha \times 1}_{\text{If goods news arrive in } t+1} + (1-p) \underbrace{\alpha P_{t+1,l-1}}_{\text{If no good news has arrived by } t+1}$$

However, consider the structure in Figure 14, denote households' valuation of debt issued by layer $l-1$ as $\tilde{P}_{t,l-1}$ (conditional on no good news arrive by time t),

$$\tilde{P}_{t,l-1} = p \times \underbrace{1}_{\text{If goods news arrive in } t+1} + (1-p) \underbrace{\alpha \tilde{P}_{t+1,l-1}}_{\text{If no good news has arrived by } t+1}$$

Notice that from period $t+1$ onward, the two structures are exactly the same. Hence $\tilde{V}_{t+1,l-1} = V_{t+1,l-1}$.

$$\tilde{P}_{t,l-1} - P_{t,l-1} = p(1-\alpha) > 0$$

Hence if there is any layer bearing more than 1 period maturity mismatch, then adding a layer and reducing the maturity mismatch strictly increases debt value. As a result, the structure that yields the highest debt value must be the one in Figure 12. The mechanism is the same as in the 3-period example: adding layers increases the liquidation value (here it is $P_{t,l-1}$) in the bad states of the world.

C Proofs and Derivations

C.1 Proof for Proposition 1

If the total chain length is L, the liquidation value if the chain breaks at l is,

$$\begin{aligned}
B_l(y,L) =& \beta V_L(L) + (1-\beta)\Big\{\lambda_y \min(y,F_y) + (1-\lambda_y)\Big[(1-\lambda_d)^l \mathbb{E}[\alpha V_L(y',L)] \\
& + \sum_{i=0}^{l-1} \lambda_d (1-\lambda_d)^i (e(1-H) + H(\alpha \mathbb{E}[B_i(y,L)|y<F_y] - c(l-i)))\Big]\Big\} \\
b_l(L) =& \beta v_L(L) + (1-\beta)(1-\lambda_y)\Big[(1-\lambda_d)^l \mathbb{E}[\alpha V_L(y',L)] \\
& + \sum_{i=0}^{l-1} \lambda_d (1-\lambda_d)^i (e(1-H) + H(\alpha \mathbb{E}[B_i(y,L)|y<F_y] - c(l-i)))\Big]
\end{aligned} \tag{59}$$

As will be shown in Appendix C.4, when $L < L^*$, $\frac{dv_L}{dL} > 0$, where $\frac{dv_L(L)}{dL} \equiv v_{L+1}(L+1) - v_L(L)$. Furthermore, define $\frac{\partial b_l(L)}{\partial L} = b_l(L+1) - b_l(L)$. Using Eq. (59),

$$\frac{\partial b_l(L)}{\partial L} = \beta \frac{dv_L(L)}{dL} + (1-\beta)(1-\lambda_y)\Big[(1-\lambda_d)^l \alpha \frac{dv_L(L)}{dL} + \sum_{i=0}^{l-1} \lambda_d m_l \alpha \frac{\partial \mathbb{E}[B_i(y,L)|y \le F_y]}{\partial L}\Big]$$

where $\frac{\partial \mathbb{E}[B_i(y,L)|y\le F_y]}{\partial L} = \frac{\partial b_i(L)}{\partial L}$. From induction, it is straightforward to show that $\frac{\partial b_l(L)}{\partial L} > 0$ when $\frac{dv_L}{dL} > 0$.

Next, we can write $b_{L-j+1}(L+1) - b_{L-j}(L)$ as

$$b_{L-j+1}(L+1) - b_{L-j}(L) = b_{L-j+1}(L+1) - b_{L-j+1}(L) + b_{L-j+1}(L) - b_{L-j}(L)$$

Since $b_l(L)$ is increasing in L when $L \le L^*$, $b_{L-j+1}(L+1) - b_{L-j+1}(L) \ge 0$. Furthermore,

$$\begin{aligned}
b_{l+1}(L) - b_l(L) =& (1-\beta)(1-\lambda_y)[\lambda_d(1-\lambda_d)^l e(1-H)(1-\alpha) \\
& + \lambda_d m_l H \alpha (v_L - b_l(L)) + cH(1-m_{1+l})] > 0
\end{aligned} \tag{60}$$

Eq. (60) implies $b_{L-j+1}(L) - b_{L-j}(L) > 0$. Together, we have $b_{L-j+1}(L+1) - b_{L-j}(L) > 0$.

C.2 Proof for Lemma 1

We want to show that $F_{y,l,t} = \min(F_{y,l-1,t}, \hat{y}_{l,t})$, given the face value of the debt equals to e.

At time t, for a given sequence of future payments $\{F_{y,l,t+j}\}_{j=1}^{\infty}$, there exists $\hat{y}_{l,t}$ such that

$$P_l = V_L(\{\hat{y}_{l,t}, \{F_{y,l,t+j}\}_{j=1}^{\infty}\}, L) = e$$

$\hat{y}_{l,t}$ is the run threshold. Because y_t is i.i.d. across periods, $\hat{y}_{l,t}$ does not depend on the history of y. If layer-l's debt matures in period t, then it must be the case that

$$F_{y,l,t} = \min\{\hat{y}_{l,t}, F_{y,l,t}\}$$

Otherwise, layer-l would not be able to rollover. Since layer-$(L-1)$ issues one period debt, this is always the case, i.e.

$$F_{y,L-1,t} = \min\{\hat{y}_{L-1,t}, F_{y,L-2,t}\} \qquad \forall t$$

For layer-$l < L-1$, consider the case when debt is issued in period s, where $s < t$. Since the fund manager can always renegotiate with the households, it must be the case that

$$F_{y,l,t} \leq \min\{\hat{y}_{l,t}, F_{y,l,t}\}$$

If $F_{y,L-1,t} < \min(\hat{y}_{L-1,t}, F_{y,L-2,t})$, then by setting $\tilde{F}_{y,L-1,t} = \min(\hat{y}_{L-1,t}, F_{y,L-2,t})$ and setting $\tilde{F}_{y,L-1,t+1} = F_{y,L-1,t+1} - \alpha(\min(\hat{y}_{L-1,t}, F_{y,L-2,t}) - F_{y,L-1,t})$, both the borrowing fund and the lending fund remain indifferent. So without loss of generality, we can assume

$$F_{y,l,t} = \min(\hat{y}_{l,t}, F_{y,l,t})$$

Next, we proceed to show $\hat{y}_{l,t}$ must be a constant.

For layer-0, since y_t is i.i.d., $\hat{y}_{0,t} = \hat{y}_0$ is a constant over time and the distribution of $F_{y,0,t}$ is stationary. Suppose for any layer-l where $1 \leq l \leq L-1$, $\hat{y}_{l-1,t} = \hat{y}_{l-1}$ is a constant, then $F_{y,l-1,t}$ is stationary. If $\hat{y}_{l,t} < \hat{y}_{l,t+1}$, then it must exist j, such that

$$\begin{aligned}
&\mathbb{E}_t[F_{y,l,t+j}] > \mathbb{E}_{t+1}[F_{y,l,t+j+1}] \\
\Rightarrow &\mathbb{E}_t[\min(\hat{y}_{l,t+j}, F_{y,l-1,t+j})] > \mathbb{E}_{t+1}[\min(\hat{y}_{l,t+j+1}, F_{y,l-1,t+j+1})] \\
\Rightarrow &\hat{y}_{l,t+j} > \hat{y}_{l,t+j+1}
\end{aligned} \tag{61}$$

However, at time $t+j$, the problem faced by the fund is exactly the same as at time t because of stationarity: at both point t and $t+j$, the manager is trying to find the best subsequent of payment such the debt is worth e to households. The two problems are identical. Hence it must be the case that

$$\hat{y}_{l,t+j} < \hat{y}_{l,t+j+1} \tag{62}$$

This contradicts Eq. (61). So $\hat{y}_{l,t} = \hat{y}_l$, i.e. it must be a constant over time. By induction, this is true for all $0 \le l \le L-1$.

We have now established stationarity. We move on to show $F_{y,l} = F_y$, i.e. layer-independence.

By the definition of $F_{y,l}$,

$$e = P_{l+1} + V_{l+1}(F_{y,l+1}; F_{y,l}) \tag{63}$$

In perfect competition, $P_{l+1} = e$ and $V_{l+1} = 0$. From the HJB of V_{l+1} (Eq. (11)-(14)), we can see that it is proportional to $F_{y,l} - F_{y,l+1}$. Hence for $V_{l+1} = 0$, it must be the case that

$$F_{y,l} = F_{y,l+1} = F_y \tag{64}$$

C.3 Proof for Corollary 1

We prove the equilibrium chain length is infinity by showing that the entrepreneur's payoff is always higher with more layers of financial intermediaries, for a given set of contract parameters.

From the proof of optimal contract, it is straightforward that rollover fails when $y < F_y$. This is true for any layer l. Suppose $V_L(\{F_y\}, L) = e$, we will show that $V_{L+1}(\{F_y\}, L+1) > e$, which implies the equilibrium $F_y(L+1) < F_y(L)$. In the following proof, unless specified otherwise, $F_y = F_y(L)$ and $H = H(F_y)$.

We can write households' value function as,

$$\begin{aligned} V_L &= \lambda_y \min(y, F_y) + (1-\lambda_y)(1-H)e \\ &+ (1-\lambda_y)H\Big[\sum_{l=0}^{L-2} m_l \lambda_d \alpha \mathbb{E}[B_l(y, L)|y < F_y] + m_{L-1}\alpha \mathbb{E}[B_{L-1}(y, L)|y < F_y]\Big] \end{aligned}$$

consider adding a layer, households' value function becomes

$$\begin{aligned} V_{L+1} &= \lambda_y \min(y, F_y) + (1-\lambda_y)(1-H)e \\ &+ (1-\lambda_y)H \times \Big[\sum_{l=0}^{L-1} m_l \lambda_d \alpha \mathbb{E}[B_l(y, L)|y < F_y] + m_L \alpha \mathbb{E}[B_L(y, L+1)|y < F_y]\Big] \end{aligned}$$

Note that we define

$$v_L \equiv (1-\lambda_y)(1-H)e + (1-\lambda_y)H\Big[\sum_{l=0}^{L-2} m_l\lambda_d\alpha\mathbb{E}[B_l(y,L)|y<F_y] + m_{L-1}\alpha\mathbb{E}[B_{L-1}(y,L)|y<F_y]\Big]$$

To compare the two,

$$\begin{aligned}
&V_{L+1} - V_L = v_{L+1} - v_L \\
=&(1-\lambda_y)H\alpha\Big[\sum_{l=0}^{L-1} m_l\lambda_d\mathbb{E}[B_l(y,L+1)|y<F_y] + m_L\mathbb{E}[B_L(y,L+1)|y<F_y] \\
&-\sum_{l=0}^{L-2} m_l\lambda_d\mathbb{E}[B_l(y,L)|y<F_y] - m_{L-1}\mathbb{E}[B_L(y,L)|y<F_y]\Big] \\
=&(1-\lambda_y)H\alpha\Big[\sum_{l=0}^{L-1} m_l\lambda_d(b_l(L+1)-b_l(L)) + m_L(b_L(L+1)-b_{L-1}(L))\Big] \qquad (65)
\end{aligned}$$

Where $b_l(L) = B_l(F_y, L) - \lambda_y \min(y, F_y)$. It is clear that the increase in debt value purely comes from the increase in liquidation value. Next, we show that the liquidation value is indeed higher when the chain length is longer.
First of all,

$$b_l(L+1) - b_l(L) = \beta(v_{L+1}-v_L) + (1-\beta)(1-\lambda_y)[m_l\alpha(v_{L+1}-v_L) + \alpha H\sum_{i=0}^{l-1}\lambda_d m_i(b_i(L+1)-b_i(L))]$$

and

$$b_0(L+1) - b_0(L) = [\beta + (1-\beta)(1-\lambda_y)\alpha](v_{L+1}(L+1) - v_L(L))$$

Denote $\sum_{l=0}^{n-1}\lambda_d m_l(b_l(L+1)-b_l(L))$ by $K_n \times (v_{L+1}-v_L)$, and $K_1 = \lambda_d[\beta+(1-\beta)(1-\lambda_y)\alpha]$. For $n \geq 1$, K_n is defined recursively by $(1 \leq n \leq L)$,

$$K_{n+1} - K_n = \lambda_d m_n[\beta + (1-\beta)(1-\lambda_y)(\alpha m_n + \alpha H K_n)] \qquad (66)$$

and $K_1 \leq \lambda_d = 1 - m_1$. Suppose $K_n \leq 1 - m_n$, plug into Eq. (66) we get

$$\begin{aligned}
K_{n+1} &\leq K_n + \lambda_d m_n[\beta + (1-\beta)(1-\lambda_y)(\alpha m_n + \alpha H K_n)] \\
&\leq K_n + \lambda_d m_n[\beta + (1-\beta)(1-\lambda_y)] \leq 1 - m_n + \lambda_d m_n = 1 - m_{n+1}
\end{aligned}$$

By induction, we have proved that $K_n \leq 1 - m_n$ for $1 \leq n \leq L$. Next,

$$
\begin{aligned}
&b_L(L+1) - b_{L-1}(L) = \beta(v_{L+1} - v_L) + (1-\beta)(1-\lambda_y)\Big[m_L \mathbb{E}[\alpha V_{L+1}(y', L+1)] \\
&+ \sum_{i=0}^{L-1} \lambda_d m_i (e(1-H) + H\alpha\mathbb{E}[B_i(y, L+1)|y < F_y]) \\
&- m_{L-1}\mathbb{E}[\alpha V_L(y', L)] - \sum_{i=0}^{L-2} \lambda_d m_i (e(1-H) + H(\alpha \mathbb{E}[B_i(y, L)|y < F_y]))\Big] \\
&= \beta(v_{L+1} - v_L) + (1-\beta)(1-\lambda_y)\Big[- \lambda_d m_{L-1}\alpha\mathbb{E}[V_{L+1}(y', L+1)] + m_{L-1}\alpha(v_{L+1} - v_L) \\
&+ \lambda_d m_{L-1} e(1-H) + \lambda_d m_{L-1} H\alpha\mathbb{E}[B_{L-1}(y, L+1)|y < F_y]\Big]
\end{aligned} \tag{67}
$$

Plug Eq. (67) into Eq. (65), we get

$$
\begin{aligned}
&v_{L+1} - v_L = (1-\lambda_y)H\alpha\Big[(K_L + m_L\beta + m_L(1-\beta)(1-\lambda_y)m_{L-1}\alpha)(v_{L+1} - v_L) \\
&+ m_L(1-\beta)(1-\lambda_y)\lambda_d m_{L-1} e(1-H) \\
&+ m_L(1-\beta)(1-\lambda_y)\lambda_d m_{L-1} H\alpha(\mathbb{E}[B_{L-1}(y, L+1)|y < F_y] - \mathbb{E}[V_{L+1}(y', L+1)|y < F_y])\Big]
\end{aligned} \tag{68}
$$

From Eq. (68), we can write $v_{L+1} - v_L$ as

$$
\begin{aligned}
&v_{L+1} - v_L = (1-\beta)\alpha H(1-\lambda_y)^2 \lambda_d m_L m_{L-1} \\
&\times \frac{e(1-H) + H\alpha(\mathbb{E}[B_{L-1}(y, L+1)|y < F_y] - \mathbb{E}[V_{L+1}(y', L+1)|y < F_y])}{1 - (1-\lambda_y)H\alpha(K_L + m_L\beta + m_L(1-\beta)(1-\lambda_y)m_{L-1}\alpha)}
\end{aligned} \tag{69}
$$

Plug Eq. (69) back into Eq. (67), we can see that $B_L(y, L+1) - B_{L-1}(y, L) = b_L(L+1) - b_{L-1}(L)$ is proportional to $(1-\beta)$

$$
\begin{aligned}
&b_L(L+1) - b_{L-1}(L) = [\beta + (1-\beta)(1-\lambda_y)m_{L-1}\alpha](v_{L+1} - v_L) \\
&+ (1-\beta)(1-\lambda_y)\Big[- \lambda_d m_{L-1}\alpha\mathbb{E}[V_{L+1}(y', L+1)] + \lambda_d m_{L-1} e(1-H) + \lambda_d m_{L-1} H\alpha\mathbb{E}[B_{L-1}(y, L+1)|y < F_y]\Big] \\
=&(1-\beta)(1-\lambda_y)\Big\{[\beta + (1-\beta)(1-\lambda_y)m_{L-1}\alpha]\alpha H(1-\lambda_y)\lambda_d m_L m_{L-1} \\
&\times \frac{e(1-H) + H\alpha(\mathbb{E}[B_{L-1}(y, L+1)|y < F_y] - \mathbb{E}[V_{L+1}(y', L+1)|y < F_y])}{1 - (1-\lambda_y)H\alpha(K_L + m_L\beta + m_L(1-\beta)(1-\lambda_y)m_{L-1}\alpha)} - \lambda_d m_{L-1}\alpha\mathbb{E}[V_{L+1}(y', L+1)] \\
&+ \lambda_d m_{L-1} e(1-H) + \lambda_d m_{L-1} H\alpha\mathbb{E}[B_{L-1}(y, L+1)|y < F_y]\Big\}
\end{aligned} \tag{70}
$$

Furthermore, we can write $\mathbb{E}[V_{L+1}(y', L+1)] = (1-H)e + H\mathbb{E}[V_{L+1}(y', L+1)|y < F_y]$,

$$\begin{aligned}
&\mathbb{E}[B_{L-1}(y, L+1)|y < F_y] - \mathbb{E}[V_{L+1}(y', L+1)|y < F_y] \\
=&(1-\beta)(1-\lambda_y)\Big[m_{L-1}\mathbb{E}[\alpha V_{L+1}(y', L+1)] - m_{L-1}e(1-H) - H\lambda_d m_{L-1}\alpha\mathbb{E}[B_{L-1}(y, L+1)|y < F_y] \\
&- Hm_L\alpha\mathbb{E}[B_L(y, L+1)|y < F_y]\Big] \\
=&(1-\beta)(1-\lambda_y)\Big[\lambda_d m_{L-1}H\alpha(\mathbb{E}[V_{L+1}(y', L+1)|y < F_y] - \mathbb{E}[B_{L-1}(y, L+1)|y < F_y]) \\
&+ Hm_L\alpha(\mathbb{E}[V_{L+1}(y', L+1)|y < F_y] - \mathbb{E}[B_L(y, L+1)|y < F_y])\Big] \\
&\mathbb{E}[B_{L-1}(y, L+1)|y < F_y] - \mathbb{E}[V_{L+1}(y', L+1)|y < F_y] \\
&= \frac{(1-\beta)(1-\lambda_y)\alpha Hm_L}{1+(1-\beta)(1-\lambda_y)\lambda_d m_{L-1}H\alpha}(\mathbb{E}[V_{L+1}(y', L+1)|y < F_y] - \mathbb{E}[B_L(y, L+1)|y < F_y]) \\
&\mathbb{E}[V_{L+1}(y', L+1)|y < F_y] - \mathbb{E}[B_L(y, L+1)|y < F_y] = (1-\beta)(1-\lambda_y) \\
&\times\Big[m_L e(1-H)(1-\alpha) + Hm_L\alpha(\mathbb{E}[B_L(y, L+1)|y < F_y] - \mathbb{E}[V_{L+1}(y', L+1)|y < F_y])\Big] \\
&\mathbb{E}[V_{L+1}(y', L+1)|y < F_y] - \mathbb{E}[B_L(y, L+1)|y < F_y] = \frac{(1-\beta)(1-\lambda_y)m_L(1-H)(1-\alpha)e}{1+(1-\beta)(1-\lambda_y)\alpha Hm_L}
\end{aligned}$$

Plug the expressions back into Eq. (68) to get,

$$\begin{aligned}
&v_{L+1} - v_L = (1-\lambda_y)H\alpha\Big[(K_L + m_L\beta + m_L(1-\beta)(1-\lambda_y)m_{L-1}\alpha)(v_{L+1} - v_L) \\
&+ m_L(1-\beta)(1-\lambda_y)\lambda_d m_{L-1}e(1-H) \\
&+ m_L(1-\beta)(1-\lambda_y)\lambda_d m_{L-1}H\alpha\frac{(1-\beta)(1-\lambda_y)\alpha Hm_L}{1+(1-\beta)(1-\lambda_y)\lambda_d m_{L-1}H\alpha}\frac{(1-\beta)(1-\lambda_y)m_L(1-H)(1-\alpha)e}{1+(1-\beta)(1-\lambda_y)\alpha Hm_L}\Big]
\end{aligned} \tag{71}$$

Rearrange terms in Eq. (71), we get

$$\begin{aligned}
&[1-(1-\lambda_y)H\alpha(K_L + m_L\beta + m_L(1-\beta)(1-\lambda_y)m_{L-1}\alpha)](v_{L+1} - v_L) \\
=&(1-\lambda_y)H\alpha m_L(1-\beta)(1-\lambda_y)\lambda_d m_{L-1}e \\
&\times\left[1 - H + H\alpha\frac{(1-\beta)(1-\lambda_y)\alpha Hm_L}{1+(1-\beta)(1-\lambda_y)\lambda_d m_{L-1}H\alpha}\frac{(1-\beta)(1-\lambda_y)m_L(1-H)(1-\alpha)}{1+(1-\beta)(1-\lambda_y)\alpha Hm_L}\right]
\end{aligned}$$

The right hand side of the equation is positive. Furthermore, using $K_L \leq 1 - m_L$, we can show

$$1-(1-\lambda_y)H\alpha(K_L + m_L\beta + m_L(1-\beta)(1-\lambda_y)m_{L-1}\alpha)] > 0$$

Hence, $v_{L+1}(L+1) - v_L(L) > 0$, which means to make the households break-even, $F_y(L+1) < F_y(L)$.

Layer $L-1$ manager's value is decreasing in $F_y(L+1)$. Hence, in equilibrium, the chain length is infinity.

C.3.1 The Case of State-Contingent Contract

Suppose that in our dynamic model the last-layer fund—as opposed to extending the chain from L to $L+1$—designs the maturity of its issued debt to depend on the realization of y_t. Specifically, if y_t is above the rollover threshold, then debt matures with probability 1; otherwise, debt matures with probability λ_d. When y_t is below the rollover threshold and debt does not mature in period t, last-layer's debt matures with probability λ_d in period $t+1$ regardless of y_{t+1}. Furthermore, we also assume $\beta = 0$ in both the state contingent case and the non-state contingent case.

Denote the value of debt held by the households by $\widetilde{V}_L(L)$,

$$\widetilde{V}_L(L) = \lambda_y \min(y, F_y) + (1-\lambda_y)(1-H)e + (1-\lambda_y)H\Big[\sum_{l=0}^{L-1} m_l \lambda_d \alpha \mathbb{E}[\tilde{B}_l(y,L)|y<F_y] + m_L \alpha \mathbb{E}[\tilde{B}_L(y,L)|y<F_y]\Big]$$

where $\tilde{B}_l(L)$ is the liquidation value in this case. When $\beta = 0$, for $l < L$

$$\tilde{B}_l(y,L) = \lambda_y \min(y,F_y) + (1-\lambda_y)\Big[(1-\lambda_d)^l \mathbb{E}[\alpha \tilde{V}_L(y',L)] + \sum_{i=0}^{l-1} \lambda_d (1-\lambda_d)^i (e(1-H) + H\alpha \mathbb{E}[\tilde{B}_i(y,L)|y<F_y])\Big]$$

Furthermore,

$$\tilde{B}_L(y,L) = \lambda_y \min(y,F_y) + (1-\lambda_y)\Big[(1-\lambda_d)^L \mathbb{E}[\alpha \tilde{V}_L(y',L)] + \sum_{i=0}^{L-1} \lambda_d (1-\lambda_d)^i (e(1-H) + H\alpha \mathbb{E}[\tilde{B}_i(y,L)|y<F_y])\Big]$$

We can then see $\tilde{B}_l(L) = B_l(L+1)$ for $l < L$, $\tilde{B}_L(L) = B_L(L+1)$ and $\widetilde{V}_L(L) = V_{L+1}(L+1)$.

C.4 Proof for Proposition 2

C.4.1 Existence and Uniqueness of F_y

A given cohort of household's strategy (run threshold) is $F_y = \frac{e - v_L(F_y')}{\lambda_y}$, where F_y' is other cohort's strategy. A symmetric equilibrium is where $F_y = F_y'$. Moreover, $\frac{d\frac{e-v_L(F_y')}{\lambda_y}}{dF_y'} \leq 1$ at the equilibrium point.

Given $\frac{e-v_L(0)}{\lambda_y} > 0$ and $\lim_{x\to\infty} \frac{e-v_L(x)}{\lambda_y} - x < 0$, there exists at least one intersection of $y = \frac{e-v_L(x)}{\lambda_y}$ with $y = x$ from above. So equilibrium exists.

We next solve for F_y. We write the equations defining $b_l(L)$ and v_L in matrix form

$$\Psi \begin{bmatrix} b_0(L) \\ b_1(L) \\ ... \\ b_{L-1}(L) \\ v_L(L) \end{bmatrix} = \eta \tag{72}$$

where

$$\Psi = \begin{bmatrix} 1 & 0 & 0 & ... & 0 & -\beta-(1-\beta)(1-\lambda_y)\alpha m_0 \\ 0 & 1 & 0 & ... & 0 & -\beta-(1-\beta)(1-\lambda_y)\alpha m_1 \\ ... \\ 0 & 0 & 0 & ... & 1 & -\beta-(1-\beta)(1-\lambda_y)\alpha m_{L-1} \\ 0 & 0 & 0 & ... & 0 & 1 \end{bmatrix}$$
$$-(1-\lambda_y)H(F_y)\alpha\lambda_d \begin{bmatrix} 0 & 0 & 0 & ... & 0 & 0 & 0 \\ (1-\beta)m_0 & 0 & 0 & ... & 0 & 0 & 0 \\ (1-\beta)m_0 & (1-\beta)m_1 & 0 & ... & 0 & 0 & 0 \\ (1-\beta)m_0 & (1-\beta)m_1 & (1-\beta)m_2 & ... & 0 & 0 & 0 \\ ... \\ (1-\beta)m_0 & (1-\beta)m_1 & (1-\beta)m_2 & ... & (1-\beta)m_{L-2} & 0 & 0 \\ m_0 & m_1 & m_2 & ... & m_{L-2} & \frac{m_{L-1}}{\lambda_d} & 0 \end{bmatrix}$$

and

$$
\begin{aligned}
\eta =&(1-\lambda_y)[\alpha\lambda_y H(y)\underline{X}(F_y) + (1-H(F_y))e]\begin{bmatrix} 0 \\ 1-\beta \\ ... \\ 1-\beta \\ 1 \end{bmatrix} + (1-\lambda_y)(1-\beta)\begin{bmatrix} \alpha\lambda_y(H\underline{X}(F_y)+F_y(1-H)) \\ cH\frac{1-\lambda_d}{\lambda_d} \\ ... \\ cH\frac{1-\lambda_d}{\lambda_d} \\ 0 \end{bmatrix} \\
&+ (1-\lambda_y)[\alpha\lambda_y F_y(1-H) - e(1-H) + cH(1-\frac{1}{\lambda_d})]\begin{bmatrix} 0 \\ (1-\beta)m_1 \\ ... \\ (1-\beta)m_{L-1} \\ 0 \end{bmatrix} \\
&- (1-\lambda_y)cH(F_y)\begin{bmatrix} 0 \\ (1-\beta) \\ ... \\ (1-\beta)(L-1) \\ L-\frac{1-\lambda_d}{\lambda_d} - (1-\frac{1}{\lambda_d})m_{L-1} \end{bmatrix}
\end{aligned}
$$

Next, to argue uniqueness, we just need to show that $\frac{d\frac{e-v_L(F_y)}{\lambda_y}}{dF_y} \leq 1 \Leftrightarrow \lambda_y + \frac{dv_L(F_y)}{dF_y} \geq 0$. We can express

$$
\begin{aligned}
&v_L = (0,0,...,0,1)\Psi^{-1}\eta \\
&\frac{dv_L}{dF_y} = -(0,0,...,0,1)\Psi^{-1}\frac{\partial\Psi}{\partial F_y}\Psi^{-1}\eta + (0,0,...,0,1)\Psi^{-1}\frac{\partial\eta}{\partial F_y}
\end{aligned}
$$

We need

$$
\lambda_y - (0,0,...,0,1)\Psi^{-1}\frac{\partial\Psi}{\partial F_y}\Psi^{-1}\eta + (0,0,...,0,1)\Psi^{-1}\frac{\partial\eta}{\partial F_y} \geq 0
$$

which is satisfied by Assumption 3.

C.4.2 Characterizing Equilibrium Chain Length

In equilibrium, $F_y(L)$ is determined by $e = \lambda_y F_y + v_L(L)$

$$
\begin{aligned}
v_L =&(1-\lambda_y)(1-H)e + (1-\lambda_y)H \times \Big[\sum_{l=0}^{L-2} m_l\lambda_d(\alpha\mathbb{E}[B_l(y,L)|y<F_y] - c(L-l)) \\
&+ m_{L-1}(\alpha\mathbb{E}[B_{L-1}(y,L)|y<F_y] - c)\Big]
\end{aligned}
$$

In equilibrium, $L^* = \arg\max_L v_L(L)$. Taking difference with respect to L ($dv_L(L) = v_L(L) - v_{L-1}(L-1)$ and $\frac{\partial b_l(L)}{\partial L} = b_l(L+1) - b_l(L)$),

$$\frac{dv_L(L)}{dL} = (1-\lambda_y)H\Big[\sum_{l=0}^{L-1} m_l\lambda_d\alpha\frac{\partial b_l(L)}{\partial L} - (1-m_L)c + m_L\alpha(b_L(L+1) - b_{L-1}(L))\Big] \tag{73}$$

To examine $\frac{\partial b_l(L)}{\partial L}$,

$$\frac{\partial b_l(L)}{\partial L} = \beta\frac{dv_L(L)}{dL} + (1-\beta)(1-\lambda_y)\Big[(1-\lambda_d)^l\alpha\frac{dv_L(L)}{dL} + \sum_{i=0}^{l-1}\lambda_d m_l\alpha\frac{\partial\mathbb{E}[B_i(y,L)|y\le F_y]}{\partial L}\Big]$$

At $\frac{dv_L(L)}{dL} = 0$, $\frac{\partial b_l(L)}{\partial L} = 0$ for all $1 \le i \le L-1$. Furthermore,

$$\begin{aligned}
&b_L(L+1) - b_{L-1}(L) = \beta\frac{dv_L(L)}{dL} + (1-\beta)(1-\lambda_y)\Big[(1-\lambda_d)^{L-1}\alpha\frac{dv_L(L)}{dL} + \\
&\lambda_d(1-\lambda_d)^{L-1}e(1-H)(1-\alpha) + \lambda_d(1-\lambda_d)^{L-1}H\alpha(b_{L-1}(L+1) - v_{L+1}) - (1-m_L+\lambda_d m_{L-1})Hc\Big] \quad (74)\\
&b_{L-1}(L+1) - v_{L+1} = \frac{(1-\beta)(1-\lambda_y)}{1+(1-\beta)(1-\lambda_y)\alpha(1-\lambda_d)^{L-1}\lambda_d H}\\
&\times\Big\{cH(2-m_L) - m_{L-1}e(1-H)(1-\alpha) - \frac{H\lambda_d m_{L-1}\alpha(1-\beta)(1-\lambda_y)}{1+(1-\beta)(1-\lambda_y)\alpha m_L H}\Big[m_L e(1-H)(1-\alpha) - cH\Big]\Big\} \quad (75)
\end{aligned}$$

Plug Eq. (75) into Eq. (74) to get $b_L(L+1) - b_{L-1}(L)$ as a function of $\frac{dv_L}{dL}$. Plug $b_L(L+1) - b_{L-1}(L)$ back into Eq. (73) and set $\frac{dv_L}{dL} = 0$ at the optimal point. We get,

$$\begin{aligned}
&-(1-m_L)c + m_L\alpha(1-\beta)(1-\lambda_y)\Big\{\lambda_d m_{L-1}e(1-H)(1-\alpha) - Hc(1-m_L)\\
&+\frac{\lambda_d m_{L-1}\alpha H(1-\beta)(1-\lambda_y)}{1+(1-\beta)(1-\lambda_y)\alpha m_{L-1}\lambda_d H}\times\Big[cH(2-m_L) - m_{L-1}(1-H)(1-\alpha)e\\
&+\frac{(1-\beta)(1-\lambda_y)m_L\alpha H}{1+(1-\beta)(1-\lambda_y)\alpha m_L H}(m_L(1-H)(1-\alpha)e - cH)\Big]\Big\} = 0
\end{aligned}$$

Rearranging terms and denote $\Omega = (1-\beta)(1-\lambda_y)\alpha H$, we get

$$\begin{aligned}
FOC_{L,prv} =& \frac{\lambda_d m_{L-1}m_L(1-\beta)(1-\lambda_y)\alpha}{(1+\lambda_d m_{L-1}\Omega)(1+m_L\Omega)}e(1-H)(1-\alpha)\\
&-\Big\{(1-m_L)\frac{1+m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega} - \frac{m_L\Omega}{1+m_L\Omega}\frac{\lambda_d m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega}\Big\}c
\end{aligned} \tag{76}$$

To show the second order condition is satisfied, take derivative with respect to L in Eq.

(76), we get

$$
\begin{aligned}
& log(1-\lambda_d)\frac{\lambda_d m_{L-1}m_L(1-\beta)(1-\lambda_y)\alpha}{(1+\lambda_d m_{L-1}\Omega)(1+m_L\Omega)}e(1-H)(1-\alpha) \\
& - log(1-\lambda_d)\frac{\lambda_d m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega}\frac{\lambda_d m_{L-1}m_L(1-\beta)(1-\lambda_y)\alpha}{(1+\lambda_d m_{L-1}\Omega)(1+m_L\Omega)}e(1-H)(1-\alpha) \\
& - log(1-\lambda_d)\frac{m_L\Omega}{1+m_L\Omega}\frac{\lambda_d m_{L-1}m_L(1-\beta)(1-\lambda_y)\alpha}{(1+\lambda_d m_{L-1}\Omega)(1+m_L\Omega)}e(1-H)(1-\alpha) \\
& - \Big\{ - log(1-\lambda_d)m_L\frac{1+m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega} + (1-m_L)log(1-\lambda_d)\left[\frac{m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega} - \frac{\lambda_d\Omega m_{L-1}(1+m_{L-1}\Omega)}{(1+\lambda_d m_{L-1}\Omega)^2}\right] \\
& - log(1-\lambda_d)\frac{m_L\Omega}{1+m_L\Omega}\frac{\lambda_d m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega}\left(\frac{1}{1+m_L\Omega}+\frac{1}{1+\lambda_d m_{L-1}\Omega}\right)\Big\}c
\end{aligned}
$$

Use the first order condition to substitute out $\frac{\lambda_d m_{L-1}m_L(1-\beta)(1-\lambda_y)\alpha}{(1+\lambda_d m_{L-1}\Omega)(1+m_L\Omega)}e(1-H)(1-\alpha)$, the above expression can be rewritten as

$$
\begin{aligned}
& c\times log(1-\lambda_d)\Big\{\left[(1-m_L)\frac{1+m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega} - \frac{m_L\Omega}{1+m_L\Omega}\frac{\lambda_d m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega}\right]\left(1-\frac{\lambda_d m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega}-\frac{m_L\Omega}{1+m_L\Omega}\right) \\
& + m_L\frac{1+m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega} - (1-m_L)\frac{m_L\Omega}{(1+\lambda_d m_{L-1}\Omega)^2} + \frac{m_L\Omega}{1+m_L\Omega}\frac{\lambda_d m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega}\left(\frac{1}{1+m_L\Omega}+\frac{1}{1+\lambda_d m_{L-1}\Omega}\right)\Big\} \\
& = \frac{c\times log(1-\lambda_d)}{1+\lambda_d m_{L-1}\Omega}\Big\{\frac{m_L\Omega}{1+m_L\Omega}\lambda_d m_{L-1}\Omega + \frac{(1-m_L)(1-\lambda_d m_{L-1}\Omega)}{1+\lambda_d m_{L-1}\Omega} - (1-m_L)\frac{1+m_{L-1}\Omega}{1+m_L\Omega}m_L\Omega + m_L(1+m_{L-1}\Omega)\Big\}
\end{aligned}
$$

Consider the last two terms in the above equation,

$$
-(1-m_L)\frac{1+m_{L-1}\Omega}{1+m_L\Omega}m_L\Omega + m_L(1+m_{L-1}\Omega) = m_L(1+m_{L-1}\Omega)[1-\frac{(1-m_L)\Omega}{1+m_L\Omega}] > 0
$$

Since $log(1-\lambda_d) < 0$, the second order condition is satisfied.

C.5 Proof for Proposition 3

C.5.1 Comparative statics with respect to c

We first consider the comparative statics with respect to the per layer bankruptcy cost c,

$$
\begin{aligned}
\frac{\partial FOC_{L,prv}}{\partial c} &= \frac{\partial FOC_{L,prv}}{\partial F_y}\frac{\partial F_y}{\partial c} \\
&- \left\{(1-m_L)\frac{1+m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega} - \frac{m_L\Omega}{1+m_L\Omega}\frac{\lambda_d m_{L-1}\Omega}{1+\lambda_d m_{L-1}\Omega}\right\}
\end{aligned}
$$

where $\frac{\partial F_y}{\partial c} = -\frac{\frac{\partial v_L(L)}{\partial c}}{\lambda_y + \frac{\partial v_L(L)}{\partial F_y}}$. Since $\frac{\partial v_L(L)}{\partial c} < 0$, $\frac{\partial F_y}{\partial c} > 0$. Furthermore,

$$
\frac{\partial FOC_{L,prv}}{\partial F_y} = \frac{\partial FOC_{L,prv}}{\partial H}h(F_y) < 0
$$

Hence $\frac{\partial FOC_{L,prv}}{\partial c} < 0$. By implicit function theorem, we have $\frac{\partial L^*}{\partial c} < 0$.

C.6 Proof for Proposition 4

First, we show that when $c = 0$, there is always gains from trade, i.e. $w_L > \hat{v}_L$. We want to show

$$(1-H)(1-\alpha)e > H\sum_{l=0}^{L-1}(m_l - m_{l-1})(v_L - b_l)$$

From Eq. (60), we know that b_l is increasing in l, hence $H\sum_{l=0}^{L-1}(m_l - m_{l-1})(v_L - b_l) < H(v_L - b_0)$. Plug in $b_0 = \beta v_L + (1-\beta)(1-\lambda_y)\alpha\mathbb{E}[\alpha V_L]$, we get

$$\begin{aligned}
v_L - b_0 &= (1-\beta)(1-\lambda_y)[(1-H)(1-\alpha)e - \alpha H\sum_{l=0}^{L-1}(m_l - m_{l+1})(v_L - b)l)] \\
&< (1-\beta)(1-\lambda_y)(1-H)(1-\alpha)e
\end{aligned}$$

Hence the benefit always dominates the cost of forming chains.

Next, we show that the social first order condition is the same as the private first order condition with respect to L. The social welfare can be written as

$$\begin{aligned}
&W(F_y(L), L) = e + \lambda_y y + (1-\lambda_y)\Big\{(1-H(F_y))((1-\alpha)e + \alpha\mathbb{E}[W|y \geq F_y]) \\
&+ \lambda_d H(F_y)\sum_{l=0}^{L-2} m_l\left[\alpha\mathbb{E}[W|y < F_y] - \alpha(v_L(L) - b_l) - c(L-l)\right] \\
&+ m_{L-1}H(F_y)\left[\alpha\mathbb{E}[W|y \leq F_y] - \alpha(v_L(L) - b_{L-1}) - c\right]\Big\}
\end{aligned}$$

Define $W(F_y(L), L) = w_L + \lambda_y y$, where w_L is independent of the realization of y in this period. Rewriting the above equation in terms of w_L, we get

$$\begin{aligned}
&w_L = e + (1-\lambda_y)\Big\{(1-H(F_y))((1-\alpha)e + \alpha(\lambda_y\mathbb{E}[y|y \geq F_y] + w_L)) \\
&+ \lambda_d H(F_y)\sum_{l=0}^{L-2} m_l\left[\alpha(\lambda_y\mathbb{E}[y|y < F_y] + w_L) - \alpha(v_L(L) - b_l) - c(L-l)\right] \\
&+ m_{L-1}H(F_y)\left[\alpha(\lambda_y\mathbb{E}[y|y < F_y] + w_L) - \alpha(v_L(L) - b_{L-1}) - c\right]\Big\}
\end{aligned}$$

Consider $w_{L+1} - w_L$ for a given F_y,

$$\begin{aligned} w_{L+1} - w_L = (1-\lambda_y)\Big\{ &(1 - H(F_y))(w_{L+1} - w_L) - (1 - m_L)H(F_y)c \\ &+ \lambda_d H(F_y) \sum_{l=0}^{L-1} m_l \Big[\alpha(w_{L+1} - w_L) - \alpha(v_{L+1} - v_L) + \alpha \frac{\partial b_l(L)}{\partial L}) \Big] \\ &+ m_L H(F_y) \left[\alpha(w_{L+1} - w_L) - \alpha(v_{L+1} - v_L) + \alpha(b_L(L+1) - b_{L-1}(L)) \right] \Big\} \end{aligned}$$

Moving $w_{L+1} - w_L$ to the left hand side, we get

$$\begin{aligned} w_{L+1} - w_L \propto (1-\lambda_y)H(F_y)\Big\{ &\underbrace{\lambda_d \sum_{l=0}^{L-1} m_l \alpha \frac{\partial b_l(L)}{\partial L} - (1 - m_L)c + m_L \alpha(b_L(L+1) - b_{L-1}(L))}_{\propto (v_{L+1} - v_L)} \\ &- \alpha(v_{L+1} - v_L) \Big\} \end{aligned}$$

From Eq. (65), we see that

$$v_{L+1} - v_L \propto \lambda_d \sum_{l=0}^{L-1} m_l \alpha \frac{\partial b_l(L)}{\partial L} - (1 - m_L)c + m_L \alpha(b_L(L+1) - b_{L-1}(L))$$

Hence at the decentralized equilibrium where $v_{L+1} - v_L = 0$, we have $w_{L+1} - w_L = 0$. In other words, the decentralised condition coincides with the social planner's first order condition with respect to L.

D Extension: Optimal Contracts

D.1 Setting

In this section, we endogenize the face value of the debt contract as well and derive the condition under which the face value equals e. Denote by π_t a generic debt contract; we assume that it takes the form of $\pi_t = \{\tilde{F}_{y,s}, F_{d,s+1}\}_{s=t}^{T}$, with an exogenously given debt maturity parameter λ_d as in the main text.

$$\tilde{F}_{y,s} \cdot \mathbf{1}_{\text{project matures at period } s, \text{ w.p. } \lambda_y} + F_{d,s+1} \cdot \mathbf{1}_{\text{debt matures at period } s+1, \text{ w.p. } \lambda_d},$$

where $\{F_{d,s+1}\}$ is $\mathcal{F}_s$-measurable for any $s \geq t$. Importantly, it cannot depend on tomorrow's fundamental y_{s+1}. The space of debt contracts is now $\Pi \equiv \mathbb{R}_+^{T-t+1} \times \mathbb{R}_+^{T-t}$. Same as before, we allow debtors, after knowing the realization of y_t, to renegotiate by "prepaying" the debt contract. In other words, they can pay the lender F_d and eliminate all future obligations.

Because the layer-l fund is essentially using its asset holding with a market value of $F_{d,l-1}$ to back its debt issuance with a market value of $F_{d,l}$, and fund managers have no initial wealth, we impose the following condition throughout the paper:

$$F_{d,l} \leq F_{d,l-1} \leq e \text{ for } \forall l. \tag{77}$$

The first part of the condition (77) essentially rules out the "Ponzi" scheme by any fund in which a fund maintains a debt that is underfunded relative to its asset holdings but keeps rolling over this debt from OLG households. A side benefit of this assumption is that it simplifies the prepayment process, as the cash-flows trickle down to the bottom. The second part $F_{d,l} \leq e$ in condition (77) captures the fact that households can only afford to pay e.

Similar to the main text, if rollover fails at layer-l, the departing households recover

$$\min(\alpha B_l(y, L), F_{d,l}) - c \cdot (L - l).$$

The value functions are similar to before except that the face value is $F_{d,l,t}$, instead of e. We adjust the feasible contract space to account for the endogenous face values.

$$\hat{\Pi} \equiv \{\pi \in \Pi : V_L(\{F_{y,s}, F_{d,s+1}\}_{s=t}^{T}, L) \leq F_{d,t} \leq e \quad \text{for} \quad \forall t\}. \tag{78}$$

Finally, the definition of equilibrium now includes the optimal design of F_d's

Definition 3 *The equilibrium credit chain is a set of contracts $\{\pi_{l,t}\}_{0 \leq l \leq L-1}$ and credit chain length L^* such that*

1. *When layer-l's liability matures,*[32]

$$\pi_l = \arg\max_{\pi \in \hat{\Pi}} \quad \mathbf{1}^l_{rollover}(P_l(y, \pi; \pi_{l-1}, L^*) + V_l(y, \pi; \pi_{l-1}, L^*)), \tag{79}$$

$$s.t. \quad F_{y,l} \leq F_{y,l-1} \leq y \text{ in } (7) \qquad F_{d,l} \leq F_{d,l-1} \leq e \text{ in } (77). \tag{80}$$

2. *The equilibrium L^* is such that the final layer of fund manager $(L^* - 1)$ prefers to borrow directly from households than to borrow via other fund managers:*

$$P_{L^*-1}(L^*) + V_{L^*-1}(L^*) \geq P_{L^*-1}(L^* + l) + V_{L^*-1}(L^* + l) \quad \text{for} \quad l \geq 1. \tag{81}$$

[32] When $t = 0$, $\mathbf{1}^l_{rollover} = 1$ for all l.

Furthermore, for all other funds $0 < l < L^* - 1$,

$$P_l(L^*) + V_l(L^*) \geq P_l(l+1) + V_l(l+1). \tag{82}$$

In other words, the funds in intermediary layers prefer to borrow via other funds than to borrow from the households.

3. *Due to perfect competition,*

$$P_l - P_{l-1} + V_l = 0. \tag{83}$$

D.2 Optimal Contracts

In this section, we derive conditions under which in equilibrium, $F_{d,l,t} = e$.

Assumption 4 *Denote* L^* *as the equilibrium chain length. The primitives of our model satisfy:*

$$(1-\alpha)(1-H(F_y)) - \frac{h(F_y)}{\lambda_y}e - \sum_{l=0}^{L^*-2} m_l \lambda_d h(F_y)\frac{1}{\lambda_y}c(L^*-l) - m_{L^*-1}h(F_y)\frac{1}{\lambda_y}c \geq 0$$

Under Assumption 4, the optimal contract in our economy is independent of history and $F_{d,l,t} = e$. Assumption 4 guarantees that inequality (77) always binds (so that in the optimal contract $F_{d,l,t} = e$), and it is more likely to be true when e is relatively small.

We first show that $F_{d,l,t} = F_{d,l}$, i.e. the optimal F_d for each layer is constant over time if the managers do not face rollover issues in this period. We start from the problem between layer $(L-1)$ and the households. Layer $(L-1)$ is given a contract π_{L-2} by layer $(L-2)$; the contract specifies a sequence of payments if debt matures $\{F_{d,L-2,t}\}_{t=0}^{T}$ and a payment if project matures $F_{y,L-2}$. T is the stopping time, either when the contract or when the project matures. Plugging in P_{L-1}, layer $L-1$ maximizes the following,

$$\begin{aligned}
&\max_{F_{d,l-1}} -P_{L-2} + \lambda_y F_{y,L-2} + (1-\lambda_y)\mathbb{E}\Big[\sum_{i=0}^{L-2}(1-\lambda_d)^i[\lambda_d \mathbf{1}^i_{rollover}(\alpha V_{L-1}(y,\pi'_{L-1};\pi'_{L-2},L) + \alpha F_{d,L-2} + (1-\alpha)F_{d,l-1}) \\
&+ (1-\mathbf{1}^i_{rollover})(\alpha B_i(y,L) - c(L-i-1))] + (1-\lambda_d)^{L-1}\mathbf{1}^{L-1}_{rollover}(\alpha V_{L-1}(y,\pi'_{L-1};\pi_{L-2},L) + (1-\alpha)F_{d,L-1}) \\
&+ (1-\lambda_d)^{L-1}(1-\mathbf{1}^{L-1}_{rollover})(\alpha B_{L-1}(y,L) - c)\Big] \\
&s.t. \quad F_{d,L-1} \leq F_{d,L-2}
\end{aligned}$$

The first order condition with respect to $F_{d,L-1,t}$ is

$$0 = -\mu^{\lambda_d}_{L-1,t} + (1-\alpha)\mathbb{E}[\sum_{i=0}^{L-2}(1-\lambda_d)^i\lambda_d\mathbf{1}^i_{rollover} + (1-\lambda_d)^{L-1}\mathbf{1}^{L-1}_{rollover}]$$
$$+ (1-\lambda_d)^{L-1}\frac{d\Pr(\text{rollover at layer } L-1)}{dF_{d,l-1,t}}(F_{d,l-1,t} - \alpha B_{L-1}(y,L) + c)$$

where $\mu^{\lambda_d}_{L-1,t}$ is the Lagrangian Multiplier in front of $F_{d,L-2,t} - F_{d,L-1,t} \geq 0$.

If $\pi_{L-2} = \pi^*_{L-2}$ is stationary and $F_{d,L-2,t}$ is constant over time, then $F^*_{d,l-1,t} = F_{d,l-1}$.

The same logic applies to $F^*_{d,l,t} = F_{d,l}$ for all $0 \leq l \leq L-1$. For $0 \leq l < L-1$, its objective can be written as

$$\max_{F_{d,l}} -P_{l-1} + \lambda_y F_{y,l-1} + (1-\lambda_y)\alpha\Big\{(1-\lambda_d)^{l+2}\mathbb{E}V_l(y',\pi_l;\pi_{l-1},L) + (1-\lambda_d)^{l+1}\lambda_d\mathbb{E}(1-\mathbf{1}^{l+1}_{rollover})V_l(y',\pi'_l;\pi_{l-1},L)$$
$$+\sum_{i=0}^{l-1}(1-\lambda_d)^i\lambda_d\mathbb{E}[\mathbf{1}^i_{rollover}(F_{d,l-1} - F_{d,l+1} - P'_{l-1} - P'_l + \max_{\pi'_l}(P'_l + V_l(y',\pi'_l;\pi'_{l-1},L)) + \max_{\pi'_{l+1}}(P'_{l+1} + V_{l+1}(y',\pi'_{l+1};\pi_l,L)))]$$
$$+ (1-\lambda_d)^l\lambda_d\mathbb{E}[\mathbf{1}^l_{rollover}(-F_{d,l+1} - P'_l + \max_{\pi'_l}(P'_l + V_l(y',\pi'_l;\pi_{l-1},L))) + \max_{\pi_{l+1'}}(P'_{l+1} + V_{l+1}(y',\pi'_{l+1};\pi_l,L))]$$
$$+ (1-\lambda_d)^{l+2}\mathbb{E}V_{l+1}(y',\pi_{l+1};\pi_l,L) + (1-\lambda_d)^{l+1}\lambda_d\mathbb{E}[\mathbf{1}^{l+1}_{rollover}(-F_{d,l+1} + \max_{\pi'_{l+1}}V_{l+1}(y',\pi'_{l+1};\pi_l,L))]\Big\} + P_{l+1}$$

we know in equilibrium $P'_{l-1} = \max_{\pi'_l}(P'_l + V_l(y',\pi'_l;\pi'_{l-1},L))$ and $P'_l = \max_{\pi'_{l+1}}(P'_{l+1} + V_{l+1}(y',\pi'_{l+1};\pi'_l,L))$, so the above can be simplified as

$$\max_{F_{d,l}} -P_{l-1} + \lambda_y F_{y,l-1} + (1-\lambda_y)\alpha\Big\{(1-\lambda_d)^{l+2}\mathbb{E}V_l(y',\pi_l;\pi_{l-1},L) + (1-\lambda_d)^{l+1}\lambda_d\mathbb{E}(1-\mathbf{1}^{l+1}_{rollover})V_l(y',\pi'_l;\pi_{l-1},L)$$
$$+\sum_{i=0}^{l-1}(1-\lambda_d)^i\lambda_d\mathbb{E}[\mathbf{1}^i_{rollover}(F_{d,l-1} - F_{d,l+1})] + (1-\lambda_d)^l\lambda_d\mathbb{E}[\mathbf{1}^l_{rollover}(-F_{d,l+1} + V_l(y',\pi'_l;\pi_{l-1},L) + V_{l+1}(y',\pi'_{l+1};\pi'_l,L) + P'_{l+1})]$$
$$+ (1-\lambda_d)^{l+2}\mathbb{E}V_{l+1}(y',\pi_{l+1};\pi_l,L) + (1-\lambda_d)^{l+1}\lambda_d\mathbb{E}[\mathbf{1}^{l+1}_{rollover}(-F_{d,l+1} + P'_{l+1} + V_{l+1}(y',\pi'_{l+1};\pi_l,L))]\Big\} + P_{l+1}$$

subject to $F_{d,l,t} \leq F_{d,l-1,t}$. Denote the Lagrangian multiplier as $\mu^{\lambda_d}_{l,t}$. The first order condition with respect to $F_{d,l,t}$ is

$$\begin{aligned}0 =& -\mu^{\lambda_d}_{l,t} + \mu^{\lambda_d}_{l+1,t} + \frac{dP_{l+1}}{dF_{d,l,t}}\\ =& -\mu^{\lambda_d}_{l,t} + \mu^{\lambda_d}_{l+1,t} + (1-\lambda_d)^l\lambda_d\frac{d\Pr(\text{rollover at layer } l)}{dF_{d,l,t}}(F_{d,l-1} - \alpha B_{l-1}(y,L) + c)\end{aligned}$$

If π^*_{l-1} does not depend on history and is stationary, then it is straightforward that $F^*_{d,l,t} = F_{d,l}$.

Next, we show that $F_{d,l} = F_d$ across layers. Since the problem is identical over time, we loose the time subscript. The first order condition with respect to $F_{d,L-1}$ in equilibrium is

$$
\begin{aligned}
0 = & -\mu_{L-1}^{\lambda_d} + (1-\alpha)\sum_{l=0}^{L-2}(1-\lambda_d)^l\lambda_d \Pr(\text{rollover at layer } l) + (1-\alpha)(1-\lambda_d)^{L-1}\Pr(\text{rollover at layer } l) \\
& (1-\lambda_d)^{L-1}\frac{d\Pr(\text{rollover at layer } L-1)}{dF_{d,L-1}}[F_{d,L-1} - \alpha B_{L-1}(y,L) + c]
\end{aligned}
$$

The first order condition with respect to $F_{d,l}$ for $0 < l < L-1$ is,

$$
0 = -\mu_l^{\lambda_d} + \mu_{l+1}^{\lambda_d} + (1-\lambda_d)^l\lambda_d\frac{d\Pr(\text{rollover at layer L-1})}{dF_{d,l}}(F_{d,l-1} - \alpha B_l(y,L) + c(L-l))
$$

For $l = 0$, the first order condition is

$$
0 = -\mu_0^{\lambda_d} + \mu_1^{\lambda_d} + \lambda_d\frac{d\Pr(\text{rollover at layer l})}{dF_{d,0}}(F_{d,l-1} - \alpha B_l(y,L) + c(L-l))
$$

Substituting in all the Lagrangian multipliers.

$$
\begin{aligned}
0 = & -\mu_0^{\lambda_d} + (1-\alpha)\sum_{l=0}^{L-2}(1-\lambda_d)^l\lambda_d \Pr(\text{rollover at layer } l) + (1-\lambda_d)^{L-1}\Pr(\text{rollover at layer } l) \\
& + \sum_{l=0}^{L-1}(1-\lambda_d)^l\lambda_d\frac{d\Pr(\text{rollover at layer } l)}{dF_{d,l}}(F_{d,l-1} - \alpha B_l(y,L) + c(L-l)) \\
& + (1-\lambda_d)^{L-1}\frac{d\Pr(\text{rollover at layer } L-1)}{dF_{d,L-1}}(F_{d,L-2} - \alpha B_{L-1}(y,L) + c) \qquad (84)
\end{aligned}
$$

Denote layer-0's choice as $F_{d,0} = F_d$, satisfying equation (84). If $\mu_0^{\lambda_d} > 0$, then $F_d = e$, and since $\mu_{L-1}^{\lambda_d} \geq \mu_{L-2}^{\lambda_d} \geq ... \geq \mu_0^{\lambda_d} > 0$, all the constraints are binding, i.e. $F_{d,l-1} = F_{d,L-2} = ... = F_d$.

If $\mu_0^{\lambda_d} = 0$, then $F_d < e$, it must be the case that $\frac{d\Pr(\text{rollover at layer } l)}{dF_{d,l}} < 0$ holds for at least one l. Denote $\hat{l}$ as the smallest l such that $\frac{d\Pr(\text{rollover at layer } l)}{dF_{d,l}} < 0$. This implies that for $l < \hat{l}$, $\frac{d\Pr(\text{rollover at layer l})}{dF_{d,l}} = 0$, so the first order conditions for $F_{d,l}$ $(l \geq \hat{l})$ are the same as that for $F_{d,0}$. In other words, $F_{d,l} = F_d$. For $l < \hat{l}$, we have $\mu_l^{\lambda_d} > 0$, so the constraint is binding, i.e. $F_{d,l-1} = F_{d,L-2} = ... = F_{d,\hat{l}-1} = F_d$.

So far we have shown that when there is no rollover concerns, we have $F_{d,l,t} = F_d$ being constant over time and across layers. Now we just to show when y is small, and when the money raised from the unconstrained optimal contract is smaller than the amount owed, the managers cannot deviate and set higher F_d. For managers in layer 1 to layer $L-1$, because $F_{d,l} \leq F_{d,l-1}$ is binding, they cannot set higher F_d. For layer 0, we will next show that

Assumption 4 ensures that $F_{d,0} \leq e$ is binding. Hence the entrepreneur at layer 0 cannot deviate and set higher F_d either. As a result, $F_{d,l,t} = F_d$ for all layer l and time t.

The proof for $F_{y,l,t} = F_y$ is the same as in Appendix C.2. In equilibrium, F_y is the minimal payment if project matures such that the new households are willing to rollover debt, for a given F_d. By definition

$$\begin{aligned} &F_d = V_L(\{F_y, F_d\}, L) \qquad \text{for} \quad y \geq F_y \\ \Rightarrow &F_d = \lambda_y F_y + v_L(\{F_y, F_d\}, L) \end{aligned}$$

Since all layers have the same F_y and rollover fails when $y < F_y$, we have $\Pr(\text{rollover at layer } l) = 1 - H(F_y)$. Plug this expression in the first order condition of F_d, we get

$$\begin{aligned} &-\mu_0^{\lambda_d} + (1-\alpha)(1-H(F_y)) - \sum_{l=0}^{L-2} m_l \lambda_d h(F_y) \frac{dF_y}{dF_d}(F_d - \alpha B_l(F_y, L) + c(L-l)) \\ &-(1-\lambda_d)^{L-1} h(F_y) \frac{dF_y}{dF_d}(F_d - \alpha B_{L-1}(F_y, L) + c) = 0 \\ &\mu_0^{\lambda_d}(e - F_d) = 0 \qquad \mu_0^{\lambda_d} \geq 0 \end{aligned}$$

When $F_d \leq e$ is binding, $\frac{dF_y}{dF_d} = \frac{1}{\lambda_y}$. Hence

$$\begin{aligned} &(1-\alpha)(1-H(F_y)) - \sum_{l=0}^{L-2} m_l \lambda_d h(F_y) \frac{1}{\lambda_y}(F_d - \alpha B_l(F_y, L) + c(L-l)) - m_{L-1} h(F_y) \frac{1}{\lambda_y}(F_d - \alpha B_{L-1}(F_y, L) + c) \\ \geq &(1-\alpha)(1-H(F_y)) - \sum_{l=0}^{L-2} m_l \lambda_d h(F_y) \frac{1}{\lambda_y}(F_d + c(L-l)) - m_{L-1} h(F_y) \frac{1}{\lambda_y}(F_d + c) \end{aligned}$$

Under Assumption 4, the above equation is greater than or equal to 0. Hence $F_d = e$.