Computing Nash equilibria Gets Harder — New Results Show Hardness Even for Parameterized Complexity

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Abstract

In this paper we show that some decision problems regarding the computation of Nash equilibria are to be considered particularly hard. Most decision problems regarding Nash equilibria have been shown to be NP-complete. While some NP-complete problems can find an alternative to tractability with the tools of Parameterized Complexity Theory, it is also the case that some classes of problems do not seem to have fixed-parameter tractable algorithms. We show that k-UNIFORM NASH and k-MINIMAL NASH SUPPORT are $W[2]$-hard. Given a game $\mathcal{G}=(A,B)$ and a non-negative integer k, the k-UNIFORM NASH problem asks whether $\mathcal{G}$ has a uniform Nash equilibrium of size k. The k-MINIMAL NASH SUPPORT asks whether $\mathcal{G}$ has Nash equilibrium such that the support of each player's Nash strategy has size equal to or less than k. First, we show that k-UNIFORM NASH (with k as the parameter) is $W[2]$-hard even when we have 2 players, or fewer than 4 different integer values in the matrices. Second, we illustrate that even in zero-sum games k-MINIMAL NASH SUPPORT is $W[2]$-hard (a sample Nash equilibrium in a zero-sum 2-player game can be found in polynomial time (von Stengel 2002)). Thus, it must be the case that other more general decision problems are also $W[2]$-hard. Therefore, the possible parameters for fixed parameter tractability in those decision problems regarding Nash equilibria seem elusive.

Keywords: Parameterized complexity, Game theory, Nash equilibrium.

1 Introduction

The modern mathematical treatment of the study of decisions taken by participants whose interests are in conflict is now generally labeled as "game theory" and its origins are attributed to von Neumann who developed the mini-max theorem. Although Borel had formalized earlier the concept of pure and mixed strategies, the first book in the field (von Neumann & Morgenstern 1947) established most of the area. One of the core concepts is now known as Nash equilibria after John Nash. Although Nash's theorem is the generalization of more than two participants regarding an earlier known result for two players, his approach provided a series of new insights. Nash established that every finite game with a finite number of players had to have a stable outcome (maybe with mixed strategies), where no participant would change their decision-making process even if it were to know the decision-making process of other players. Nash proved this result using Brouwer's fixed-point theorem and in doing so provided a simpler proof to von Neumann's mini-max theorem (although the original proof also used Brouwer's fixed-point theorem (Luce & Raiffa 1957)). Brouwer's fixed-point theorem requires topological notions regarding continuous functions and its proof is non-constructive. Therefore, this existence result did not provide an algorithm to find such Nash equilibria.

Nash equilibria are important because they represent the outcome of many scenarios, many of which have been previously used to model the behavior of participants (governments, unions, individuals) in many social models. However, scenarios where game theory applies now emerge in many multi-agent interactions. Tardos & Vazirani (2007) illustrate that the canonical example of a game (the Prisoner's Dilemma) actually can be the setting for making decisions by two Internet service providers (ISPs) who must simultaneously choose between two routing schemes. Given a game computation of Nash equilibria (or designing algorithms for obtaining them) has been labeled the most important complexity problem of this time (Papadimitriou 2001). While finding Nash equilibria in two-player zero-sum games can be performed in polynomial time using tools from linear programming (von Stengel 2002), the situation becomes much more complex as soon as three or more players are involved (Tardos & Vazirani 2007) (replacing zero-sum, that is complementary pay-offs, with common pay-offs results in NP-Completeness results even for 2 players (Chu & Halpern 2001)). The computation of Nash equilibria is not a decision problem, and therefore, an alternative tool has been introduced to show that it is unlikely that Nash equilibria can be found in polynomial time (Papadimitriou 2007). In particular, the class PPAD [1] was used to show that the finding of Nash equilibria is at least as hard as the finding of fixed points in settings where Brouwer's theorem applies. However, the first complexity results for computing Nash equilibria used classic notions of complexity theory. In particular, NP-Completeness proofs for decision problems regarding Nash equilibria are interesting (Gilboa & Zemel 1989). Those proofs were achieved from reductions from CLIQUE and SET COVER. After this early complexity results, several researchers have introduced different types of equilibria and games. Typically, the same NP-hard re-

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[1] polynomial parity argument (decision case).

sults have been achieved (Bonifaci et al. 2008, Abbott et al. 2005, Codenotti & Stefankovic 2005, Conitzer & Sandholm 2003), perhaps with more specialized games. In this paper we establish the parameterized complexity of finding uniform Nash equilibria in imitation games and computing the minimal support of Nash equilibria for normal form games for the following reasons:

- Parameterized complexity has advanced the algorithm design for many NP-complete problems (Downey & Fellows 1998, Niedermeier 2006).
- "A uniform mixed strategy is probably the simplest way of mixing pure strategies" (Bonifaci et al. 2008).
- There is a corresponding one-to-one relation between Nash equilibria of two-player games and Nash strategies for the row player in an imitation game (Codenotti & Stefankovic 2005).
- Finding the support of a Nash equilibrium leads to fining a sample Nash equilibrium in polynomial time, hence the first trivial parameter for computing a sample Nash equilibrium can be considered the size of the support of Nash equilibrium.

The rest of the paper is organized as follows. In Section 2 we give formal definitions for game, graph and parameterized complexity and review some theorems. In Section 3, we prove the main parameterized hardness results of the paper. In Section 4 we discuss further the implications of our results for bi-matrix games.

2 Preliminaries

We review notation and concepts in three areas: game theory, graph theory, and parameterized complexity theory, since these preliminaries are necessary to present our main result.

2.1 Game theory

A two-player *normal form* game (also bi-matrix game) $\mathcal{G}$ consists of two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$, where a_{ij} denotes the payoff for the first player and b_{ij} denotes the payoff for second player when the first player plays his i-th strategy and the second player plays his j-th strategy. We will identify the first player as the row player and second player as the column player. Therefore, a row is a pure strategy for the row player while a column is a pure strategy for the column player. Players select their strategy without knowledge of their opponent's choices and the objective of each player is to individually maximize their payoff.

While in general, there may be different types of games, here we study *imitation games* and *Zero-Sum* games and we will explain them explicitly in the rest of this section.[2] A mixed strategy for a player is a probability distribution over his pure strategy space.

Definition 2.1 *A mixed strategy $\vec{x}$ for a player is an ordered n-tuple $(x_1, ..., x_n)$ where $\sum_{i=1}^{i=n} x_i = 1$ and $x_i \geq 0$. We denote the probability space over the set of rows of A and columns of B by $\Delta(A)$ and $\Delta(B)$, respectively.*

[2]Note that when presenting negative/hardness complexity results, choosing a more special case of the problem is useful because the hardness of the special problem implies the hardness of the general problem as the special problem reduces trivially to the general problem.

The *support* of mixed strategy $\vec{x}$ is the set of pure strategies which are played with positive probability, that is $\{i : 1 \leq i \leq n, x_i > 0\}$, and we will denote it by $supp(\vec{x})$. A *mixed strategy profile* is an ordered pair $(\vec{x}, \vec{y})$ where $\vec{x}$ is a mixed strategy of the row player and $\vec{y}$ is a mixed strategy for the column player. A pure strategy can be considered a deterministic algorithm, while a mixed strategy can be considered a way to participate in the game with a randomized algorithm. The difference is important because not every game has each player using a pure strategy and remaining this way even with the knowledge of the other player's strategy. However, every game has at least one stable outcome with mixed strategies.

Definition 2.2 *In a bi-matrix game $\mathcal{G} = (A, B)$ a strategy profile $(\vec{x}^*, \vec{y}^*)$ is a Nash equilibrium if*

$$\forall \vec{x} \in \Delta(A) \qquad \vec{x}^{*T} A\vec{y}^* \geq \vec{x}^T A\vec{y}^*, \text{ and}$$
$$\forall \vec{y} \in \Delta(B) \qquad \vec{x}^{*T} B\vec{y}^* \geq \vec{x}^{*T} B\vec{y}.$$

Clearly, by definition, in a Nash equilibrium no player wants to deviate from the equilibrium points, even when the opponent's strategies become known. In other words, a strategy profile $(\vec{x}^*, \vec{y}^*)$ is a Nash equilibrium if and only if the strategy $\vec{x}^*$ of the row player is a best response to the strategy $\vec{y}^*$ of the column player and vice versa. The following result (von Stengel 2002) shows that for Nash equilibria, if we are provided with the strategy of one player, we can easily compute the other player's best strategy to complete the profile for a Nash equilibrium.

Theorem 2.1 *In a bi-matrix game $\mathcal{G}$=(A,B), the strategy $\vec{x}$ of the row player is the best response to the column player's $\vec{y}$ strategy if and only if*

$$\forall i \left[i \in supp(\vec{x}) \Longrightarrow \vec{e_i}^T A\vec{y} \stackrel{def}{=} (A\vec{y})_i = \max_{j=1,\ldots,n} (A\vec{y})_j \right].$$

To find the support of the other player given the Nash equilibrium component of one player, we find maximum values in the vector of payoffs that results from the opponent's Nash equilibrium's strategy being applied to the matrix of the player. By linear programming we can compute the other mixed strategy of the Nash strategy profile. We focus on *Zero-Sum* games (games whose matrices are the negative of each other, i.e. $\mathcal{G}$=(A,−A)). The following theorem is one of the central theorems for of *Zero-Sum* games, and it is known as Minimax theorem.

Theorem 2.2 *In* Zero-Sum *game $\mathcal{G}$=(A,−A), a strategy profile ($\vec{x}^*$,$\vec{y}^*$) is a Nash equilibrium if and only if it satisfies the following conditions*

$$a) \qquad \vec{x}^{*T} A\vec{y}^* = \max_{\vec{x}\in\Delta(A)} \min_{\vec{y}\in\Delta(A)} \vec{x}^T A\vec{y} = \min_{\vec{y}\in\Delta(A)} \max_{\vec{x}\in\Delta(A)} \vec{x}^T A\vec{y}.$$

$$b1) \qquad \vec{x}^* \in \arg\max_{\vec{x}\in\Delta(A)} \min_{\vec{y}\in\Delta(A)} \vec{x}^T A\vec{y},$$

$$b2) \qquad \vec{y}^* \in \arg\min_{\vec{y}\in\Delta(A)} \max_{\vec{x}\in\Delta(A)} \vec{x}^T A\vec{y}.$$

As we mentioned in the introduction we are interested in imitation games. In an imitation game, the row player receives a payoff of 1 if he plays the same strategy as the column player but otherwise he receives 0. In other words, an imitation game can be expressed as a pair of matrices (I, M), where I is the identity matrix and M is a square matrix.

Theorem 2.3 *Let ($\vec{x}^*$,$\vec{y}^*$) be a Nash equilibrium of the imitation game (I,M), then supp($\vec{x}^*$) $\subseteq$ supp($\vec{y}^*$).*

Proof: By contradiction, we let $i \in supp(\vec{x}^*)$ and $i \notin supp(\vec{y}^*)$ (thus, $y_i^* = 0$). Since $(\vec{x}^*, \vec{y}^*)$ is a Nash equilibrium, we know $\vec{x}^*$ is a best response to $\vec{y}^*$. That is,

$$(I\vec{y}^*)_i = \max_{j=1,\dots,n} (I\vec{y}^*)_j \quad \text{or,}$$

$$y_i^* = \max_{j=1,\dots,n} y_j^* \neq 0.$$

This contradicts $y_i^* = 0$. □

The study of imitation games is also justified, because any bi-matrix game $\mathcal{G}$ can be transformed into an imitation game with a one to one relation between Nash equilibria of $\mathcal{G}$ and the Nash strategies of the row player in the corresponding imitation game (Codenotti & Stefankovic 2005).

Lemma 2.1 *Let $\mathcal{G}$=(A,B) be a bi-matrix game and consider the matrix C defined by blocks as $C = \begin{pmatrix} 0 & B \\ A^T & 0 \end{pmatrix}$. If C does not have a zero row, then there is a one-to-one relation between Nash equilibria of $\mathcal{G}$ and Nash strategies for the row player in the imitation game (I,C), where I is the identity matrix of size $2n$.*

Definition 2.3 *A mixed strategy $\vec{x}$ is called a uniform mixed strategy if for all $i \in supp(\vec{x})$, we have $x_i = 1/\|supp(\vec{x})\|$.*

That is, $x_i = x_j$, for all $i, j \in supp(\vec{x})$. A Nash equilibrium ($\vec{x}$,$\vec{y}$) is called a *uniform* Nash equilibrium if both mixed strategies $\vec{x}$ and $\vec{y}$ are uniform mixed strategies.

2.2 Graph theory

Definition 2.4 *A graph G is a pair (V,E) of sets where $E \subseteq \{\{u, v\} : u, v \in V\}$. The members of V are called vertices and the sets $\{u, v\} \in E$ are called edges of G. An* induced *subgraph of $G = (V, E)$ is a graph G'=(V',E') such that $V' \subseteq V, E' \subseteq E$ and we have $\forall v_1, v_2 \in V' [\{v_1, v_2\} \in E \Longrightarrow \{v_1, v_2\} \in E']$.*

For a given subset V' of V, the induced subgraph by V' is denoted by $G_{V'}$.

Definition 2.5 *A graph G=(V,E) is called* complete *if for all $v_1, v_2 \in V$, $\{v_1, v_2\} \in E$. An induced complete subgraph of a graph $G = (V, E)$ is called a* clique.

Definition 2.6 *A graph G=(V,E) is called* regular *if for all $v_1, v_2 \in V$, $d(v_1) = d(v_2)$ where $d(v)$ is the* degree *of the vertex v and is given by $d(v) = \|\{u \in V : \{u, v\} \in E\}\|$.*

Lemma 2.2 *Let $V' \subseteq V$ so that $G_{V'}$ has at least one edge. Consider $u \in V \setminus V'$. If $G_{V'}$ and $G_{V' \cup \{u\}}$ are regular, then both are cliques.*

Proof: Let $d \geq 1$ be the degree of $G_{V'}$. Then, from the regularity of $G_{V' \cup \{u\}}$, it must have a positive degree and there must be an edge $\{u, v\}$ involving u. Then, the degree of u must be the same as the degree of v which is $d + 1$. That is, u is a neighbor of all the vertices of V'. This implies that any vertex of $G_{V' \cup \{u\}}$ should have the same degree as u. Hence every vertex in $G_{V'}$ has degree d because in this graph u is not involved. Thus, both graphs are complete. □

2.3 Parameterized complexity theory

R. Downey and M. Fellows introduced the field of parameterized complexity theory (Downey & Fellows 1998). In contrast to classical complexity, in parameterized complexity the decision problem is organized in two parts, namely, the input and the parameter.

Definition 2.7 *A parameterized problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet. The second component of the problem is called the parameter.*

We illustrate the distinction with an example of classical decision problem before reviewing the complexity classes in this theory.

Classical Decision Problem
MAX CLIQUE

Instance :	Graph G=(V,E) and positive integer k.
Question :	Is there a subset V' of V such that $G_{V'}$ constitutes a maximal clique of size k?.

The parameterized version is usually referred to as the p-version (Chen & Flum 2008).

Parameterized Decision Problem
p-MAX CLIQUE

Instance :	Graph G=(V,E) and positive integer k.
Parameter :	Positive integer k.
Question :	Is there a subset V' of V such that $G_{V'}$ constitutes a maximal clique of size k?.

Parameterized complexity aims at providing an alternative to exponential algorithms for NP-complete problems by identifying a formulation where the parameter would take small values in practice, and shifting the exponential explosion to this parameter while the rest of the computation is polynomial in the size of the input. A parameterized problem L is fixed-parameter tractable if there is an algorithm that decides in $f(k)n^{O(1)}$ time whether $(x, k) \in L$, where f is an arbitrary computable function depending only on k. FPT denotes the complexity class that contains all fixed-parameter tractable problems. Similar to classical complexity theory, Downey and Fellows (Downey & Fellows 1998) advanced parameterized reduction and completeness notions for the fact that some problems did not appear to be fixed parameter tractable.

Definition 2.8 *Let $L, L' \subseteq \Sigma^* \times \mathbb{N}$ be two parameterized problems. We say L* reduces *to L' by a standard parameterized (many-to-one) reduction if there are functions $k \mapsto k'$ and $k \mapsto k''$ from $\mathbb{N}$ to $\mathbb{N}$ and a function $(x, k) \mapsto x'$ from $\Sigma^* \times \mathbb{N}$ to Σ^* such that*

- *$(x, k) \mapsto x'$ is computable in $k'' \cdot |(x, k)|^c$ time for some constant c and*
- *$(x, k) \in L$ if and only if $(x', k') \in L'$.*

Classical complexity theory has CNF-SATISFIABILITY as a core problem. The question is to decide whether a given Boolean formula in conjunctive normal form (CNF) has a truth assignment or not. Deciding whether a given Boolean formula in conjunctive normal form has an assignment with a certain number of true variables is called WEIGHTED CNF-SATISFIABILITY. Moreover, if the length of each clause in the CNF Boolean formula is equal to or less than t, then the CNF formula is said to be a t-CNF formula. Naturally, the corresponding decision problems are called t-CNF-SATISFIABILITY and t-WEIGHTED

CNF-SATISFIABILITY, respectively. Parameterized tractability of WEIGHTED 2-CNF-SATISFIABILITY is not known (Niedermeier 2006) and $W[1]$ is a basic class for fixed intractability.

Definition 2.9 *The class W[1] contains all problems that can be reduced to* WEIGHTED 2-CNF-SATISFIABILITY *by a parameterized reduction.*

Definition 2.10 *A parameterized problem (L, k) is called W[1]-hard if the parameterized problem* WEIGHTED 2-CNF-SATISFIABILITY *can be reduced to (L, k) by a parameterized reduction.*

A problem in $W[1]$ that satisfies both of the above properties is $W[1]$-complete. Analogously, the class $W[2]$ is defined by swapping WEIGHTED 2-CNF-S with WEIGHTED CNF-SATISFIABILITY (Niedermeier 2006).

Theorem 2.4 *$FPT \subseteq W[1] \subseteq W[2]$.*

There are many problems in $W[1]$ and $W[2]$. No FPT algorithms have been found for any problem in $W[1]$. The above classes may be equal (if NP=P, for example); however, there is evidence to suspect (Downey & Fellows 1998) that $W[2]$-hardness is a strong indication of intractability in the FPT sense.

3 Parameterized hardness result

In this section we prove the $W[2]$-hardness of two Nash equilibria problems. From these other Nash equilibria problems become $W[2]$-hard as well. The first proof uses MAX-CLIQUE while the second one uses a parameterized reduction from SET COVER.

3.1 Finding uniform Nash equilibrium is unlikely to be FPT

A link between Nash equilibria and the maximal cliques of a graph, as well as asymptotically stable stationary points of quadratic programming problems, was established by Bomze (1997, Theorem 9 and Theorem 10). Later, this link was extended to stationary points and Karash-Kuhn-Tucker points of certain forms (Bomze 1998). Recently, McLennan and Tourky (McLennan & Tourky 2005) advanced those ideas to introduce a reduction from MAX CLIQUE to show that several decision problems regarding Nash equilibria in imitation games are NP-Complete. Imitation games have a close relationship with symmetric games (McLennan & Tourky 2005) (and thus, the NP-hardness results for imitation games imply NP-hardness results for symmetric games). It is not unusual that a proof of NP-Completeness does not result in a proof of hardness for parameterized complexity. The VERTEX COVER PROBLEM can be reduced to the INDEPENDENT SET PROBLEM to show that the later is NP-Complete. However, the reduction is not a parameterized reduction and the VERTEX COVER PROBLEM lies in FPT while the INDEPENDENT SET PROBLEM is only known to be in $W[1]$. Some proofs of NP-Complete problems related to Nash equilibria used SATISFIABILITY as the problem that is reduced by constructing a game (Conitzer & Sandholm 2003), but they are not parameterized reductions. Another proof also used CLIQUE (Gilboa & Zemel 1989) but this reduction is also not a parameterized reduction. McLennan and Tourky (McLennan & Tourky 2005) provide additional properties of the resulting game in the transformation and its Nash equilibria. McLennan & Tourky (2005) did not work with uniform Nash equilibrium. The complexity of finding Nash equilibria that are uniform mixed strategies has received attention even more recently (Bonifaci et al. 2008).

We follow the McLennan and Tourky (McLennan & Tourky 2005) reduction to prove our hardness results. We consider the following problem.

k-UNIFORM NASH
Instance : An imitation game $\mathcal{G}=(I,M)$.
Parameter : Positive integer k.
Question : Is there a uniform Nash equilibrium $(\vec{x}, \vec{x})$ such that $\|supp(\vec{x})\| = k$?

We now prove that it is unlikely this problem has a fixed-parameter algorithm.

Theorem 3.1 *k-*UNIFORM NASH *is W[2]-hard.*

In order to prove the theorem, we will produce a parameterized reduction from p-MAX CLIQUE. The p-MAX CLIQUE problem is $W[2]$-complete (Chen & Flum 2008). We will use a specialized version of MAX CLIQUE where we know there is one vertex that has an edge to every other vertex.

Lemma 3.1 *Consider an input [graph G=(V,E), integer k] of* MAX CLIQUE. *We construct a new graph G'=(V',E') such that $V' = V \cup \{u\}$ where $u \notin V$ and $E' = E \cup \{\{u, v\} : v \in V\}$. The reduction $[G= (V,E), k] \mapsto [G' = (V', E'), k' = k + 1]$ is a parameterized reduction with the property that G has a maximal clique of size k if and only if G' has maximal clique of size k'.*

Lemma 3.2 *Consider [G=(V,E), integer k] where there is a vertex u connected to every vertex $v \in V$. We construct a new graph G'=(V',E') as follows. Its set of vertices is $V' = V \setminus \{u\}$ and its edges are $E' = E \setminus \{\{u, v\} : v \in V\}$. The reduction $[G= (V,E), k] \mapsto [G' = (V', E'), k' = k-1]$ is a parameterized reduction with the property that G has a maximal clique of size k if and only if G' has maximal clique of size k'.*

The above two lemmas follow directly from Lemma 2.2 and prove the following lemma.

Lemma 3.3 *Let G=(V,E) be a graph with a vertex connected to every vertex $v \in V$ and let k be a positive integer (the parameter). Deciding whether G has a maximal clique of size k is W[2]-complete.*

Now, we describe the reduction, assuming the instance of MAX CLIQUE (an undirected graph G=(V,E) and an integer k) has one vertex adjacent to all others. Label the vertices in the input of MAX CLIQUE with $V = \{1, \ldots, n\}$. As the corresponding instance of k-UNIFORM NASH we introduce an imitation game $\mathcal{G}$=(I,M) such that M is the column-player's payoff matrix. The matrix $M = (m_{ij})$ is defined as follows:

$$m_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in E, \\ 1/2 & \text{if } i = j, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Clearly this instance can be found in polynomial time (McLennan & Tourky 2005) in the size of G=(V,E) and more importantly, the parameter k' is set to k.

To prove this is a reduction, we need to establish the correspondence between YES-instances to YES-instances, and NO-instances to NO-instances. We denote by $G_{\vec{x}}$ the induced subgraph over the support of a uniform Nash equilibrium ($\vec{x}$,$\vec{x}$).

Lemma 3.4 *The strategy profile ($\vec{x}$,$\vec{x}$) is a k-uniform Nash equilibrium of the imitation game (I,M), if and only if $G_{\vec{x}}$ is a maximal clique of size k.*

Proof:

($\Rightarrow$) Let $(\vec{x},\vec{x})$ be k-uniform Nash equilibrium of (I,M) and assume the m-th vertex connects to all other vertices of G. Because I is the payoff matrix of the first player, in a Nash equilibrium the first player always places in its support a strategy played by the second player (it is an imitation game). The first player has payoff $1/k^2$ always and therefore will maximize its payoff by minimizing its support to the support of the second player. Because M is the payoff of the second player, for the second player, a uniform strategy has payoff $(k-1)/k + 1/(2k) = 1 - 1/(2k)$ if the support is a clique of size k (less than this if some edge is missing in the graph induced by the support). Thus, Player 2 wants to maximize the support to get a payoff as close as possible to 1 but can only do this for cliques.

Claim 1 : $m \in supp(\vec{x})$. Suppose $m \notin supp(\vec{x})$ and let $j \in supp(\vec{x})$. Then, the expected payoff when playing j by the second player is $(\vec{x}^T M)_j = 1 - 1/(2k)$. However, $(\vec{x}^T M)_m = 1$ because m is adjacent to all other vertices and $m \notin supp(\vec{x})$. But then $(\vec{x}^T M)_m > (\vec{x}^T M)_j$ when $j \in supp(\vec{x})$. This contradicts Theorem 2.1 because $(\vec{x},\vec{x})$ is a Nash equilibrium.

Claim 2: $G_{\vec{x}}$ is a clique. From Theorem 2.1, one can find that $\forall i,j \in supp(\vec{x})$ implies $(\vec{x}^T M)_i = (\vec{x}^T M)_j$. In other words, for every pair i,j in $supp(\vec{x})$ the following expressions [3] regarding their payoff are equal

$$1 \cdot 1/k \cdot d_{G_{\vec{x}}}(i) + 1/2 \cdot 1/k \quad \text{and,}$$
$$1 \cdot 1/k \cdot d_{G_{\vec{x}}}(j) + 1/2 \cdot 1/k$$

where $d_{G_{\vec{x}}}(i)$ is the degree of vertex i in $G_{\vec{x}}$. Therefore, the degree of vertex i and the degree of j in $G_{\vec{x}}$ are the same. Hence we have a regular subgraph which has a vertex that connects to all others, therefore $G_{\vec{x}}$ is a clique.

Claim 3: The clique $G_{\vec{x}}$ is maximal. Thi is similar to Claim 1. If t is connected to every vertex in $supp(\vec{x})$, but $t \notin supp(\vec{x})$, we have $1 = (\vec{x}^T M)_t > (\vec{x}^T M)_j$ for any $j \in supp(\vec{x})$. This contradicts the assertion that $(\vec{x},\vec{x})$ is Nash equilibrium because of Theorem 2.1

($\Leftarrow$) Let $G_{\vec{x}}$ be a maximal clique of size k (we have k-entries in $\vec{x}$ equal to $1/k$ and all other are zero). As we mentioned earlier, it is not hard to inspect M and find that the payoff of each i in $supp(\vec{x})$ is given by $(\vec{x}^T M)_i = 1 \cdot (k-1)/k + 1/2 \cdot 1/k$. If $k = n$, then G is the complete graph and by Theorem 2.1, $(\vec{x},\vec{x})$ is a uniform Nash equilibrium. If $k < n$, for any $t \notin supp(\vec{x})$ there is at least one $i \in supp(\vec{x})$ which $\{i,t\} \notin E$. Therefore, $(\vec{x}^T M)_t < 1 \cdot (k-1)/k + 1/2 \cdot 1/k$. But then, again from Theorem 2.1, $(\vec{x},\vec{x})$ is Nash equilibrium.

□

This completes the proof that k-Uniform Nash is $W[2]$-hard.

The claims in the above proofs enable us to consider another problem.

[3]Each term in the expressions is the product of the payoff, times the probability of playing it, times the number of repetitions.

Maximum Payoff for 2nd Player

Instance : An imitation game $\mathcal{G}$=(I,M).
Parameter : Positive integer k.
Question : Does $\mathcal{G}$ have a uniform Nash equilibrium $(\vec{x},\vec{x})$ such that the payoff for Player 2 is at least k?

Suppose we modify the payoff matrix in Equation (1) as follows. Let M now be given by

$$m_{ij} = \begin{cases} 2k & \text{if } \{i,j\} \in E, \\ k & \text{if } i = j, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Note that a clique of size t corresponds to a mixed strategy for the second player with payoff $2k(t-1)/t + k/t$ and thus, if the clique has size k, the payoff is $2k+1$. The claims in the proof of the previous theorem directly imply the following results.

Corollary 3.1 *Let [G=(V,E),integer k] be an instance of* Max Clique. *The reduction* $[G=(V,E),k] \mapsto [\mathcal{G}(I,M), k' = 2k+1]$ *is a parameterized reduction for* Max Clique *to* Maximum Payoff for 2nd Player. *Moreover, the graph G=(V,E) has maximal clique of size k if and only if the $\mathcal{G}$ has a Nash equilibrium in which the column player's payoff is at least k'.*

Corollary 3.2 *Let $\mathcal{G}$=(I,M) be an imitation game and k be an integer. Deciding whether $\mathcal{G}$ has Nash equilibrium in which the second player's payoff is at least k is W[2]-hard.*

3.2 Finding Nash equilibrium with smallest support is unlikely to be FPT

We now show a different concern that also seems very hard to resolve, even with exponential time in the parameter and polynomial time for the size of the input. This will bring more light into the challenges faced when characterizing the complexity of computing Nash equilibria. We show that deciding whether a game has a Nash equilibrium with a support of size equal to or less than a fixed integer number is hard (in the matter of parameterized complexity). Alternatively, minimizing the support of a Nash equilibrium is at least as hard as the parameterized decision problem, and we are showing that this is unlikely to be FPT. We will prove this for a more specialized type of game, those where one player's loss is the opponent's gain; those are called *Zero-Sum* games. The negative result for *Zero-Sum* games propagates to more general (arbitrary games). In other words, we will show the following decision problem is $W[2]$-hard.

k-Minimal Nash support

Instance : A *Zero-Sum* game $\mathcal{G}$=$(A,-A)$.
Parameter : Positive integer k.
Question : Does $\mathcal{G}$ have a Nash equilibrium $(\vec{x},\vec{y})$ such that $\max\{\|supp(\vec{x})\|, \|supp(\vec{y})\|\} \le k$?

Our hardness result is based on a parameterized reduction from an instance from Set Cover. Set Cover is $W[2]$-complete (Downey & Fellows 1998).

Set Cover

Instance : A family $S = \{S_1, ..., S_r\}$ of r subsets of set $N = \{1, ..., n\}$ that covers N, that is $\bigcup_{i=1,\dots,r} S_i = N$.
Parameter : Positive integer $k \le r$.
Question : Does S have a subset of size at most k such that it covers N?

Theorem 3.2 *k*-Minimal Nash support *is W[2]-hard.*

Let (N,S,r,k) be an instance of Set Cover. We construct a *Zero-Sum* game $(A,-A)$ where $A_{(r+1)\times(n+1)}$ is the payoff matrix of the row player which is defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } i \leq r,\, j \leq n,\, j \in S_i, \\ 0 & \text{if } i \leq r,\, i \leq n,\, j \notin S_i, \\ 1/k & \text{if } i \leq r,\, j = n+1, \\ 1/2r & \text{if } i = r+1. \end{cases} \tag{3}$$

Clearly, constructing this game given an instance of Set Cover requires polynomial time. The following result shows that the parameter k remains unchanged (and thus does not depend on n). Moreover, YES-instances are mapped to and only to YES-instances while NO-instances are mapped to and only to NO-instances. Therefore this is a parameterized reduction.

Theorem 3.3 *The cover S of the set N has a sub-cover of size k or less if and only if the game $\mathcal{G}$=(A,−A) defined in Equation 3 has a Nash equilibrium such that the size of the support of the Nash strategy is at most k.*

Proof: We first make some observations about the game in Equation 3.

Claim 1: The support of a mixed strategy chosen by the row player must be a cover. Consider a mixed strategy $\vec{x}$ for the row player. Otherwise, if the set $J = supp(\vec{x}) \setminus \{r+1\}$ is such that $N \nsubseteq \bigcup_{i\in J} S_i$, then the second player can improve (and therefore reduce the payoff of the first player) by playing $t \in N \setminus \bigcup_{i\in J} S_i$.

Claim 2: The first player has a guaranteed payoff of $1/r$. The mixed strategy $\vec{x}_0^T = (1/r, \ldots, 1/r, 0)$ (that is, the uniform distribution on the first r pure strategies and probability 0 for the last one) is a cover because the game was derived from an instance of Set Cover and has payoff $1/r$ (the second player plays $n+1$ with probability 1). Also, because we have an instance of Set Cover, the value k is no more than r, or $1/r \leq 1/k$.

($\Rightarrow$) Assume $\mathcal{G}$ has a Nash equilibrium ($\vec{x}^*$,$\vec{y}^*$) in which the support of both players has at most size k. From the Minimax Theorem 2.2 the expected payoff for the first player is at least $1/r$. The first player does not play strategy $r+1$ purely (the payoff for playing that strategy is at most $1/2r$ and it contradicts that the minimum payoff for the first player is at least $1/r$), therefore the set $J = supp(\vec{x}^*) - \{r+1\}$ is not empty. But by the claims above the set $T = \bigcup_{i\in J} S_i$ equals N. Furthermore our assumption on the size of supports of Nash strategies assures us that the size of J is at most k (that is a cover of size at most k).

($\Leftarrow$) Let S_{i_1},S_{i_2},...,S_{i_k} constitute a sub-cover of size k for N. We define mixed strategies $\vec{x}$ and $\vec{y}$ for the row and column player as follows:

$$x_i = \begin{cases} 1/k & \text{if } i \in \{i_1, i_2 \ldots, i_k\}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and,}$$

$$y_{n+1} = 1, y_j = 0 \text{ for } j \leq n.$$

Claim: The strategy profile ($\vec{x}$,$\vec{y}$)is a Nash equilibrium for the game $\mathcal{G}$.
In order to show that ($\vec{x}$,$\vec{y}$) is a Nash equilibrium, we recall Theorem 2.1. Note that $\vec{x}$ is the best response to $\vec{y}$ and vice versa, because for any $i \in supp(\vec{x})$ and for all $j \in supp(\vec{y})$ we have

$$\begin{aligned} (A\vec{y})_i &= \max_{t=1,\ldots,r+1} (A\vec{(y)})_t = 1/k \quad \text{and,} \\ -(\vec{x}^T A)_i &= \max_{t=1,\ldots,n+1} -(\vec{x}^T A)_t = -1/k. \end{aligned}$$

□

4 Conclusion

Parameterized complexity researchers aim at developing fixed-parameter tractable algorithms for NP-Complete problems. This provides a path to tractability, and the exact solving of very hard problems. A very important aspect of this enterprise for FPT algorithm is the identification of suitable parameters for the problems (Downey & Fellows 1998). One would be inclined to believe that finding the support for Nash strategies in the game is a milestone to finding a sample Nash equilibrium(because once is found, the actual strategies can be found in polynomial time). In finding the support, a subproblem is to find its size (if we had the support, we can trivially find its size). So, finding the size of the support is perhaps the easiest milestone in computing Nash equilibria. The support size also seems a reasonable parameter to parameterize the quest for Nash strategies. Our results show that probably there is no FPT algorithm for finding Nash equilibria with the property that the support size of each Nash strategy is equal to or less than k even in *Zero-Sum* games (the hardness result can be extended to general bi-matrix games because *Zero-Sum* games are just a special case). This implies that decision problems for computing Nash equilibria are far from being tractable.

We also illustrated that solving the k-Uniform Nash in imitation games is hard in the sense of parameterized complexity. This result reveals that finding such Nash equilibria for the general class of normal form games is $W[2]$-hard. Moreover, from Lemma 2.1 and the hardness results of k-Uniform Nash, it can be concluded that in any bi-matrix game $\mathcal{G}$=(A,B), deciding whether both players play a uniform strategy in which the sum of size of their Nash strategies support is k is $W[2]$-hard.

McLennan & Tourky (2005) introduced the notion of I-equilibrium in the class of imitation games and its relation with symmetric equilibrium of symmetric games[4]. They presented a list of NP-hard decision problems for imitation games and their corresponding decision problems in symmetric games. From McLennan & Tourky (2005, Proposition 1), it arises that every uniform Nash equilibrium ($\vec{x}$,$\vec{x}$) is an I-equilibrium, but the reverse is not true. However, the same parameterized hardness result holds for symmetric games, that is, deciding whether a symmetric game has a k-uniform Nash equilibrium ($\vec{x}$,$\vec{x}$) is $W[2]$-hard. But, our other parameterized results would not extend to its symmetric counterparts. For example, classifying the following decision problem (within the complexity classes of parameterized complexity theory) is still open.

[4]A bi-matrix $\mathcal{G}$=(A,B) is called symmetric if $A = B^T$. A Nash equilibrium of the form ($\vec{x}$,$\vec{x}$) is called a symmetric Nash equilibrium.

Instance :	A symmetric game $\mathcal{G}=(M,M)$ where M is symmetric matrix.
Parameter :	Positive integer k.
Question :	Does the game $\mathcal{G}$ have symmetric Nash equilibrium such that the payoff of each player is equal to or greater than k?

To the best of our knowledge, our results here are the first link between parameterized complexity theory and computation of Nash equilibria, after an earlier FPT result on computing an approximation Nash equilibria (Kalyanaraman & Umans 2007).

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