Testing and Reconstruction of Lipschitz Functions

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Lipschitz Continuous Functions

A function $f : D \rightarrow R$ has Lipschitz constant c
if for all x,y in *D*,
$distance_R(f(x),f(y)) \leq c \cdot distance_D(x,y)$.

A fundamental notion in

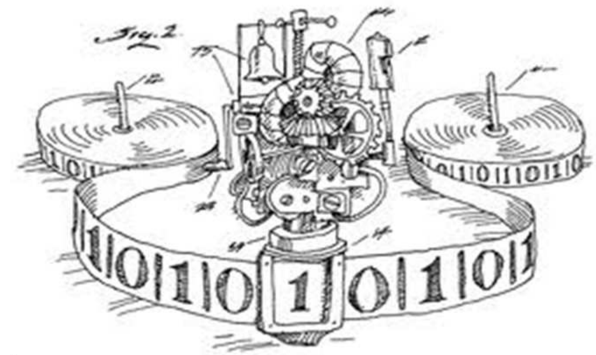
- *mathematical analysis*
- *theory of differential equations*

Example uses of a Lipschitz constant c of a given function f

- probability theory: in tail bounds via McDiarmid's inequality
- program analysis: as a measure of robustness to noise
- data privacy: to scale noise added to preserve differential privacy

Computing a Lipschitz Constant?

- Infeasible



- Undecidable to even verify if f computed by a TM has Lipschitz constant c

- NP-hard to verify if f computed by a circuit has Lipschitz constant c
  - even for finite domains

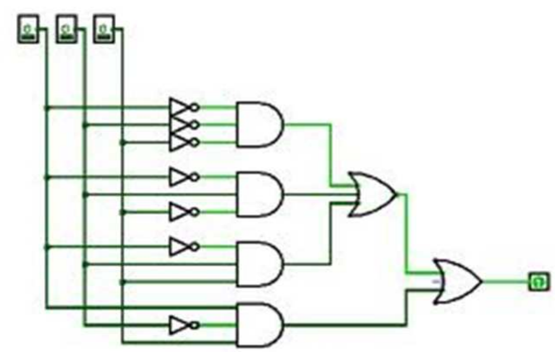


Image sources: http://www.ecs.syr.edu/faculty/fawcett/handouts/webpages/coretechnologies.htm
http://www.augustana.ab.ca/~mohrj/courses/2004.fall/csc110/assignments/lab2.html

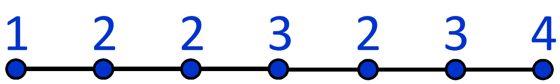
Lipschitz Functions Over Finite Domains

We call a function Lipschitz if it has Lipschitz constant 1.

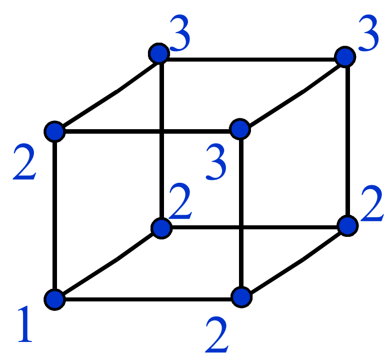
- can rescale by $1/c$ to get a Lipschitz function from a function with Lipschitz constant c

Examples

$f : \{1,\ldots,n\} \to R$



$f : \{0,1\}^d \to R$



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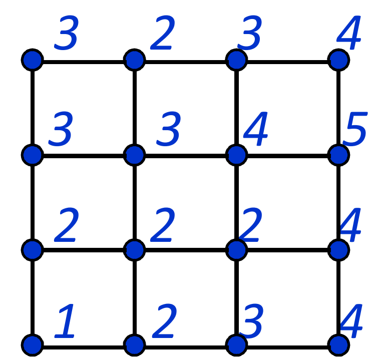


nodes = points in the domain; edges = points at distance 1

node labels = values of the function

Application 1: Program Analysis

Certifying that a program computes a Lipschitz function

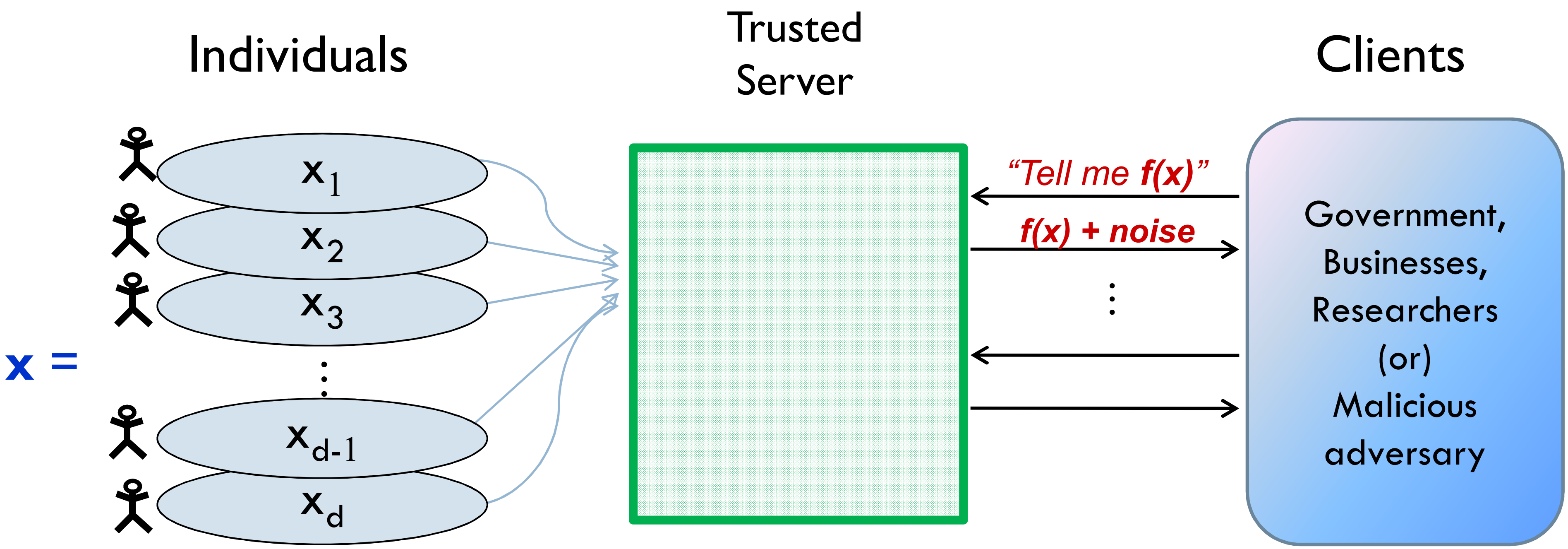
[Chaudhuri Gulwani Lublinerman Navidpour 10]

To ensure that a program

- is robust to noise in its inputs (e.g., caused by communication/ measurement errors)
- responds well to compiler optimizations that lead to an approximately equivalent program

- **Question**: Can we test if a function is Lipschitz?

Application 2: Data Privacy



Typical examples: census, civic archives, medical records,…

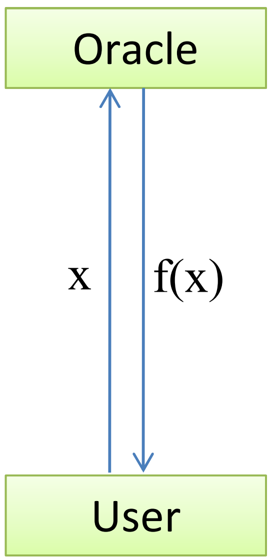
➢[Dwork McSherry Nissim Smith 06]

Lipschitz functions can be released with little noise while satisfying differential privacy.

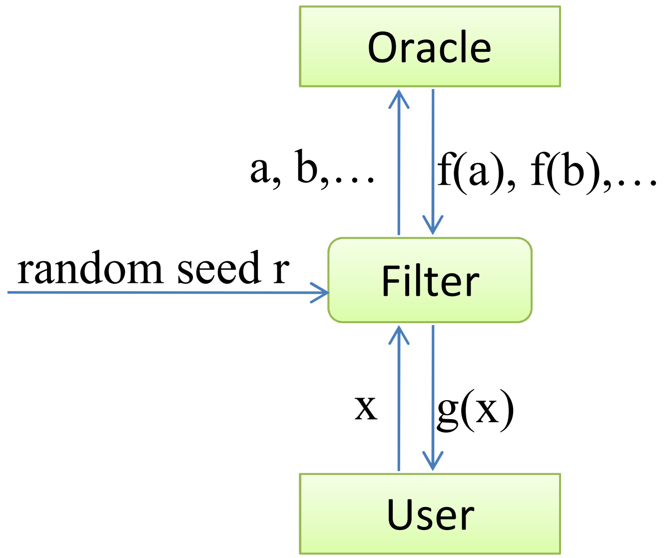
➢Question: Can we ensure that the server only answers queries about Lipschitz functions?

Local Property Reconstruction [Saks Seshadhri 10]

Extends [Ailon Chazelle Seshadhri Liu 08]



User expects f to satisfy property P



Reconstruction of property P

- for each f and r, function g satisfies property P
- w.h.p. g is close to f (in Hamming distance)
- g(x) can be computed quickly
- Local filter: g does not depend on queries x

Local Property Reconstruction [Saks Seshadhri 10]

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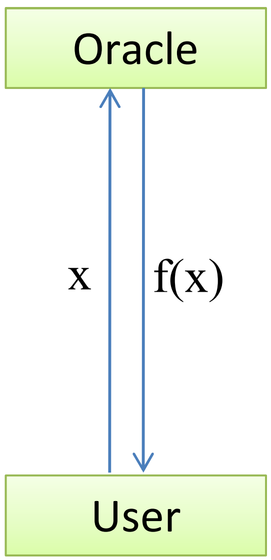

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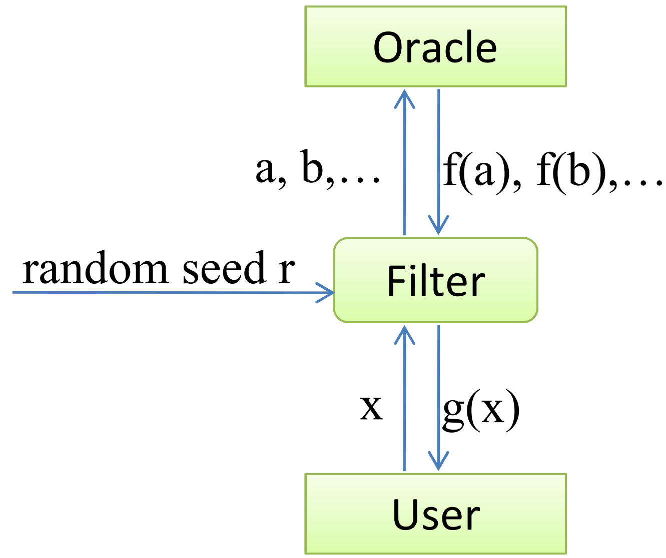
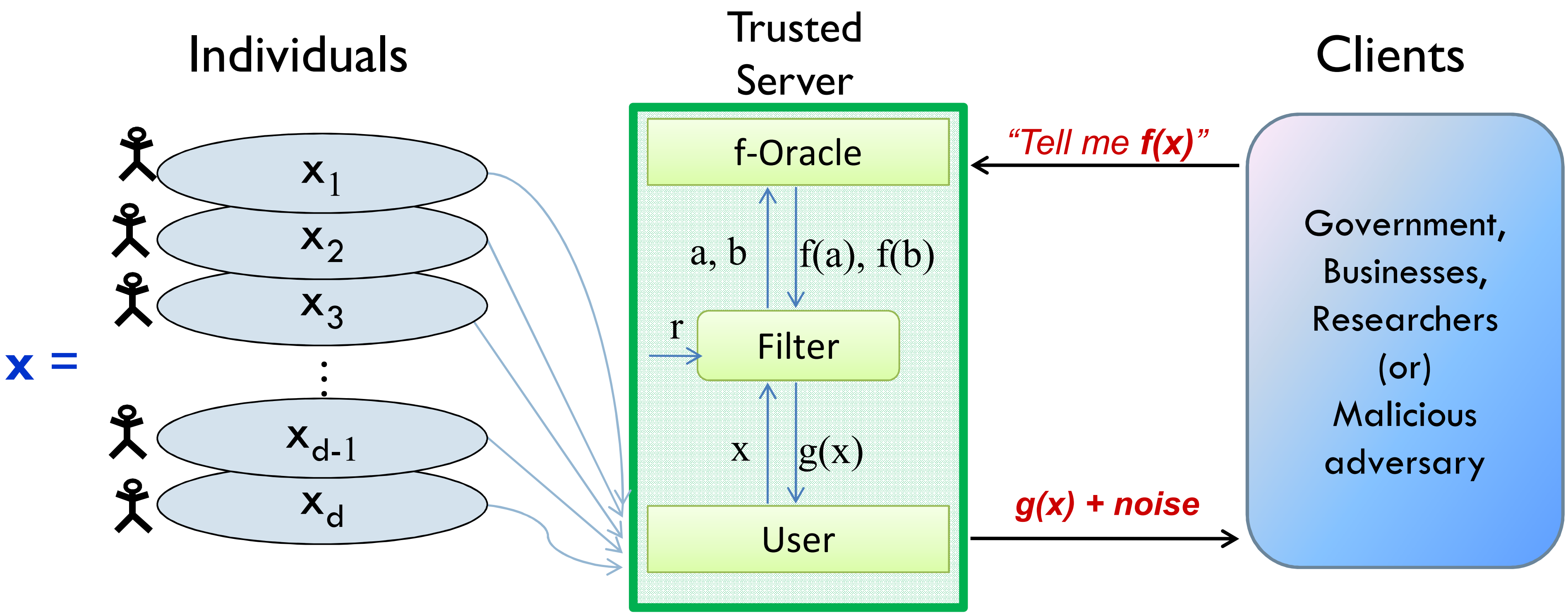


Reconstruction of property P

- for each f and r, function g satisfies property P
- g = f if f satisfies property P
- g(x) can be computed quickly
- Local filter: g does not depend on queries x

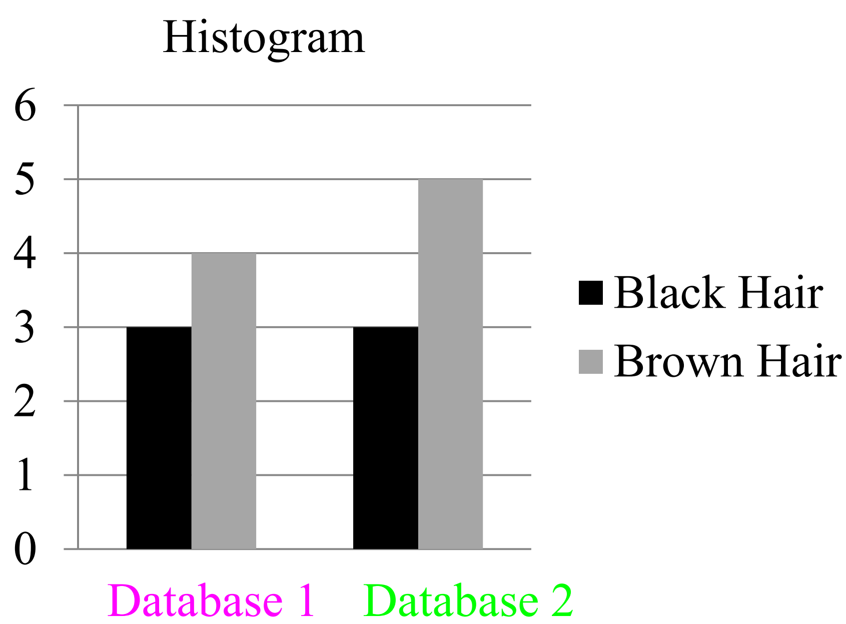
Filter Mechanism for Data Privacy

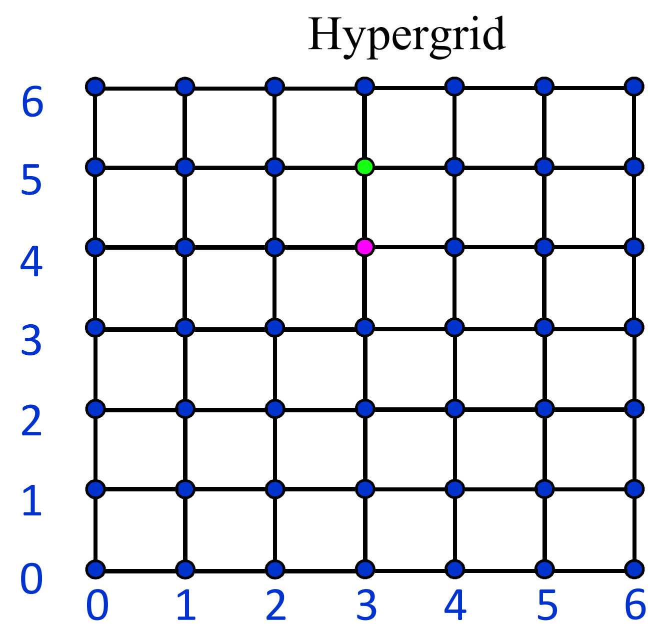


➢Question:

Can we quickly (locally) reconstruct Lipschitz property?

Using Local Lipschitz Filter on the Hypergrid



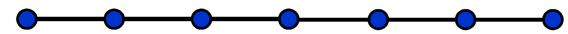


- **Question**:

  Can we quickly locally reconstruct Lipschitz property for functions on the hypergrid domains?

Our Results: Lipschitz Testers

Line $f : \{1, \ldots, n\} \to R$

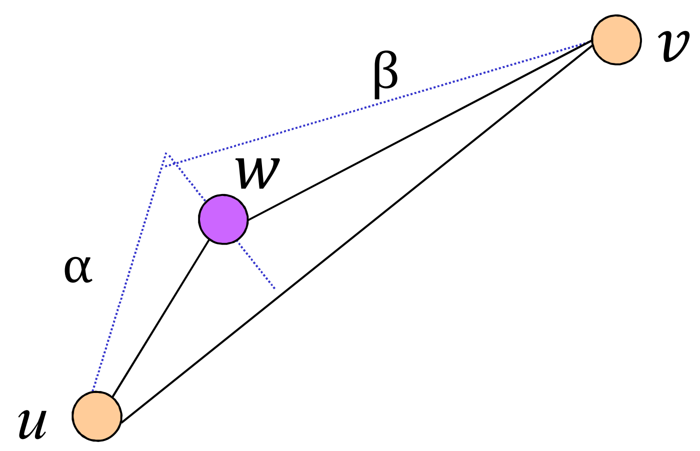


- Upper bound: $O(\log n / \varepsilon)$ time
  - applies to all discretely metrically convex spaces R
    - ✓ $(\mathbb{R}^k, \ell_p)$ for all $p \in [1, \infty)$, $(\mathbb{R}^k, \ell_\infty)$, $(\mathbb{Z}^k, \ell_1)$, $(\mathbb{Z}^k, \ell_\infty)$
    - ✓ the shortest path metric d_G for all graphs G
  - generalization of monotonicity tester via transitive-closure-spanners [Dodis Goldreich Lehman R Ron Samorodnitky 99, Bhattacharyya Grigorescu Jung R Woodruff 09]
  - applies to all edge-transitive properties that allow extension
- Lower bound: $\Omega(\log n)$ queries for nondaptive 1-sided error tests
  - holds even for range $\mathbb{Z}$

Metric Convexity

- a standard notion in geometric functional analysis

> A metric space (R, d_R) is metrically convex
> if for all $u, v \in R$ and
> all positive $\alpha, \beta \in \mathbb{R}$ satisfying $d_R(u, v) \leq \alpha + \beta$
> there exists $w \in R$ such that $d_R(u, w) \leq \alpha$ and $d_R(w, v) \leq \beta$

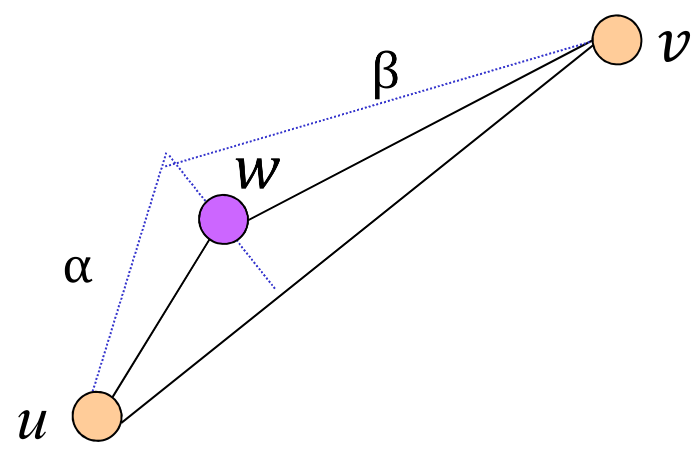


Discrete Metric Convexity

- a relaxation of

  a standard notion in geometric functional analysis

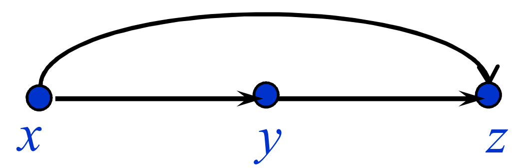
A metric space (R, d_R) is **discretely** metrically convex
if for all $u, v \in R$ and
all positive $\alpha, \beta \in \mathbb{Z}$ satisfying $d_R(u, v) \leq \alpha + \beta$
there exists $w \in R$ such that $d_R(u, w) \leq \alpha$ and $d_R(w, v) \leq \beta$



Class of Properties to Which Line Tester Applies

- A property is edge-transitive if
  1) it can be expressed in terms conditions on ordered pairs of domain points

  2) it is transitive: whenever (x, y) and (y, z) satisfy (1), so does (x, z)



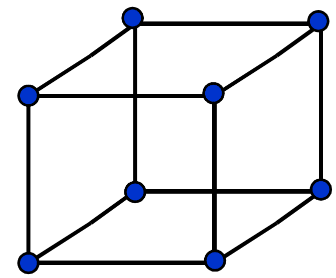
- A property allows extension if
  3) any function that satisfies (1) on a subset of the domain can be extended to a function with the property

Our Results: Lipschitz Testers

Line $f : \{1,\ldots,n\} \to R$

- Upper bound: $O(\log n / \varepsilon)$ time
  - applies to all discretely metrically convex spaces R
    - ✓ $(\mathbb{R}^k, \ell_p)$ for all $p \in [1,\infty)$, $(\mathbb{R}^k, \ell_\infty)$, $(\mathbb{Z}^k, \ell_1)$, $(\mathbb{Z}^k, \ell_\infty)$
    - ✓ the shortest path metric d_G for all graphs G
  - generalization of monotonicity tester via TC-spanners [DGLRRS99, BGJRW09]
  - applies to all edge-transitive properties that allow extension
- Lower bound: $\Omega(\log n)$ queries for nondaptive 1-sided error tests
  - holds even for range $\mathbb{Z}$

Our Results: Lipschitz Testers



Hypercube $f : \{0,1\}^d \rightarrow R$

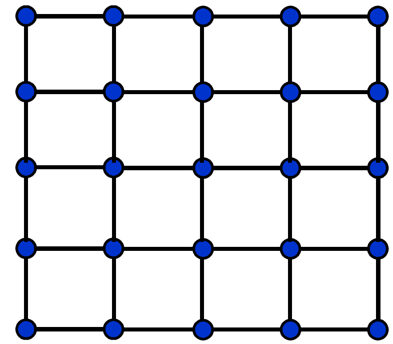
- Upper bound: $O(d \cdot \min(d, \text{ImageDiam}(f))/(\delta\varepsilon))$ time for range $\delta\mathbb{Z}$
  - same time to distinguish Lipschitz and ε-far from $(1+\delta)$-Lipschitz for range $\mathbb{R}$

Today

- Lower bound: $\Omega(d)$ queries
  - tight for range {0,1,2}
  - reduction from a communication complexity problem

    (new technique due to [Blais Brody Matulef 11])

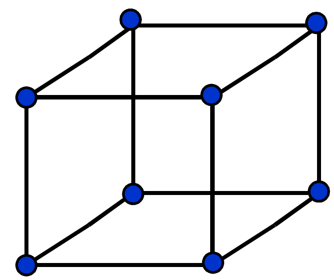
Our Results: Local Lipschitz Reconstructors



Hypergrid $f : \{1,\ldots, n\}^d \to \mathbb{R}$

- Upper bound: $O((\log n + 1)^d)$ time
- Lower bound: $\Omega\left(\frac{(\ln n - 1)^{d-1}}{d(4\pi)^d}\right)$ ueries

  for nonadaptive filters



Hypercube $f : \{0,1\}^d \to \mathbb{R}$

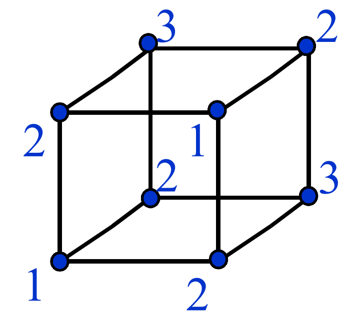
- Lower bound: $\Omega\left(2^{\alpha d}/d\right)$ ueries, where $\alpha \approx 0.1620$,

  for nonadaptive filters

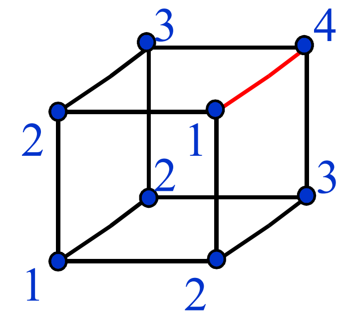
Hypercube Test: Important Special Case

Testing if $f : \{0,1\}^d \rightarrow \mathbb{Z}$ is Lipschitz

in $O(d \cdot \min(d, \text{ImageDiam}(f))/\varepsilon)$ time



- f is Lipschitz if its values on endpoints of *every* edge differ by at most 1.



- A an edge $\{x, y\}$ is violated if $|f(x) - f(y)| > 1$

Goal: Relate the number of violated edges, $V(f)$, to the distance to the Lipschitz property.

Hypercube Test: Key Lemma

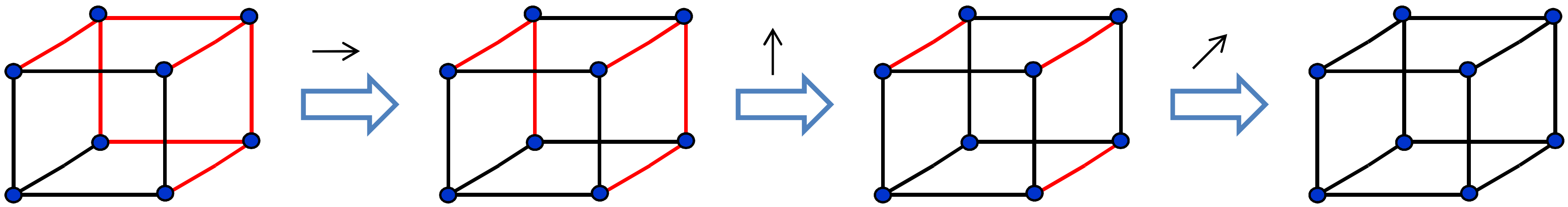
Key Lemma

If $f : \{0,1\}^d \rightarrow \mathbb{Z}$ is ε-far from Lipschitz then $V(f) \geq \frac{\varepsilon \cdot 2^{d-1}}{ImageDiam(f)}$

- Enough to show: we can make f Lipschitz by modifying $2 \cdot V(f) \cdot ImageDiam(f)$ values.
- Then $2 \cdot V(f) \cdot ImageDiam(f) \geq \varepsilon \cdot 2^d$ for ε-far f, implying Key Lemma.

Averaging Operator

Plan: Transform f into a Lipschitz function by repairing edges in one dimension at a time.



- As in the analysis of monotonicity tester in [DGLRRS99, GGLRS00]
  - Worked only for Boolean functions
  - General range was handled by induction on the size of the range
  - Function with range {0,1} are all Lipschitz,

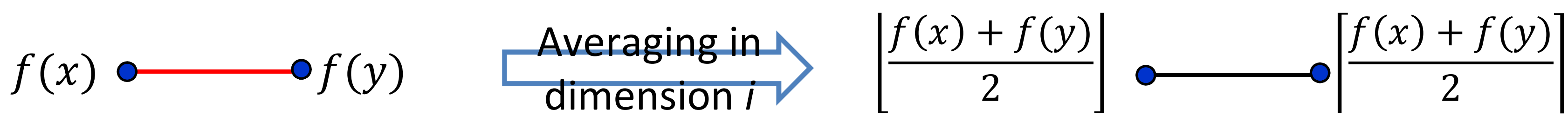
    with range {0,2} are trivially testable

Averaging Operator

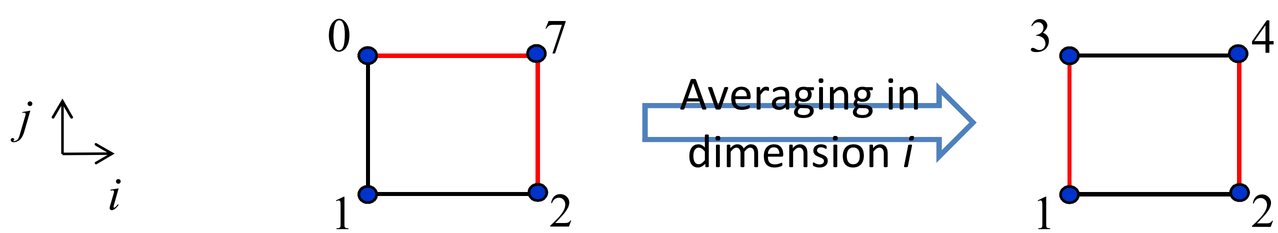
Plan: Repairing edges in one dimension at a time.

Averaging Operator

For each violated edge $\{x, y\}$ along dimension i with $f(x) < f(y) + 1$



Issue: might increase the # of violated edges in other dimensions



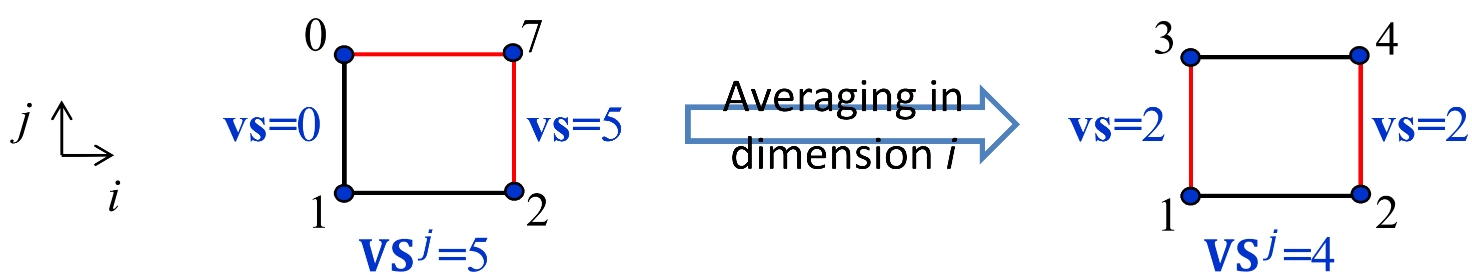
Intuition: violation is "spread" among the edges in dimension *j*

Potential Function Argument

Idea: Take into account the magnitude of violations.

Violation Score

- Violation score **vs**$(\{x, y\})$ = $\max(0, |f(x) - f(y)| - 1)$
- $\mathbf{VS}^j$ = sum of violation scores of edges along dimension j



Want to show: Averaging in dimension i does not increase $\mathbf{VS}^j$ for all dimensions $j \neq i$

Issue: averaging operator is complicated

Basic Step Operator

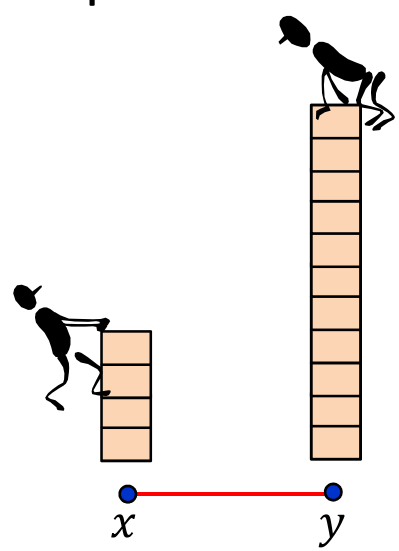
Idea: Break up the action of Averaging Operator into basic steps.

Basic Step Operator

For each violated edge $\{x, y\}$ along dimension i with $f(x) < f(y) + 1$

$f(x)$ ● —— ● $f(y)$ ⇒ Basic Step in dimension i ⇒ $\boldsymbol{f(x) + 1}$ ● —— ● $\boldsymbol{f(y) - 1}$

Averaging in dimension i = multiple Basic Steps in dimension i



Basic Step Operator

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Basic Step Operator

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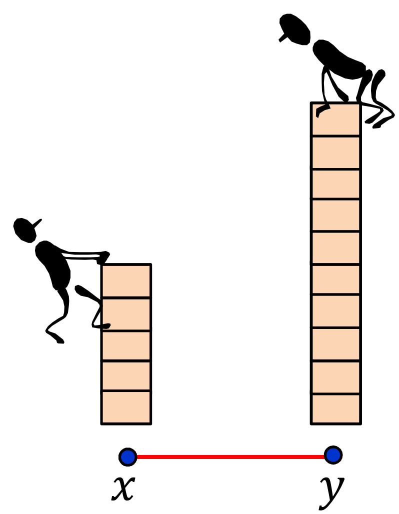
$f(x)$ ●——● $f(y)$ ⟹ (Basic Step in dimension i) $\boldsymbol{f(x)+1}$ ●——● $\boldsymbol{f(y)-1}$

Averaging in dimension i = multiple Basic Steps in dimension i



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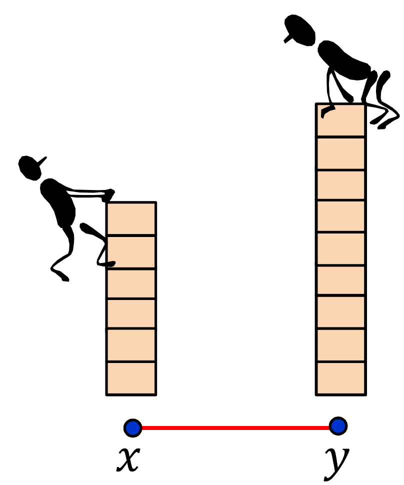
Averaging in dimension i = multiple Basic Steps in dimension i



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For each violated edge $\{x, y\}$ along dimension i with $f(x) < f(y) + 1$

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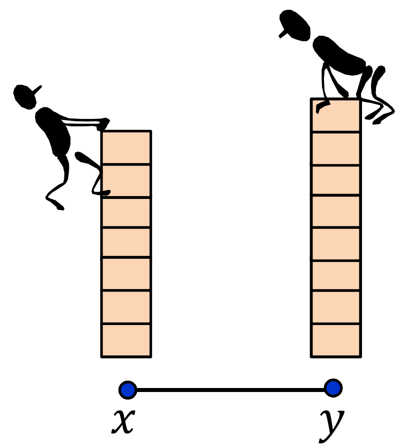
Averaging in dimension i = multiple Basic Steps in dimension i

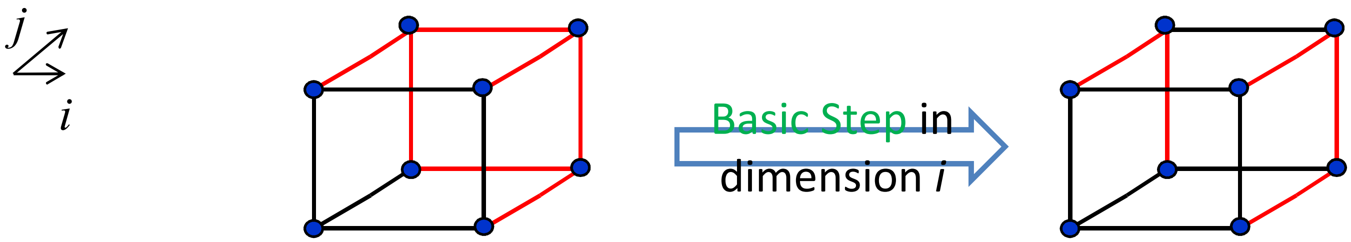


Enough to show:

Basic Step in dimension i does not increase $\mathbf{VS}^j$ $\forall$dimensions $j \neq i$

Basic Step in dimension i does not increase **VS**^{j}

Enough to prove it for squares



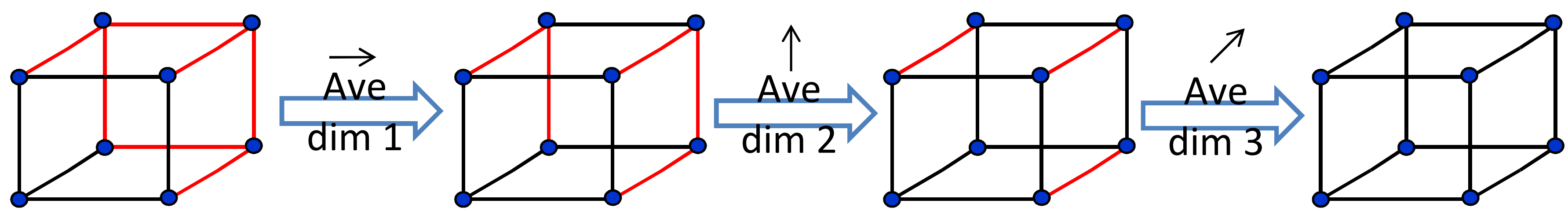
Can be proved by simple case analysis

Analysis of the Averaging Operator

Know: Averaging dimension i

1. repairs all violated edges in dimension i (brings $\mathbf{VS}^i$ down to 0)
2. doesn't increase $\mathbf{VS}^j$ $\forall$dimensions $j \neq i$

- Averaging in dimensions $i = 1, \ldots, d$ repairs all violations because $\mathbf{VS}^j = 0$ means "no violated edges in dimension i"



Analysis of the Averaging Operator

How many function values are changed when averaging dimension i?

2 · (# of violated edges in dimension i after averaging dimensions $1, \ldots, i-1$)

- Let $V^i(f)$ be the # of edges in dimension i violated by f

$$V^i(f) \leq \mathbf{VS}^i\ (f) \leq V^i(f) \cdot ImageDiam(f)$$

- Dimension i starts and ends up with $\mathbf{VS}^i \leq V^i(f) \cdot ImageDiam(f)$

- # of violated edges in dimension i never exceeds $V^i(f) \cdot ImageDiam(f)$

# of changes

= 2 · (# of violated edges in dimension i after averaging dimensions $1, \ldots, i-1$)

$\leq 2 \cdot V(f) \cdot ImageDiam(f)$

Lipschitz Test for Functions $f:\{0,1\}^d \to \mathbb{Z}$

Key Lemma

If $f:\{0,1\}^d \to \mathbb{Z}$ is ε-far from Lipschitz then $V(f) \geq \frac{\varepsilon \cdot 2^{d-1}}{ImageDiam(f)}$

- i.e., fraction of violated edges is $\geq \frac{\varepsilon}{d \cdot ImagDiam(f)}$
- Enough to sample $\Theta(d \cdot ImageDiam(f)/\varepsilon)$ edges

Issue: $ImageDiam(f)$ can be $> 2^d$

Observation: A Lipschitz function on $\{0,1\}^d$ has image diameter at most d.

Algorithm

1. Sample $\Theta(1/\varepsilon)$ domain points x
2. $r = \max_x f(x) - \min_x f(x)$
3. If $r > d$, **reject**
4. Sample $\Theta(d \cdot r/\varepsilon)$ edges, and **reject** if any edge is violated

Analysis of Lipschitz Hypercube Test

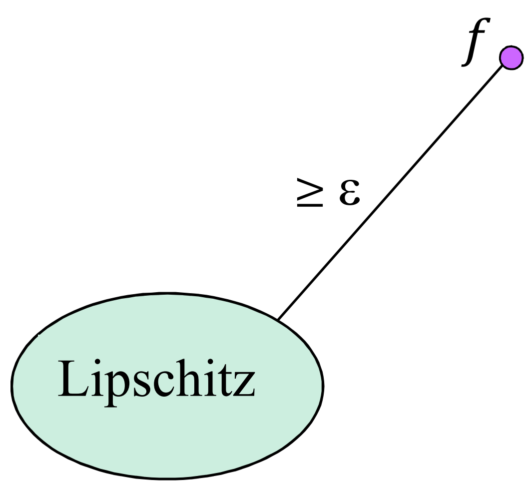
Algorithm

1. Sample $\Theta(1/\varepsilon)$ domain points x
2. $r = \max_x f(x) - \min_x f(x)$
3. If $r > d$, **reject**
4. Sample $\Theta(d \cdot r/\varepsilon)$ edges, and **reject** if any edge is violated

If f is Lipschitz, it is always accepted.

Suppose f is ε-far from Lipschitz.

- If $r > d$, the algorithm rejects.
- It remains to consider the case $r \leq d$.



Analysis of Lipschitz Hypercube Test

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2. $r = \max_x f(x) - \min_x f(x)$
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4. Sample $\Theta(d \cdot r/\varepsilon)$ edges, and **reject** if any edge is violated

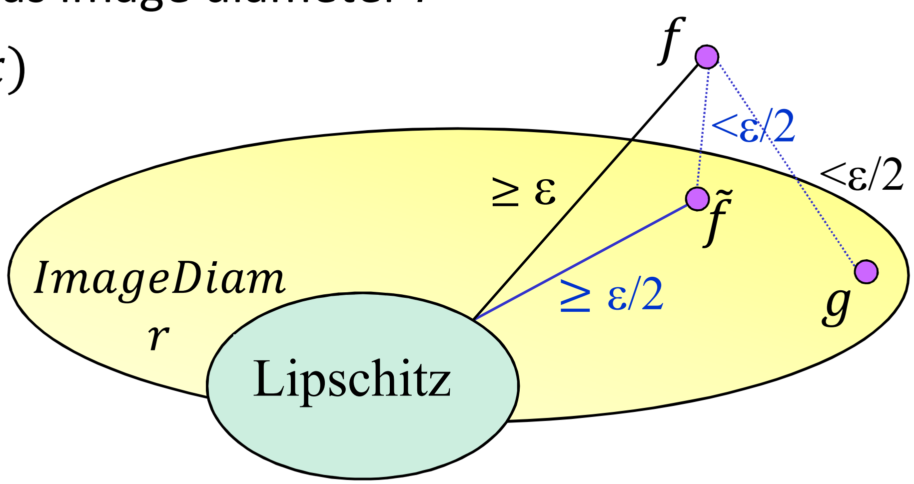
Suppose f is ε-far from Lipschitz and $r \le d$.

- W.h.p. r is such that f is $\varepsilon/2$-close to having image diameter r

  That is, some function g at distance < $\varepsilon/2$ has image diameter r
- Let $a_{min} = \min_x g(x)$ and $a_{max} = \max_x g(x)$

  Let $\tilde{f}(x) = \begin{cases} a_{min} & \text{if } f(x) < a_{min} \\ a_{max} & \text{if } f(x) > a_{max} \\ f(x) & \text{otherwise} \end{cases}$
- $\tilde{f}$ has image diameter r and

  is at distance < $\varepsilon/2$ from f $\Rightarrow$ it is $\varepsilon/2$-far from Lipschitz



Analysis of Lipschitz Hypercube Test

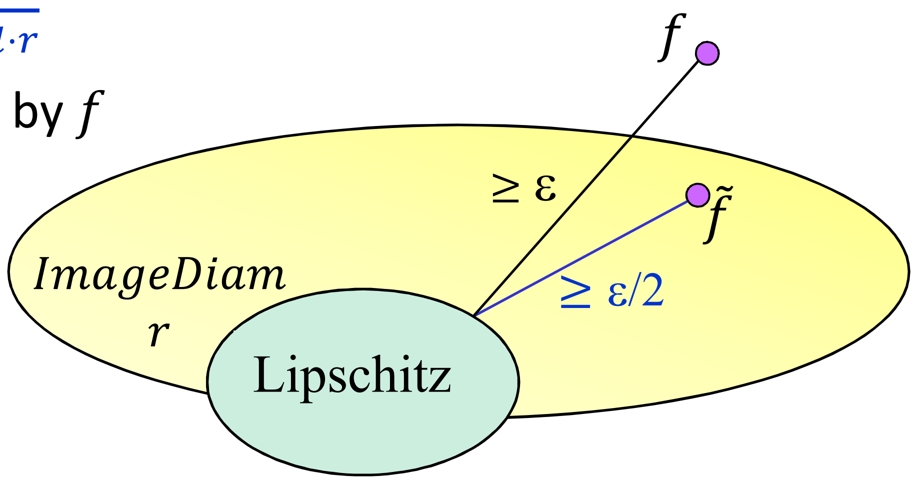
Algorithm

1. Sample $\Theta(1/\varepsilon)$ domain points x
2. $r = \max_x f(x) - \min_x f(x)$
3. If $r > d$, **reject**
4. Sample $\Theta(d \cdot r/\varepsilon)$ edges, and **reject** if any edge is violated

Suppose f is ε-far from Lipschitz and $r \leq d$.

- We have: $\tilde{f}$ has image diameter r and is ε/2-far from Lipschitz
- By Key Lemma, $V(\tilde{f}) \geq \frac{\varepsilon/2}{d \cdot ImagDiam(\tilde{f})} = \frac{\varepsilon}{2 \cdot d \cdot r}$
- An edge is violated by $\tilde{f}$ only if it is violated by f

  $V(f) \geq V(\tilde{f}) \geq \frac{\varepsilon}{2 \cdot d \cdot r}$
- Algorithm rejects w.h.p. ✓

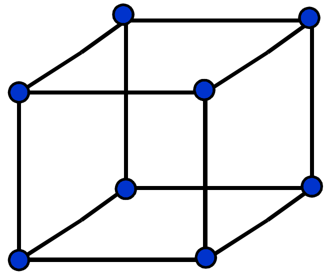


Our Results for the Lipschitz Property

TESTERS

Line $f : \{1,\ldots,n\} \to R$

Hypercube $f : \{0,1\}^d \to R$



- Upper bound: $O(d \cdot \min(d, \text{ImageDiam}(f))/ (\delta\varepsilon))$ time for range $\delta\mathbb{Z}$
  - same time to distinguish Lipschitz and ε-far from $(1+ \delta)$-Lipschitz for range $\mathbb{R}$
- Lower bound: $\Omega(d)$ queries
  - tight for range {0,1,2}

LOCAL RECONSTRUCTORS

Hypergrid $f : \{1,\ldots,n\}^d \to \mathbb{R}$

Hypercube $f : \{0,1\}^d \to \mathbb{R}$

Open Questions

Lipschitz Property

- Tight bounds for testers on the hypercube
- Tester on the hypergrid
- Adaptive lower bounds for local filters on the hypercube/hypergrid
- (Nonlocal) reconstruction
- Explore more complicated ranges than $\mathbb{R}$
  - for testers on domains other than the line
  - for reconstructors

Other Properties

- Filters for data privacy mechanisms based on local notions of sensitivity
  - smooth sensitivity [Nissim Raskhodnikova Smith 07]