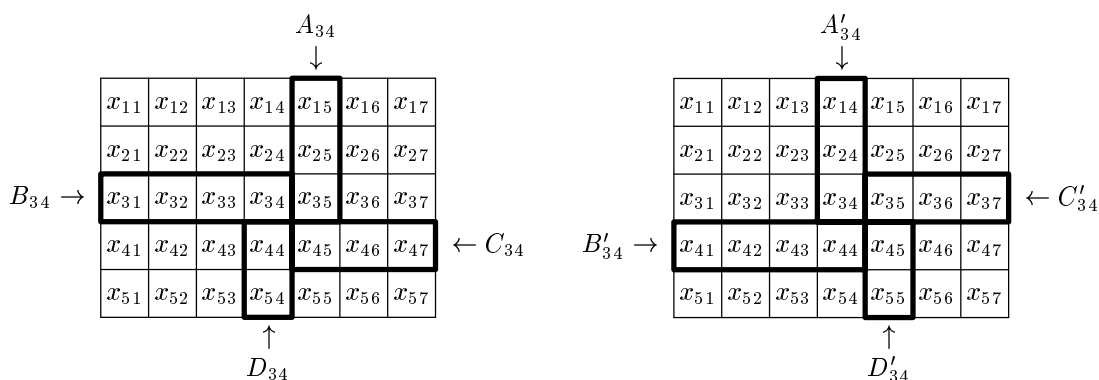# Baxter matrices

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Let's say that an $m \times n$ matrix x_{ij} of 0s and 1s is a *Baxter matrix* if it has the following properties: (i) Every row is nonzero. (ii) Every column is nonzero. (iii) At least one of the regions A_{kl}, B_{kl}, C_{kl}, D_{kl} is zero, for each $1 \le k < m$ and $1 \le l < n$. (iv) At least one of the regions A'_{kl}, B'_{kl}, C'_{kl}, D'_{kl} is zero, for each $1 \le k < m$ and $1 \le l < n$. In this definition

$$A_{kl} = \{x_{i(l+1)} \mid 1 \le i \le k\}, \ B_{kl} = \{x_{kj} \mid 1 \le j \le l\}, \ C_{kl} = \{x_{(k+1)j} \mid l < j \le n\}, \ D_{kl} = \{x_{il} \mid k < i \le m\};$$

$$A'_{kl} = \{x_{il} \mid 1 \le i \le k\}, \ B'_{kl} = \{x_{(k+1)j} \mid 1 \le j \le l\}, \ C'_{kl} = \{x_{kj} \mid l < j \le n\}, \ D'_{kl} = \{x_{i(l+1)} \mid k < i \le m\};$$

these regions form subsets of rows $\{k, k+1\}$ and columns $\{l, l+1\}$ that make a "pinwheel," illustrated here for $k = 3$, $l = 4$, $m = 5$, $n = 7$:

Left pinwheel (A_{34} above column 5, $B_{34} \to$ row 3, $\leftarrow C_{34}$ row 4, D_{34} below column 4):

x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}
x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}
x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	x_{36}	x_{37}
x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	x_{46}	x_{47}
x_{51}	x_{52}	x_{53}	x_{54}	x_{55}	x_{56}	x_{57}

Right pinwheel (A'_{34} above column 4, $B'_{34} \to$ row 4, $\leftarrow C'_{34}$ row 3, D'_{34} below column 5):

x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}
x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}
x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	x_{36}	x_{37}
x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	x_{46}	x_{47}
x_{51}	x_{52}	x_{53}	x_{54}	x_{55}	x_{56}	x_{57}

For example, the reader may verify that the 5×7 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

is *almost*, but not quite, a Baxter matrix. It satisfies all of the conditions *except* that A'_{34}, B'_{34}, C'_{34}, D'_{34} are nonzero—and it is one of exactly 41990 matrices with that peculiar property!

Notice that the left-right and top-down reflection of any Baxter matrix is also a Baxter matrix. And so is the transpose.

It turns out that there are 69 Baxter matrices of size 3×3. Here's the complete set:

```
001  001  001  001  001  001  001  001  001  001  001  001  001  001  001  001  010  010  010  010  010  010  010
001  001  010  010  010  011  011  100  100  101  110  110  110  111  111  111  001  001  010  010  010  011  011
110  111  100  101  110  100  110  010  110  010  001  010  100  001  010  100  100  110  101  111  100  110  001

010  010  010  010  010  010  010  010  010  011  011  011  011  011  011  011  100  100  100  100  100  100  100
100  101  101  101  110  110  111  111  111  001  010  010  100  100  110  110  001  001  010  010  010  011  011
011  001  010  100  001  011  001  010  100  100  100  110  010  100  010  100  011  001  011  101  001  011  101

100  100  100  100  100  100  100  100  100  101  101  101  101  110  110  110  110  110  110  110  111  111  111
011  100  100  101  110  110  111  111  111  010  010  010  010  001  001  010  010  011  011  100  001  010  100
100  011  111  010  001  011  001  010  100  001  010  100  101  001  010  001  011  001  010  001  001  010  100
```

Three of them have 8-fold symmetry. Sixteen of them have no symmetry whatsoever.

The first natural question that you might ask about Baxter matrices is, perhaps, why that name might be appropriate. Don't worry; I'll explain that soon.

The next natural question is to count them, in order to check with OEIS [1] whether they are equivalent to anything that has already been well studied. According to the statistics for m and n up to 7, the answer seems to be that these matrices are a newfangled notion:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
$m = 1$	1	1	1	1	1	1	1
$m = 2$	1	6	14	24	36	50	66
$m = 3$	1	14	69	203	463	903	1585
$m = 4$	1	24	203	972	3324	9074	21168
$m = 5$	1	36	463	3324	16355	61267	188153
$m = 6$	1	50	903	9074	61267	306352	1219598
$m = 7$	1	66	1585	21168	188153	1219598	6175181

Except for the first two rows and the first two columns, the sequences in this table (including the diagonals and antidiagonals) haven't previously been published.

It's not hard to see why there are $n^2 + 3n - 4$ Baxter matrices of size $2 \times n$, when $n \geq 2$: The columns can't all be $\binom{1}{1}$. Solutions in which a column of the form $\binom{0}{1}$ occurs before $\binom{1}{0}$ have five types, either (i) $\binom{0}{1}^k \binom{1}{0}^{n-k}$, with $0 < k < n$; or (ii) $\binom{0}{1}^k \binom{1}{0}^l \binom{0}{1}^{n-k-l}$, with $k > 0$, $l > 0$, $k + l < n$; or (iii) $\binom{1}{1}\binom{1}{0}^{n-1}$; or (iv) $\binom{0}{1}^k \binom{1}{1}\binom{0}{1}^{n-1-k}$, with $0 < k < n$; or (v) $\binom{0}{1}^k \binom{1}{1}\binom{1}{0}^{n-1-k}$, with $0 < k < n - 1$. An equal number of solutions have $\binom{1}{0}$ before $\binom{0}{1}$. So the total comes to $(2n - 2) + (n - 1)(n - 2) + 2 + (2n - 2) + (2n - 4)$.

The smallest number of 1s in an $m \times n$ Baxter matrix is obviously $\max\{m, n\}$. And the number of matrices actually attaining this minimum is comparatively small, with respect to the total number:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
$m = 1$	1	1	1	1	1	1	1
$m = 2$	1	2	6	12	20	30	42
$m = 3$	1	6	6	32	100	240	490
$m = 4$	1	12	32	22	172	744	2364
$m = 5$	1	20	100	172	92	956	5328
$m = 6$	1	30	240	744	956	422	5492
$m = 7$	1	42	490	2364	5328	5492	2074

Notice that when $m = n$, a minimum-1s matrix has just one 1 in every row and in every column; hence it's a permutation matrix. And aha! The diagonal counts in this array belong to the well-known sequence A001181, $\langle 1, 2, 6, 22, 92, 422, 2074, \ldots \rangle$, which enumerates *Baxter permutations*. A permutation matrix is a Baxter matrix if and only if the corresponding permutation is a Baxter permutation.

Indeed, that fact follows immediately from the definition of Baxter permutations, which can be phrased in the following way: Represent the permutation $p_1 \ldots p_n$ in two-line form, with i above p_i for $1 \leq i \leq n$. The permutation is non-Baxter if and only if its two-line form has four column entries

$$\binom{< l}{k + 1} \quad \binom{l}{< k} \quad \binom{l + 1}{> k + 1} \quad \binom{> l + 1}{k}$$

or four column entries

$$\binom{< l}{k} \quad \binom{l}{> k + 1} \quad \binom{l + 1}{< k} \quad \binom{> l + 1}{k + 1}.$$

And we can represent any 0–1 matrix in two-line form, with a column i above j whenever $x_{ij} = 1$. Such a matrix is non-Baxter if and only if its two-line form has four column entries

$$\binom{\leq l}{k + 1} \quad \binom{l}{\leq k} \quad \binom{l + 1}{\geq k + 1} \quad \binom{\geq l + 1}{k}$$

or four column entries

$$\binom{\leq l}{k} \quad \binom{l}{\geq k + 1} \quad \binom{l + 1}{\leq k} \quad \binom{\geq l + 1}{k + 1}.$$

What about the *maximum* number of 1s? It appears that this is exactly $m + n - 1$! At least, that's true when m and n are at most 7. Here are the counts of maximum-1s Baxter matrices:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$m=1$	1	1	1	1	1	1	1
$m=2$	1	4	8	12	16	20	24
$m=3$	1	8	26	55	96	149	214
$m=4$	1	12	55	156	354	688	1198
$m=5$	1	16	96	354	1037	2533	5383
$m=6$	1	20	149	688	2533	7632	19522
$m=7$	1	24	214	1198	5383	19522	59020

Again they're fairly small compared to the total. And again they're not (yet) in OEIS.

From this data I'm willing to conjecture that $m + n - 1$ is truly the maximum. (And also that some other interesting structure, still waiting to be discovered, will lead to a proof.)

Let's try counting the Baxter matrices in which all row sums are 1:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$m=1$	1	0	0	0	0	0	0
$m=2$	1	2	0	0	0	0	0
$m=3$	1	6	6	0	0	0	0
$m=4$	1	12	32	22	0	0	0
$m=5$	1	20	100	172	92	0	0
$m=6$	1	30	240	744	956	422	0
$m=7$	1	42	490	2364	5328	5492	2074

These may be called the *Baxter words*, of length m on an n-letter alphabet. When $m = n$ they're the Baxter permutations, of course; otherwise they seem to be previously unknown. When $m = 4$ and $n = 3$ the 32 Baxter words are

1123, 1132, 1213, 1223, 1231, 1232, 1233, 1321, 1322, 1323, 1332, 2113, 2123, 2132, 2133, 2213,

2231, 2311, 2312, 2321, 2331, 3112, 3121, 3122, 3123, 3211, 3212, 3213, 3221, 3231, 3312, 3321.

Finally, how about Baxter matrices in which every row sum is 2? These might be called *Baxter graphs*, on n vertices and with possibly-repeated edges labeled from 1 to m. According to the conjecture above, they exist only when $m < n \le 2m$. The statistics for $m \le 7$ are 1; 2, 4; 3, 24, 26; 8, 98, 284, 214; 19, 374, 1922, 3496, 2030; 44, 1342, 10620, 33398, 44674, 21174; 111, 4596, 51904, 245684, 554500, 589092, 236410. (In this list there m counts for each fixed m, shown for $n = m + 1, m + 2, \ldots, 2m$.)

Mathematical nomenclature. Something often goes wrong when a mathematical idea is named after a mathematician. We might learn that another person actually had discovered the subject long before (as in the case of Fibonacci numbers or Catalan numbers); or we might find that the eponymous mathematician had never actually considered the topic (as in the case of the Lambert function or the Pochhammer symbol). My drastic decision to propose the names Baxter matrices, Baxter words, and Baxter graphs clearly falls into the latter category. (In fact, Glen Baxter actually defined a *different* set of permutations; what we now call Baxter permutations were originally called *reduced* Baxter permutations. See [2] and [3].)

So why have I chosen those names?

When I first realized that we get an interesting class of matrices by simply replacing '<' by '≤' in the definition of Baxter permutations, I tried to imagine what name another person would have chosen for the concept, in a hypothetical paper that might have already been in print. So I googled the phrase "Baxter matrix"—and got only references to the Yang–Baxter matrix equation. I also googled "Baxter words"— and got only references to Rota–Baxter words. In both cases I ran into concepts from orthogonally different aspects of Baxter's research.

The connection between Baxter matrices and Baxter permutations is however quite strong, and time has shown that Baxter permutations correspond to a wide variety of other important concepts such as "floorplans." (See, for example, [4].) I certainly would never have thought of the concept if it hadn't been for Baxter's pioneering work.

Thus I'm quite comfortable with the terminology suggested above.

Open problems. The topic of Baxter matrices clearly raises a number of questions that cry out to be answered, including the following:

1. If the first element of a Baxter permutation is removed and the remaining elements are renumbered, the result is a Baxter permutation. Suppose we delete the first row of a Baxter matrix, and remove any columns that have become empty. Is the result a Baxter matrix? What other operations preserve Baxterhood? (Consider, for example, splitting a row of weight > 1 into two adjacent rows.)

2. Prove (or disprove) that every $m \times n$ Baxter matrix has fewer than $m + n$ 1s. Nikolai Beluhov has rephrased this conjecture in a particularly appealing way: "Show that every $m \times n$ matrix with at least $m + n$ nonzero entries contains a pinwheel with four nonzero arms."

3. Find formulas by which the numbers tabulated above for small m and n can be computed rapidly. Also count the $m \times n$ Baxter matrices of weight t, for $\max\{m, n\} \le t < m + n$.

4. Find the asymptotic behavior of the quantities in question 3.

5. Baxter matrices of a given size are partially ordered by inclusion (that is, by requiring that $x_{ij} \le x'_{ij}$ for all i and j). Study the minimal and maximal elements of this partial ordering. (For example, when $m = n = 3$, the weight-4 matrices $\begin{smallmatrix}001\\100\\110\end{smallmatrix}, \begin{smallmatrix}011\\001\\100\end{smallmatrix}, \begin{smallmatrix}100\\001\\011\end{smallmatrix}, \begin{smallmatrix}110\\100\\001\end{smallmatrix}$ are maximal; $\begin{smallmatrix}001\\001\\110\end{smallmatrix}, \begin{smallmatrix}011\\100\\100\end{smallmatrix}, \begin{smallmatrix}100\\100\\011\end{smallmatrix}, \begin{smallmatrix}110\\001\\001\end{smallmatrix}$ and $\begin{smallmatrix}001\\110\\001\end{smallmatrix}, \begin{smallmatrix}010\\010\\101\end{smallmatrix}, \begin{smallmatrix}100\\011\\100\end{smallmatrix}, \begin{smallmatrix}101\\010\\010\end{smallmatrix}$ and $\begin{smallmatrix}010\\101\\010\end{smallmatrix}$ are minimal.)

6. What happens when *other* classes of permutations, defined by pattern exclusion, are extended to matrices in a similar way?

References:

[1] http://oeis.org/, The On-Line Encyclopedia of Integer Sequences®, founded in 1964 by N. J. A. Sloane. The tables in this note became sequences A347672–A347679 on 10 September 2021; see also A347736.

[2] F. R. K. Chung, R. L. Graham, V. E. Hoggatt, Jr., and M. Kleiman, "The number of Baxter permutations," *Journal of Combinatorial Theory* **A24** (1978), 382–394.

[3] W. M. Boyce, "Baxter permutations and functional composition," *Houston Journal of Mathematics* **7** (1981), 175–189.

[4] Andrei Asinowski, Gill Barequet, Mireille Bousquet-Mélou, Toufik Mansour, and Ron Y. Pinter, "Orders induced by segments in floorplans and (2-14-3, 3-41-2)-avoiding permutations," *Electronic Journal of Combinatorics* **20**, 2 (2013), paper P35, 43 pages.

Breaking News: Xavier Viennot has recently pointed out potential connections between Baxter matrices and many other combinatorial objects, including Laguerre empilements of segments(!). See "Some remarks on Baxter permutations and Baxter matrices," http://www.viennot.org/wa_files/baxtermatrices.pdf (27 September 2021), 5 pages.

George Spahn, in "Counting Baxter matrices," arXiv:2110.09688 [math.CO] (19 October 2021), 6 pages, has constructed finite-state automata to prove that the number of m-rowed Baxter matrices with n columns is a polynomial in n of degree $2m - 2$. His construction also yields a proof of the conjecture in point 2 above.