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Two-Parameter Identities for q-Appell Polynomials

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Abstract

In this paper, by using the techniques of the q-exponential generating series, we extend a well-known two-parameter identity for the Appell polynomials to the q-Appell polynomials of type I and II. More precisely, we obtain two different q-analogues of such an identity. Then, we specialize these identities for some q-polynomials arising in combinatorics, in q-calculus or in the theory of orthogonal polynomials. In particular, we consider the generalized q-Bernoulli and q-Euler polynomials and then we deduce some further identities involving the Bernoulli and Euler numbers. In this way, as a byproduct, we derive the symmetric Carlitz identity for the Bernoulli numbers. Finally, we find a (non-symmetric) q-analogue of Carlitz's identity involving the q-Bernoulli numbers of type I and II.

1 Introduction

Appell sequences form an important and interesting class containing many classical polynomials arising in physics, in numerical analysis, in the theory of orthogonal polynomials [13], in analysis [6, 10, 36], in the modern umbral calculus [32, 33, 34] and in the theory of Sheffer sequences [38, 41] or, equivalently, in the theory of Sheffer matrices (or exponential Riordan arrays) [25, 26]. The ordinary powers, the generalized Hermite polynomials, the generalized Bernoulli and Euler polynomials, and the generalized rencontres polynomials are all examples of Appell polynomials.

An *Appell sequence* $\{a_n(x)\}_{n\in\mathbb{N}}$ can be characterized in several ways [5, 45]. The following statements are equivalent.

1. For every $n \in \mathbb{N}$, the polynomial $a_n(x)$ has degree n and $a_n'(x) = n a_{n-1}(x)$.
2. There exists a sequence $\{g_n\}_{n\in\mathbb{N}}$, with $g_0 \neq 0$, such that
$$a_n(x) = \sum_{k=0}^{n} \binom{n}{k} g_{n-k}\, x^k$$
or, equivalently,
$$g_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k\, a_{n-k}(x).$$
3. The sequence $\{a_n(x)\}_{n\in\mathbb{N}}$ has exponential generating series
$$A(x;t) = \sum_{n\geq 0} a_n(t)\, \frac{t^n}{n!} = g(t)\, \mathrm{e}^{xt},$$
where $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{n!}$ is an exponential series with $g_0 \neq 0$.
4. There exists a sequence $\{g_n\}_{n\in\mathbb{N}}$, with $g_0 \neq 0$, such that
$$a_n(x) = \sum_{k\geq 0} g_k\, \frac{\mathfrak{D}^k}{k!}\, x^n,$$
where $\mathfrak{D}$ is the derivative with respect to x.
5. The sequence $\{a_n(x)\}_{n\in\mathbb{N}}$ satisfies the Appell identity
$$\sum_{k=0}^{n} \binom{n}{k} a_k(x)\, y^{n-k} = a_n(x+y).$$

Furthermore, the Appell polynomials $a_n(x)$ and the coefficients g_n are related by the following two-parameter binomial identity [45, p. 316] [36, Formula (16′)]:

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k\, a_{m+n-k}(x) = \sum_{k=0}^{m} \binom{m}{k} g_{n+k}\, x^{m-k}\,. \tag{1}$$

It is easy to see that this is a further characterization of the Appell polynomials. More precisely, we have that the polynomials $a_n(x)$ form an Appell sequence if and only if there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$, with $g_0 \neq 0$, for which identity (1) is satisfied for every $m, n \in \mathbb{N}$.

In 1967, after a paper by Sharma and Chak [37], Al-Salam [2] introduced the *q-Appell polynomials* (of type I) and, recently, Sadjang [35] introduced the *q-Appell polynomials of type II*. In this paper, we extend identity (1) to the q-Appell polynomials of type I and II. Specifically, through the techniques of the theory the q-exponential generating series, we obtain two different q-analogues of this identity. Then, we specialize these identities for some q-polynomials arising in combinatorics, in q-calculus or in the theory of orthogonal polynomials, such as the q-Hermite polynomials, the Gaussian polynomials, the Al-Salam-Carlitz polynomials, the q-permutation polynomials, the q-rencontres and q-arrangement polynomials (and some of their generalizations), the q-Bernoulli and q-Euler polynomials. In this last case, we further specialize our identities to the Bernoulli and Euler polynomials and we deduce some other identities involving the Bernoulli and Euler numbers. In particular, as a byproduct, we derive the Carlitz symmetric identity for the Bernoulli numbers [9, 25]. Finally, we find a q-analogue of Carlitz's identity. Such an identity, however, is not symmetric and involves the q-Bernoulli numbers of type I and II.

2 q-Appell polynomials

We start by recalling some definitions [23]. For every $n \in \mathbb{N}$, we have the *q-natural number* $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ and the *q-factorial number* $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$. Then, for every $n, k \in \mathbb{N}$, the *q-binomial coefficients* (or *Gaussian coefficients*) are defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \qquad \text{for } k = 0, 1, \ldots, n$$

and by 0 otherwise. A *q-polynomial* is a polynomial with coefficients in $\mathbb{Q}_q = \mathbb{Q}(q)$, i.e., the field of quotients of $\mathbb{Q}[q]$, where q is an indeterminate. The algebra of the q-polynomials are denoted by $\mathbb{Q}_q[x]$. The *q-Pochhammer symbol* is the q-polynomial defined by

$$(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^k. \tag{2}$$

A *q-exponential series* is a formal series of the form $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{[n]_q!}$. The sum and the multiplication by a scalar are defined componentwise. The product of two q-exponential series $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{[n]_q!}$ and $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{[n]_q!}$ is defined by

$$f(t)g(t) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{n}{k}_q f_k g_{n-k} \right) \frac{t^n}{[n]_q!}. \tag{3}$$

The *q-derivative* (*Jackson's derivative*) $\mathfrak{D}_q$ of a q-exponential generating series $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{[n]_q!}$ is defined [21, 22] by the formula

$$\mathfrak{D}_q f(t) = \frac{f(qt) - f(t)}{(q-1)t} = \sum_{n\geq 0} f_{n+1} \frac{t^n}{[n]_q!}.$$

Given two q-exponential series $f(t)$ and $g(t)$, we have the q-*Leibniz formula*

$$\mathfrak{D}_q^m f(t)g(t) = \sum_{k=0}^{m} \binom{m}{k}_q \mathfrak{D}_q^k f(t) \cdot \mathcal{Q}_q^k \mathfrak{D}_q^{m-k} g(t), \tag{4}$$

where $\mathcal{Q}_q$ is the operator defined by $\mathcal{Q}_q h(t) = h(qt)$.

The q-*exponential series* (*Jackson's q-exponential*) [21]

$$E_q(t) = \sum_{n\geq 0} \frac{t^n}{[n]_q!} = \prod_{k\geq 0} \frac{1}{1+(q-1)q^k t} \tag{5}$$

is the eigenfunction of the q-derivative, that is,

$$\mathfrak{D}_q E_q(\lambda t) = \lambda E_q(t). \tag{6}$$

In particular, since $\mathfrak{D}_q E_q(t) = E_q(t)$, we have the relation

$$E_q(qt) = (1-(1-q)t)\, E_q(t). \tag{7}$$

Consequently, for every $m \in \mathbb{N}$, we have

$$E_q(q^m t) = ((1-q)t; q)_m\, E_q(t). \tag{8}$$

The inverse of the q-exponential series $E_q(t)$ is

$$E_q(t)^{-1} = \sum_{n\geq 0} (-1)^n q^{\binom{n}{2}} \frac{t^n}{[n]_q!}. \tag{9}$$

Given two q-exponential series $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{[n]_q!}$ and $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{[n]_q!}$, we have $f(t) = E_q(\alpha t) g(t)$ if and only if $g(t) = E_q(\alpha t)^{-1} f(t)$. This is equivalent to the q-binomial inversion theorem:

$$f_n = \sum_{k=0}^{n} \binom{n}{k}_q \alpha^k g_{n-k} \qquad \Longleftrightarrow \qquad g_n = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} \alpha^k f_{n-k}.$$

A q-*Appell sequence* (of type I) [2] is a sequence $\{a_n(x)\}_{n\in\mathbb{N}}$, where $a_n(x)$ is a q-polynomial of degree n and $\mathfrak{D}_q a_n(x) = [n]_q a_{n-1}(x)$ for every $n \in \mathbb{N}$. Also the q-Appell sequences can be characterized in several ways, as in the ordinary case. Indeed, the following statements are equivalent.

1. $\{a_n(x)\}_{n\in\mathbb{N}}$ is a q-Appell sequence (of type I).

2. There exists a sequence $\{g_n\}_{n\in\mathbb{N}}$, with $g_0 \neq 0$, such that

$$a_n(x) = \sum_{k=0}^{n} \binom{n}{k}_q g_{n-k}\, x^k$$

or, equivalently,

$$g_n = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^k\, a_{n-k}(x).$$

3. The sequence $\{a_n(x)\}_{n\in\mathbb{N}}$ has q-exponential generating series

$$A(x;t) = \sum_{n\geq 0} a_n(x) \frac{t^n}{[n]_q!} = g(t)\, E_q(xt), \tag{10}$$

where $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{[n]_q!}$ is a q-exponential series with $g_0 \neq 0$.

4. There exists a sequence $\{g_n\}_{n\in\mathbb{N}}$, with $g_0 \neq 0$, such that

$$a_n(x) = \sum_{k\geq 0} g_k \frac{\mathfrak{D}_q^k}{[k]_q!}\, x^n.$$

Similarly, a *q-Appell sequence of type II* [35] is a sequence $\{a_n(x)\}_{n\in\mathbb{N}}$, where $a_n(x)$ is a q-polynomial of degree n and $\mathfrak{D}_q a_n(x) = [n]_q a_{n-1}(qx)$, for every $n \in \mathbb{N}$. Also in this case, we have the following equivalent characterizations:

1. $\{a_n(x)\}_{n\in\mathbb{N}}$ is a q-Appell sequence of type II.

2. There exists a sequence $\{g_n\}_{n\in\mathbb{N}}$, with $g_0 \neq 0$, such that

$$a_n(x) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} g_{n-k}\, x^k$$

or, equivalently,

$$g_n = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k x^k\, a_{n-k}(x).$$

3. The sequence $\{a_n(x)\}_{n\in\mathbb{N}}$ has q-exponential generating series

$$A(x;t) = \sum_{n\geq 0} a_n(x) \frac{t^n}{[n]_q!} = g(t)\, E_q(-xt)^{-1}, \tag{11}$$

where $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{[n]_q!}$ is a q-exponential series with $g_0 \neq 0$.

4. There exists a sequence $\{g_n\}_{n\in\mathbb{N}}$, with $g_0 \neq 0$, such that

$$a_n(x) = \sum_{k\geq 0} g_k\, q^{\binom{n-k}{2}} \frac{\mathfrak{D}_q^k}{[k]_q!}\, x^n.$$

3 Two-parameter identities

To obtain a q-analogue of identity (1), we apply the q-Leibniz rule to the q-exponential generating series of a q-Appell sequence. By applying such a rule in two different ways, we obtain two different extensions of identity (1). First, we have the following result:

Theorem 1. *Let $\{a_n(x)\}_{n\in\mathbb{N}}$ be the q-Appell sequence (of type I) associated with the sequence $\{g_n\}_{n\in\mathbb{N}}$. Then, for every $m,n\in\mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}}x^k a_{m+n-k}(x)=\sum_{k=0}^{m}\binom{m}{k}_q q^{nk}g_{n+m-k}\,x^k\,. \tag{12}$$

Proof. By applying the q-Leibniz rule (4) to series (10), we get

$$\mathfrak{D}_q^m A(x;t)=\mathfrak{D}_q^m E_q(xt)g(t)=\sum_{k=0}^{m}\binom{m}{k}_q\mathfrak{D}_q^k E_q(xt)\cdot\mathcal{Q}_q^k\mathfrak{D}_q^{m-k}g(t).$$

Then, by identity (6) and by setting $g^{(h)}(t)=\mathfrak{D}_q^h g(t)$, we have

$$\mathfrak{D}_q^m A(x;t)=\sum_{k=0}^{m}\binom{m}{k}_q x^k E_q(xt)\,g^{(m-k)}(q^k t),$$

that is,

$$E_q(xt)^{-1}\mathfrak{D}_q^m A(x;t)=\sum_{k=0}^{m}\binom{m}{k}_q x^k\,g^{(m-k)}(q^k t).$$

Recalling formulas (3) and (9) and taking the coefficients of $\frac{t^n}{[n]_q!}$ from the first and last series, we obtain identity (12). $\square$

To state the next theorem, we need the following definition: given the q-Appell polynomials $a_n(x)$ associated with the q-numbers g_n, we define the shifted polynomials $a_n^{[m]}(x)$ as the q-Appell polynomials associated with the q-numbers g_{m+n}; namely,

$$a_n^{[m]}(x)=\sum_{k=0}^{n}\binom{n}{k}_q g_{m+n-k}\,x^k.$$

This means that their q-exponential generating series is

$$\sum_{n\geq 0}a_n^{[m]}(x)\frac{t^n}{[n]_q!}=g^{(m)}(t)\,E_q(xt),$$

where $g^{(m)}(t)=\mathfrak{D}_q^m g(t)$.

Theorem 2. *Let $\{a_n(x)\}_{n\in\mathbb{N}}$ be the q-Appell sequence (of type I) associated with the sequence $\{g_n\}_{n\in\mathbb{N}}$. Then, for every $m,n\in\mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}}x^k a_{m+n-k}(x)=\sum_{k=0}^{m\wedge n}\binom{m}{k}_q\binom{n}{k}_q[k]!(q-1)^k q^{\binom{k}{2}}x^k a_{m-k}^{[n]}(x), \tag{13}$$

where $m\wedge n$ denotes the minimum between m and n.

Proof. By applying the q-Leibniz rule (4) to series (10), we have

$$\mathfrak{D}_q^m A(x;t)=\mathfrak{D}_q^m g(t)E_q(xt)=\sum_{k=0}^{m}\binom{m}{k}_q\mathfrak{D}_q^k g(t)\cdot\mathcal{Q}_q^k\mathfrak{D}_q^{m-k}E_q(xt).$$

Then, by identity (6), we have

$$\mathfrak{D}_q^m A(x;t)=\sum_{k=0}^{m}\binom{m}{k}_q g^{(k)}(t)\,x^{m-k}E_q(q^k xt)$$

and, by Eq. (8), we have

$$\mathfrak{D}_q^m A(x;t)=\sum_{k=0}^{m}\binom{m}{k}_q g^{(k)}(t)\,x^{m-k}((1-q)xt;q)_k E_q(xt),$$

that is,

$$E_q(xt)^{-1}\mathfrak{D}_q^m A(x;t)=\sum_{k=0}^{m}\binom{m}{k}_q g^{(k)}(t)\,x^{m-k}((1-q)xt;q)_k.$$

Hence, by Eq. (2), we have

$$\begin{aligned}
E_q(xt)^{-1}\mathfrak{D}_q^m A(x;t)&=\sum_{k=0}^{m}\binom{m}{k}_q g^{(k)}(t)\,x^{m-k}\sum_{i=0}^{k}\binom{k}{i}_q(q-1)^i q^{\binom{i}{2}}x^i t^i\\
&=\sum_{i,k\geq 0}^{m}\binom{m}{k}_q\binom{k}{i}_q(q-1)^i q^{\binom{i}{2}}x^{i+m-k}t^i g^{(k)}(t)\\
&=\sum_{i,k\geq 0}^{m}\binom{m}{i}_q\binom{m-i}{k-i}_q(q-1)^i q^{\binom{i}{2}}x^{i+m-k}\sum_{n\geq 0}\binom{n}{i}_q[i]_q!g_{n+k-i}\frac{t^n}{[n]_q!}\\
&=\sum_{n\geq 0}\left(\sum_{i,k\geq 0}\binom{m}{i}_q\binom{m-i}{k-i}_q\binom{n}{i}_q[i]_q!(q-1)^i q^{\binom{i}{2}}g_{n+k-i}x^{m-k+i}\right)\frac{t^n}{[n]_q!}\\
&=\sum_{n\geq 0}\left(\sum_{i\geq 0}\binom{m}{i}_q\binom{n}{i}_q[i]_q!(q-1)^i q^{\binom{i}{2}}x^i\sum_{k=i}^{m}\binom{m-i}{k-i}_q g_{n+k-i}x^{m-k}\right)\frac{t^n}{[n]_q!}
\end{aligned}$$

$$= \sum_{n\geq 0}\left(\sum_{i\geq 0}\binom{m}{i}_q\binom{n}{i}_q[i]_q!(q-1)^i q^{\binom{i}{2}}x^i\sum_{k=0}^{m-i}\binom{m-i}{k}_q g_{n+k}x^{m-i-k}\right)\frac{t^n}{[n]_q!}$$
$$= \sum_{n\geq 0}\left(\sum_{i\geq 0}\binom{m}{i}_q\binom{n}{i}_q[i]_q!(q-1)^i q^{\binom{i}{2}}x^i\sum_{k=0}^{m-i}\binom{m-i}{k}_q g_{n+m-i-k}x^{k}\right)\frac{t^n}{[n]_q!}$$
$$= \sum_{n\geq 0}\left(\sum_{i\geq 0}\binom{m}{i}_q\binom{n}{i}_q[i]_q!(q-1)^i q^{\binom{i}{2}}x^i a^{[n]}_{m-i}(x)\right)\frac{t^n}{[n]_q!}.$$

Taking the coefficients of $\frac{t^n}{[n]_q!}$ in the first and last series, we obtain identity (13). □

Given a sequence $\{g_n\}_{n\in\mathbb{N}}$, we denote by $a_n(x)$ the associated q-Appell polynomials of type I and by $a^*_n(x)$ the associated q-Appell polynomials of type II. To establish a relation between these two sequences, we define the umbral map $\psi : \mathbb{Q}_q[x] \to \mathbb{Q}_q[x]$ by setting

$$\psi(x^n) = q^{\binom{n}{2}}x^n$$

and by extending it by linearity. Then, we immediately have the following first result:

Lemma 3. *The umbral map ψ transforms the q-Appell polynomials of type I into the q-Appell polynomials of type II, that is, for every $n \in \mathbb{N}$, we have*

$$\psi(a_n(x)) = a^*_n(x).$$

This result can be extended as follows.

Lemma 4. *For every $m \in \mathbb{N}$ and for every polynomial $p(x)$, we have*

$$\psi(x^m p(x)) = q^{\binom{m}{2}}x^m\, p^*(q^m x), \tag{14}$$

where $p^*(x) = \psi(p(x))$.

Proof. Suppose $p(x) = \sum_{k=0}^{n} p_k x^k$. Then, by the linearity of the umbral map, we have

$$\psi(x^m p(x)) = \sum_{k=0}^{n}\binom{n}{k}_q p_k\,\psi(x^{m+k}) = \sum_{k=0}^{n}\binom{n}{k}_q p_k\, q^{\binom{m+k}{2}}x^{m+k}$$
$$= q^{\binom{m}{2}}x^m\sum_{k=0}^{n}\binom{n}{i}_q q^{\binom{k}{2}}p_k\,(q^m x)^k = q^{\binom{m}{2}}x^m p^*(q^m x).$$

This establishes identity (14). □

By this lemma, we can easily find the following analogues of identities (12) and (13) for the q-Appell polynomials of type II:

Theorem 5. *Let $\{a_n^*(x)\}_{n\in\mathbb{N}}$ be the q-Appell sequence of type II associated with the sequence $\{g_n\}_{n\in\mathbb{N}}$. Then, for all $m,n \in \mathbb{N}$, we have the identities*

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{k^2-k} x^k a^*_{m+n-k}(q^k x) = \sum_{k=0}^{m} \binom{m}{k}_q q^{\binom{k}{2}+nk} g_{n+m-k}\, x^k \tag{15}$$

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{k^2-k} x^k a^*_{m+n-k}(q^k x) = \sum_{k=0}^{m\wedge n} \binom{m}{k}_q \binom{n}{k}_q [k]!(q-1)^k q^{k^2-k} x^k a^{[n]*}_{n+m-k}(q^k x). \tag{16}$$

Proof. By Lemma 4, we have $\psi\big(x^m a_n^{[r]}(x)\big) = q^{\binom{m}{2}} x^m a_n^{[r]*}(q^m x)$. Hence, by applying ψ to identities (12) and (13) we obtain at once identities (15) and (16). □

4 Examples (of type I)

4.1 q-Hermite polynomials

The *q-Hermite polynomials* (or *Rogers-Szegö polynomials*) [8, 3] [32, p. 180] are the q-Appell polynomials associated with the numbers $g_n = 1$; namely,

$$H_n(q;x) = \sum_{k=0}^{n} \binom{n}{k}_q x^k\,, \tag{17}$$

and have q-exponential generating series

$$\sum_{n\geq 0} H_n(q;x) \frac{t^n}{[n]_q!} = E_q(t)\, E_q(xt).$$

Since $H_n^{[m]}(q;x) = H_n(q;x)$, identities (12) and (13) become

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^k H_{m+n-k}(q;x) = \sum_{k=0}^{m} \binom{m}{k}_q q^{nk}\, x^k, \tag{18}$$

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^k H_{m+n-k}(q;x) = \sum_{k=0}^{m\wedge n} \binom{m}{k}_q \binom{n}{k}_q [k]!(q-1)^k q^{\binom{k}{2}} x^k H_{m-k}(q;x). \tag{19}$$

In particular, for $x=1$, we have the *Galois numbers* $G_n(q) = H_n(q;1) = \sum_{k=0}^{n} \binom{n}{k}_q$, [18, 28], and the two previous identities become

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} G_{m+n-k}(q) = \sum_{k=0}^{m} \binom{m}{k}_q q^{nk}, \tag{20}$$

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} G_{m+n-k}(q) = \sum_{k=0}^{m\wedge n} \binom{m}{k}_q \binom{n}{k}_q [k]!(q-1)^k q^{\binom{k}{2}} G_{m-k}(q). \tag{21}$$

4.2 Gaussian polynomials

The *Gaussian polynomials* [18, 19] are the q-Appell polynomials associated with the q-numbers $g_n = (-1)^n q^{\binom{n}{2}}$; namely,

$$g_n(q;x) = (x-1)(x-q)\cdots(x-q^{n-1}) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{\binom{n-k}{2}} x^k \,,$$

and have q-exponential generating series

$$\sum_{n\geq 0} g_n(q;x)\frac{t^n}{[n]_q!} = E_q(t)^{-1}E_q(xt).$$

In this case, we have

$$\begin{aligned} g_n^{[m]}(q;x) &= \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{m+n-k} q^{\binom{m+n-k}{2}} x^k \\ &= (-1)^m q^{\binom{m}{2}} q^{mn} \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{\binom{n-k}{2}} \left(\frac{x}{q^m}\right)^k \\ &= (-1)^m q^{\binom{m}{2}} q^{mn} g_n(q;x/q^m) = (-1)^m q^{\binom{m}{2}} \frac{g_{m+n}(q;x)}{g_n(q;x)}. \end{aligned}$$

Hence, identities (12) and (13) become

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^k g_{m+n-k}(q;x) = \sum_{k=0}^{m} \binom{m}{k}_q (-1)^{n+m-k} q^{\binom{n+m-k}{2}+nk} x^k, \tag{22}$$

$$\begin{aligned} &g_n(q;x) \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^k g_{m+n-k}(q;x) = \\ &\quad = (-1)^n q^{\binom{n}{2}} \sum_{k=0}^{m\wedge n} \binom{m}{k}_q \binom{n}{k}_q [k]!(q-1)^k q^{\binom{k}{2}} x^k g_{n+m-k}(q;x). \end{aligned} \tag{23}$$

These two identities can be rewritten in terms of the Pochhammer symbol. More precisely, since $(x;q)_n = x^n g_n(q;x^{-1})$, we have the identities

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} (x;q)_{m+n-k} = \sum_{k=0}^{m} \binom{m}{k}_q (-1)^{n+m-k} q^{\binom{n+m-k}{2}+nk} x^{n+m-k}, \tag{24}$$

$$\begin{aligned} &(x;q)_n \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{\binom{k}{2}} (x;q)_{m+n-k} \\ &\quad = q^{\binom{n}{2}} x^n \sum_{k=0}^{m\wedge n} \binom{m}{k}_q \binom{n}{k}_q [k]!(q-1)^k q^{\binom{k}{2}} (x;q)_{n+m-k}. \end{aligned} \tag{25}$$

4.3 Al-Salam-Carlitz polynomials

The *Al-Salam-Carlitz polynomials* (or *q-Carlitz polynomials*) ([4] [13, p. 195] [20]) are defined by the formula

$$U_n^{(\alpha)}(q;x) = \sum_{k=0}^{n} \binom{n}{k}_q (-\alpha)^{n-k} g_k(x)$$

and consequently they have q-exponential generating series

$$U_q^{(\alpha)}(x;t) = \sum_{n\geq 0} U_n^{(\alpha)}(q;x)\frac{t^n}{[n]_q!} = \frac{E_q(xt)}{E_q(t)E_q(\alpha t)} = E_q(t)^{-1}E_q(\alpha t)^{-1}E_q(xt).$$

This means that they form the q-Appell sequence associated with the q-numbers

$$u_n^{(\alpha)}(q) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} q^{\binom{n-k}{2}} \alpha^k$$

defined by the q-exponential generating series

$$u_q^{(\alpha)}(t) = \sum_{n\geq 0} u_n^{(\alpha)}(q)\frac{t^n}{[n]_q!} = E_q(t)^{-1}E_q(\alpha t)^{-1}.$$

In this case, identity (12) become

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^k U_{m+n-k}^{(\alpha)}(q;x) = \sum_{k=0}^{m} \binom{m}{k}_q q^{nk} u_{n+m-k}^{(\alpha)}(q)\, x^k\,. \tag{26}$$

4.4 q-Factorial polynomials

The *q-factorial polynomials* are the q-Appell polynomials associated with the q-factorial numbers $g_n = [n]_q!$; namely,

$$F_n(q;x) = \sum_{k=0}^{n} \binom{n}{k}_q [n-k]_q!\, x^k.$$

Hence, they have q-exponential generating series

$$F_q(x,t) = \sum_{n\geq 0} P_n(q;x)\frac{t^n}{[n]_q!} = \frac{E_q(xt)}{1-t}.$$

Moreover, the polynomials

$$F_n^{[m]}(q;x) = \sum_{k=0}^{n} \binom{n}{k}_q [m+n-k]_q!\, x^k$$

have q-exponential generating series

$$F_q^{[m]}(x,t) = \sum_{n\geq 0} P_n^{[m]}(q;x)\frac{t^n}{[n]_q!} = F_q^{[m]}(t)E_q(xt),$$

where

$$F_q^{[m]}(t) = \sum_{n\geq 0}[m+n]_q!\frac{t^n}{[n]_q!} = [m]_q!\sum_{n\geq 0}\frac{[m+n]_q!}{[m]_q![n]_q!}t^n = [m]_q!\sum_{n\geq 0}\binom{m+n}{m}_q t^n,$$

that is,

$$F_q^{[m]}(t) = \frac{[m]_q!}{(t;q)_{m+1}} = \frac{[m]_q!}{(1-t)(1-qt)\cdots(1-q^m t)}.$$

Hence, we have

$$F_q^{[m]}(x,t) = \frac{[m]_q!\,E_q(xt)}{(1-t)(1-qt)\cdots(1-q^m t)}.$$

Identities (12) and (13) become

$$\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}}x^k F_{m+n-k}(x) = \sum_{k=0}^{m}\binom{m}{k}_q q^{nk}[n+m-k]_q!\,x^k, \tag{27}$$

$$\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}}x^k F_{m+n-k}(x) = \sum_{k=0}^{m\wedge n}\binom{m}{k}_q\binom{n}{k}_q[k]!(q-1)^k q^{\binom{k}{2}}x^k F_{n+m-k}^{[n]}(x). \tag{28}$$

4.5 q-rencontres and q-arrangement polynomials

The *q-derangement numbers* $d_n(q)$ [46, 11, 27] and the *q-arrangement numbers* $a_n(q)$ are respectively defined by the formulas

$$d_n(q) = \sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}}[n-k]_q! \qquad \text{and} \qquad a_n(q) = \sum_{k=0}^{n}\binom{n}{k}_q[n-k]_q!$$

and have q-exponential generating series

$$D_q(t) = \sum_{n\geq 0} d_n(q)\frac{t^n}{[n]_q!} = \frac{E_q(t)^{-1}}{1-t} \qquad \text{and} \qquad A_q(t) = \sum_{n\geq 0} a_n(q)\frac{t^n}{[n]_q!} = \frac{E_q(t)}{1-t}.$$

For $q = 1$, we recover the derangement numbers d_n ([31, p. 65] and A000166 and the arrangement numbers a_n ([31, p. 16] and A000522).

The *q-rencontres polynomials* $D_q(x;t)$ and the *q-arrangement polynomials* $A_q(x;t)$ are the q-Appell polynomials associated respectively with the q-numbers $d_n(q)$ and $a_n(q)$; namely,

$$D_n(q;x) = \sum_{k=0}^{n}\binom{n}{k}_q d_{n-k}(q)\,x^k \qquad \text{and} \qquad A_n(q;x) = \sum_{k=0}^{n}\binom{n}{k}_q a_{n-k}(q)\,x^k,$$

and have q-exponential generating series

$$D_q(x;t) = \sum_{n\geq 0} D_n(q;x)\frac{t^n}{[n]_q!} = \frac{E_q(t)^{-1}}{1-t}\,E_q(xt),$$
$$A_q(x;t) = \sum_{n\geq 0} A_n(q;x)\frac{t^n}{[n]_q!} = \frac{E_q(t)}{1-t}\,E_q(xt).$$

Then, by identity (12), we have

$$\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}}x^k D_{m+n-k}(q;x) = \sum_{k=0}^{m}\binom{m}{k}_q q^{nk}d_{n+m-k}(q)\,x^k, \tag{29}$$
$$\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}}x^k A_{m+n-k}(q;x) = \sum_{k=0}^{m}\binom{m}{k}_q q^{nk}a_{n+m-k}(q)\,x^k. \tag{30}$$

More generally, given $m\in\mathbb{N}$, we can consider the *generalized q-derangement numbers* $d_n^{(m)}(q)$ and the *generalized q-arrangement numbers* $a_n^{(m)}(q)$ defined by

$$d_n^{(m)}(q) = \sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}}[m+n-k]_q! \qquad \text{and} \qquad a_n^{(m)}(q) = \sum_{k=0}^{n}\binom{n}{k}_q[m+n-k]_q!$$

and having q-exponential generating series

$$D_q^{(m)}(t) = \sum_{n\geq 0} d_n^{(m)}(q)\frac{t^n}{[n]_q!} = \frac{[m]_q!\,E_q(t)^{-1}}{(1-t)(1-qt)\cdots(1-q^m t)},$$
$$A_q^{(m)}(t) = \sum_{n\geq 0} a_n^{(m)}(q)\frac{t^n}{[n]_q!} = \frac{[m]_q!\,E_q(t)}{(1-t)(1-qt)\cdots(1-q^m t)}.$$

Then, the associated q-Appell polynomials are the *generalized q-rencontres polynomials* $D_q^{(m)}(x;t)$ and the *generalized q-arrangement polynomials* $A_q^{(m)}(x;t)$ defined by

$$D_n^{(m)}(q;x) = \sum_{k=0}^{n}\binom{n}{k}_q d_{n-k}^{(m)}(q)\,x^k \qquad \text{and} \qquad A_n^{(m)}(q;x) = \sum_{k=0}^{n}\binom{n}{k}_q a_{n-k}^{(m)}(q)\,x^k$$

and with q-exponential generating series

$$D_q^{(m)}(x;t) = \sum_{n\geq 0} D_n^{(m)}(q;x)\frac{t^n}{[n]_q!} = \frac{[m]_q!\,E_q(t)^{-1}}{(1-t)(1-qt)\cdots(1-q^m t)}\,E_q(xt),$$
$$A_q^{(m)}(x;t) = \sum_{n\geq 0} A_n^{(m)}(q;x)\frac{t^n}{[n]_q!} = \frac{[m]_q!\,E_q(t)}{(1-t)(1-qt)\cdots(1-q^m t)}\,E_q(xt).$$

In this way, when $q = 1$, we recover the generalized derangement numbers $d_n^{(m)}$ and the generalized arrangement numbers $a_n^{(m)}$, and the associated Appell polynomials $D_n^{(m)}(x)$ and $A_n^{(m)}(x)$, [7, 15, 16].

For these generalized polynomials, by identity (12), we have

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^k D_{m+n-k}^{(\mu)}(q;x) = \sum_{k=0}^{m} \binom{m}{k}_q q^{nk} d_{n+m-k}^{(\mu)}(q)\, x^k, \tag{31}$$

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^k A_{m+n-k}^{(\mu)}(q;x) = \sum_{k=0}^{m} \binom{m}{k}_q q^{nk} a_{n+m-k}^{(\mu)}(q)\, x^k. \tag{32}$$

In particular, for $x = 1$, we have $D_n^{(\mu)}(q;1) = [\mu + n]_q!$. Hence, the first identity becomes

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} [\mu + m + n - k]_q! = \sum_{k=0}^{m} \binom{m}{k}_q q^{nk} d_{n+m-k}^{(\mu)}(q). \tag{33}$$

4.6 Generalized q-Bernoulli and q-Euler polynomials

The *generalized q-Bernoulli numbers* $B_n^{(\nu)}(q)$ and the *generalized q-Bernoulli polynomials* $B_n^{(\nu)}(q;x)$, [1], are defined, respectively, by the q-exponential generating series

$$B_q^{(\nu)}(t) = \sum_{n\geq 0} B_n^{(\nu)}(q)\, \frac{t^n}{[n]_q!} = \left(\frac{t}{E_q(t) - 1} \right)^{\nu},$$

$$B_q^{(\nu)}(x;t) = \sum_{n\geq 0} B_n^{(\nu)}(q;x)\, \frac{t^n}{[n]_q!} = \left(\frac{t}{E_q(t) - 1} \right)^{\nu} E_q(xt).$$

Similarly, the q-numbers $\widetilde{E}_n^{(\nu)}(q)$ and the *generalized q-Euler polynomials* $E_n^{(\nu)}(q;x)$ are defined, respectively, by the q-exponential generating series

$$\widetilde{E}_q^{(\nu)}(t) = \sum_{n\geq 0} \widetilde{E}_n^{(\nu)}(q)\, \frac{t^n}{[n]_q!} = \left(\frac{2}{E_q(t) + 1} \right)^{\nu},$$

$$E_q^{(\nu)}(x;t) = \sum_{n\geq 0} E_n^{(\nu)}(q;x)\, \frac{t^n}{[n]_q!} = \left(\frac{2}{E_q(t) + 1} \right)^{\nu} E_q(xt).$$

Hence, by identity (12), we have

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^k B_{m+n-k}^{(\nu)}(q;x) = \sum_{k=0}^{m} \binom{m}{k}_q q^{nk} B_{n+m-k}^{(\nu)}(q)\, x^k, \tag{34}$$

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^k E_{m+n-k}^{(\nu)}(q;x) = \sum_{k=0}^{m} \binom{m}{k}_q q^{nk} \widetilde{E}_{n+m-k}^{(\nu)}(q)\, x^k. \tag{35}$$

Clearly, when $\nu = 1$ and $q = 1$, we have the *Bernoulli polynomials* $B_n(x)$ and the *Euler polynomials* $E_n(x)$, [24, 29]. Moreover, we have the *Bernoulli numbers* $B_n = B_n(0)$ and $\widetilde{E}_n = E_n(0) = (2-2^{n+2})\frac{B_{n+1}}{n+1}$, the *Euler numbers* $E_n = 2^n E_n(1/2)$, ([14, p. 49] and A122045, A000364, A028296) and the *Springer numbers* $S_n = (-1)^{\lceil n/2 \rceil} 4^n E_n(1/4)$ ([40] and A001586). Hence, the above identities can be rewritten as

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^{n-k} B_{m+k}(x) = \sum_{k=0}^{m} \binom{m}{k} B_{n+k}\, x^{m-k}, \tag{36}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^{n-k} E_{m+k}(x) = \sum_{k=0}^{m} \binom{m}{k} \widetilde{E}_{n+k}\, x^{m-k}. \tag{37}$$

In particular, when $x = 1/2$, we have $B_n(1/2) = (2^{1-n} - 1)B_n$ and identities (36) and (37) become

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (2 - 2^{m+k})\, B_{m+k} = \sum_{k=0}^{m} \binom{m}{k} 2^{n+k}\, B_{n+k}, \tag{38}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E_{m+k} = \sum_{k=0}^{m} \binom{m}{k} 2^{n+k} (2 - 2^{n+k+2})\, \frac{B_{n+k+1}}{n+k+1}. \tag{39}$$

Similarly, when $x = 1/4$, we have $B_n(1/4) = \frac{1}{2^n}\left(\frac{1}{2^{n-1}} - 1\right) B_n - \frac{1}{4^n} n E_{n-1}$ and identities (36) and (37) become

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Big((2 - 2^{m+k})\, B_{m+k} - (m+k)\, E_{m+k-1} \Big) = \sum_{k=0}^{m} \binom{m}{k} 4^{n+k}\, B_{n+k}, \tag{40}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k+\lceil \frac{m+k}{2} \rceil} S_{m+k} = \sum_{k=0}^{m} \binom{m}{k} 4^{n+k} (2 - 2^{n+k+2})\, \frac{B_{n+k+1}}{n+k+1}. \tag{41}$$

From identities (38) and (40), we also have

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (m+k)\, E_{m+k-1} = \sum_{k=0}^{m} \binom{m}{k} 2^{n+k} (1 - 2^{n+k})\, B_{n+k}. \tag{42}$$

Finally, when $x = 1$, we have $B_n(1) = (-1)^n B_n$ and $E_n(1) = (-1)^n E_n(0) = (-1)^n \widetilde{E}_n$ and identities (36) and (37) become

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} B_{m+k} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} B_{n+k} \tag{43}$$

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} (2 - 2^{m+k+2})\, \frac{B_{m+k+1}}{m+k+1}, = (-1)^m \sum_{k=0}^{m} \binom{m}{k} (2 - 2^{n+k+2})\, \frac{B_{n+k+1}}{n+k+1}. \tag{44}$$

The first identity (43) is the symmetric Carlitz identity for the Bernoulli numbers [9]. Such an identity can be proved (and generalized) in several ways [12, 17, 30, 42, 44]. In [25], we proved it by using a general method based on umbral calculus. Here, we proved that it is a simple consequence of the general two-parameter identity for the Appell polynomials (specialized to the Bernoulli polynomials). Similarly, the second identity (44) is the analogue symmetric identity for the numbers $\widetilde{E}_n$.

We conclude by noticing that the previous symmetric identities can be extended to the generalized Bernoulli numbers. Indeed, we have $B_n^{(\nu)}(\nu) = (-1)^n B_n^{(\nu)}$ and $E_n^{(\nu)}(\nu) = (-1)^n \widetilde{E}_n^{(\nu)}$. So, setting $q = 1$ and $x = \nu$, we obtain

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} \nu^{n-k} B_{m+k}^{(\nu)} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} \nu^{m-k} B_{n+k}^{(\nu)}, \tag{45}$$

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} \nu^{n-k} \widetilde{E}_{m+k}^{(\nu)} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} \nu^{m-k} \widetilde{E}_{n+k}^{(\nu)}. \tag{46}$$

However, Carlitz's identity (43) cannot be immediately extended to the q-Bernoulli numbers. To obtain such an extension, we need to consider the q-Bernoulli numbers $B_n^*(q)$ of type II defined by the q-exponential generating series

$$B_q^*(t) = \sum_{n \geq 0} B_n^*(q) \frac{t^n}{[n]_q!} = \frac{t}{E_q^*(t) - 1},$$

where

$$E_q^*(t) = E_q(-t)^{-1} = \sum_{n \geq 0} q^{\binom{n}{2}} \frac{t^n}{[n]_q!}.$$

Since the q-exponential generating series of the q-Bernoulli numbers can be written as follows

$$B_q(t) = \frac{t}{E_q(t) - 1} = \frac{-t\, E_q(t)^{-1}}{E_q(t)^{-1} - 1} = B_q^*(-t)\, E_q(t)^{-1},$$

we have $B_q(1;t) = B_q^*(-t)$, that is, $B_n(q;1) = (-1)^n B_n^*(q)$. Therefore, for $\nu = 1$ and $x = 1$, identity (34) can be rewritten as

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{n-k}{2}} B_{m+k}^*(q) = (-1)^m \sum_{k=0}^{m} \binom{m}{k}_q q^{n(m-k)} B_{n+k}(q). \tag{47}$$

This is the q-analogue of Carlitz's identity (43), even though it is not symmetric and involves the q-Bernoulli numbers of type I and II. The q-Bernoulli numbers of type II, however, can be expressed in terms of the q-Bernoulli numbers of type I. Indeed, since

$$B_q^*(t) = \frac{t}{E_q(-t)^{-1} - 1} = \frac{-t\, E_q(-t)}{E_q(-t) - 1} = \frac{-t\,(1 + E_q(-t) - 1)}{E_q(-t) - 1} = B_q(-t) - t,$$

we have $B_n^*(q) = (-1)^n B_n(q) - \delta_{n,1} = (-1)^n(B_n(q) + \delta_{n,1})$. Moreover, by the identity [43]

$$E_q(-t)\,E_{q^{-1}}(t) = 1\,, \quad \text{or} \qquad E_q(-t)^{-1} = E_{q^{-1}}(t)\,,$$

we also have

$$B_q^*(t) = \frac{t}{E_q(-t)^{-1} - 1} = \frac{t}{E_{q^{-1}}(t) - 1} = B_{q^{-1}}(t).$$

Since $[n]_{q^{-1}}! = q^{-\binom{n}{k}}[n]_q!$, we have

$$B_q^*(t) = \sum_{n\geq 0} B_n(q^{-1})\frac{t^n}{[n]_{q^{-1}}!} = \sum_{n\geq 0} q^{\binom{n}{k}} B_n(q^{-1})\frac{t^n}{[n]_q!},$$

that is,

$$B_n^*(q) = q^{\binom{n}{k}} B_n(q^{-1}).$$

So, replacing this expression in (47), we obtain the identity

$$(-1)^n q^{\binom{n}{2}} \sum_{k=0}^{n} \binom{n}{k}_q q^{k(m-n+k)} B_{m+k}(q^{-1}) = (-1)^m q^{-\binom{m}{k}} \sum_{k=0}^{m} \binom{m}{k}_q q^{n(m-k)} B_{n+k}(q). \qquad (48)$$

In a completely similar way, defining the q-numbers $\widetilde{E}_n^*(q)$ of type II by the q-exponential generating series

$$\widetilde{E}_q^*(t) = \sum_{n\geq 0} \widetilde{E}_n^*(q)\frac{t^n}{[n]_q!} = \frac{2}{E_q^*(t) + 1} = \frac{2}{E_{q^{-1}}(t) + 1} = \widetilde{E}_{q^{-1}}(t)\,,$$

we can derive the following identity from identity (35):

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{n-k}{2}} \widetilde{E}_{m+k}^*(q) = (-1)^m \sum_{k=0}^{m} \binom{m}{k}_q q^{n(m-k)} \widetilde{E}_{n+k}(q). \qquad (49)$$

Finally, since $\widetilde{E}_n^*(q) = q^{\binom{n}{k}} \widetilde{E}_n(q^{-1})$, we also have

$$(-1)^n q^{\binom{n}{2}} \sum_{k=0}^{n} \binom{n}{k}_q q^{k(m-n+k)} \widetilde{E}_{m+k}(q^{-1}) = (-1)^m q^{-\binom{m}{k}} \sum_{k=0}^{m} \binom{m}{k}_q q^{n(m-k)} \widetilde{E}_{n+k}(q). \qquad (50)$$

References

[1] W. A. Al-Salam, q-Bernoulli numbers and polynomials, *Math. Nachr.* **17** (1959), 239–260.

[2] W. A. Al-Salam, q-Appell polynomials, *Ann. Mat. Pura Appl.* **77** (1967), 31–45.

[3] W. A. Al-Salam and L. Carlitz, A q-analog of a formula of Toscano, *Boll. Un. Mat. Ital.* **12** (1957), 414–417.

[4] W. A. Al-Salam and L. Carlitz, Some orthogonal q-polynomials, *Math. Nachr.* **30** (1965), 47-61.

[5] P. Appell, Sur une classe de polynômes, *Ann. Sci. École Norm. Sup. (Paris)* **9** (1880), 119–144.

[6] R. P. Boas Jr. and R. C. Buck, *Polynomial Expansions of Analytic Functions*, Academic Press, 1964.

[7] S. Capparelli, M. M. Ferrari, E. Munarini, and N. Zagaglia Salvi, A generalization of the "problème des rencontres", *J. Integer Sequences* **21** (2018), Article 18.2.8.

[8] L. Carlitz, Some polynomials related to the theta functions, *Ann. Mat. Pura Appl.* **41** (1955), 359–373.

[9] L. Carlitz, Problem 795, *Math. Mag.* **44** (1971), 107.

[10] B. C. Carlson, Polynomials satisfying a binomial theorem, *J. Math. Anal. Appl.* **32** (1970), 543–558.

[11] W. Y. C. Chen and G.-C. Rota, q-analogs of the inclusion-exclusion principle and permutations with restricted position, *Discrete Math.* **104** (1992), 7–22.

[12] W. Y. C. Chen and L. H. Sun, Extended Zeilberger's algorithm for identities on Bernoulli and Euler polynomials, *J. Number Theory* **129** (2009), 2111–2132.

[13] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, 1978.

[14] L. Comtet, *Advanced Combinatorics*, Reidel, 1974.

[15] M. M. Ferrari and E. Munarini, Decomposition of some Hankel matrices generated by the generalized rencontres polynomials, *Linear Algebra Appl.* **567** (2019), 180-–201.

[16] M. M. Ferrari, E. Munarini, and N. Zagaglia Salvi, Some combinatorial properties of the generalized derangement numbers, *Riv. Mat. Univ. Parma* **11** (2020).

[17] I. M. Gessel, Applications of the classical umbral calculus, *Algebra Universalis* **49** (2003), 397–434.

[18] J. Goldman and G.-C. Rota, The number of subspaces of a vector space, *Recent Progress in Combinatorics* (Proc. Third Waterloo Conf. on Combinatorics, 1968) Academic Press, 1969, pp. 75–83.

[19] J. Goldman and G.-C. Rota, On the foundations of combinatorial theory. IV. Finite vector spaces and Eulerian generating functions, *Studies in Appl. Math.* **49** (1970), 239–258.

[20] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge University Press, 2005.

[21] F. H. Jackson, A basic-sine and cosine with symbolical solutions of certain differential equations, *Proc. Edinburgh Math. Soc.* **22** (1904), 28–39.

[22] F. H. Jackson, On q-functions and a certain difference operator, *Trans. Roy. Soc. Edin.* **46** (1908), 253–281.

[23] V. Kac and P. Cheung, *Quantum Calculus*, Springer-Verlag, 2002.

[24] L. M. Milne-Thomson, *The Calculus of Finite Differences*, Macmillan, 1951.

[25] E. Munarini, Combinatorial identities for Appell polynomials, *Appl. Anal. Discrete Math.* **12** (2018), 362–388.

[26] E. Munarini, Combinatorial identities involving the central coefficients of a Sheffer matrix, *Appl. Anal. Discrete Math.* **13** (2019), 495–517.

[27] E. Munarini, q-derangement identities, *J. Integer Sequences* **23** (2020), Article 20.3.8.

[28] A. Nijenhuis, A. E. Solow, and H. S. Wilf, Bijective methods in the theory of finite vector spaces, *J. Combin. Theory Ser. A* **37** (1984), 80–84.

[29] N. E. Nörlund, Mémoire sur les polynomes de Bernoulli, *Acta Math.* **43** (1922), 121–196.

[30] H. Prodinger, A short proof of Carlitz's Bernoulli number identity, *J. Integer Sequences* **17** (2014), Article 14.4.1.

[31] J. Riordan, *An Introduction to Combinatorial Analysis*, Wiley, 1958.

[32] S. Roman, *The Umbral Calculus*, Academic Press, 1984.

[33] S. Roman and G.-C. Rota, The umbral calculus, *Adv. Math.* **27** (1978), 95–188.

[34] G.-C. Rota, D. Kahaner, and A. Odlyzko, On the foundations of combinatorial theory. VIII: finite operator calculus, *J. Math. Anal. Appl.* **42** (1973), 684–760.

[35] P. N. Sadjang, On a new q-analogue of Appell polynomials, arXiv:1801.08859v1 [math.CA]. Available at `https://arxiv.org/abs/1801.08859`.

[36] C. Scaravelli, Sui polinomi di Appell, *Riv. Mat. Univ. Parma* **6** (1965), 103–116.

[37] A. Sharma and A. Chak, The basic analogue of a class of polynomials, *Riv. Mat. Univ. Parma* **5** (1954), 325–337.

[38] I. M. Sheffer, Some properties of polynomial sets of type zero, *Duke Math. J.* **5** (1939), 590–622.

[39] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, Available at `https://oeis.org/`.

[40] T. A. Springer, Remarks on a combinatorial problem, *Nieuw Arch. Wisk.* **19** (1971), 30–36.

[41] J. F. Steffensen, The poweroid, an extension of the mathematical notion of power, *Acta Math.* **73** (1941), 333–366.

[42] Z.-W. Sun, Combinatorial identities in dual sequences, *European J. Combin.* **24** (2003), 709–718.

[43] M. R. Ubriaco, Time evolution in quantum mechanics on the quantum line, *Phys. Lett. A* **163** (1992), 1–4.

[44] P. Vassilev and M. V. Missana, On one remarkable identity involving Bernoulli numbers, *Notes on Number Theory and Discrete Mathematics* **11** (2005), 22–24.

[45] R. Vein and P. Dale, *Determinants and Their Applications in Mathematical Physics*, Springer-Verlag, 1999.

[46] M. L. Wachs, On q-derangement numbers, *Proc. Amer. Math. Soc.* **106** (1989), 273–278.

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