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Takagi Function Identities on Dyadic Rationals

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Abstract

The number of unbalanced interior nodes of divide-and-conquer trees on n leaves is known to form a sequence of dilations of the Takagi function on dyadic rationals. We use this fact to derive identities on the Takagi function and on the Hamming weight of an integer in terms of the Takagi function.

1 Introduction

The Takagi function is a widely-studied self-similar continuous nowhere-differentiable function on $[0, 1]$, identified by Takagi in 1901 [11]. It has connections to many areas of mathematics, including number theory, combinatorics, probability theory, and analysis, and many people have studied this function over the past century. Two recent comprehensive surveys on the subject have been published: one by Lagarias [8], and one by Allaart and Kawamura [1]. The On-Line Encyclopedia of Integer Sequences (OEIS) sequence A268289 [10] is also related to the Takagi function. There are a number of identities linking this sequence with the Takagi function [8, 1, 2].

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There is an interesting connection between binary trees representing divide-and-conquer calculations and the Takagi function, shown by Coronado et al. [4]: the number of unbalanced interior nodes of a divide-and-conquer tree is a sequence of dilations of the Takagi function on dyadic rationals.

In this paper we derive several formulas for counting such nodes on a divide-and-conquer tree, and use the dilations to extend them to identities on the Takagi function on dyadic rationals. These identities are in Section 5, and include the following:

- Recursive and closed formulas for the Takagi function on dyadic rationals. The recursive formulas are in Theorems 24 and 26, and the closed are in Theorems 25 and 27.

- A formula for the Hamming weight of an integer in terms of the Takagi function on dyadic rationals, in Theorem 28, derived from the formulas for counting D-nodes. As a consequence, we provide another proof of Trollope's theorem on cumulative binary digit sums [12] in Corollary 30, and give three succinct forms of the cumulative binary digit sum in Corollary 31.

2 Notation

We assume throughout the paper that $n > 0$ is an integer, and use the following representations. If additional assumptions are made in individual theorems on these variables, we state them explicitly; otherwise, the definitions in use are as stated in Notation 1.

Notation 1. A positive integer n may be represented as:

- $n = 2^k + r$, with integers k and r such that $k \geq 0$ and $0 \leq r < 2^k$.
- $n = 2^k \cdot (1 + x)$, with an integer $k \geq 0$ and a dyadic rational x with $0 \leq x < 1$.
- $n = \sum_{i=0}^{k} n_i \cdot 2^i$, where $k = \lfloor \log_2(n) \rfloor$ (binary decomposition). This may also be represented as $n = n_k n_{k-1} \cdots n_0$.

Lemma 2. *Let n, k, r, x be as in Notation 1. Then $x = \frac{r}{2^k}$ and $1 + x = \frac{n}{2^k}$.*

3 The Takagi function and divide-and-conquer trees

The Takagi function, shown in Figure 1(d), is a continuous nowhere-differentiable function on $[0, 1]$. We recast the original definition of the Takagi function on dyadic rationals from [11] to be consistent with notation and numbering convenient to the study of binary trees.

Definition 3 (**Takagi function**). [11] Let n be an integer. The *Takagi function* on dyadic rationals is defined as

$$\tau\left(\frac{n}{2^{k+1}}\right) = \sum_{i=1}^{\infty} \frac{\ell_i(n)}{2^i}, \text{ where } \ell_{i+1}(n) = \begin{cases} \sum_{j=0}^{i-1} n_{k-j}, & \text{if } n_{k-i} = 0; \\ i - \sum_{j=0}^{i-1} n_{k-j}, & \text{if } n_{k-i} = 1, \end{cases}$$

where $n_i = 0$ for every negative integer i.

Theorem 8 gives a simple formula for the dilations of the Takagi function on dyadic rationals in [0,1], in terms of certain interior nodes of full binary trees, called D-nodes.

Definition 4 (**S-nodes and D-nodes**). [9] An interior node of a rooted full binary tree is called an *S-node* if its two subtrees have the **S**ame number of leaves, and a *D-node* if its subtrees have **D**ifferent numbers of leaves. Such a labeling gives an *SD-tree*.

The SD-tree was defined to express the notion of balance in a binary commutative non-associative product [9].

The next lemma is true because an n-leaf binary tree has $n-1$ interior nodes [6], and these are partitioned into S-and D-nodes.

Lemma 5. *Let $a(n)$ be the number of S-nodes in a binary tree having n leaves. Then the number of D-nodes is $n-1-a(n)$.*

Definition 6 (**Divide-and-conquer tree**). A *divide-and-conquer tree* is a full binary tree in which the number of leaf descendants of the left and right children of any node differ by at most 1.

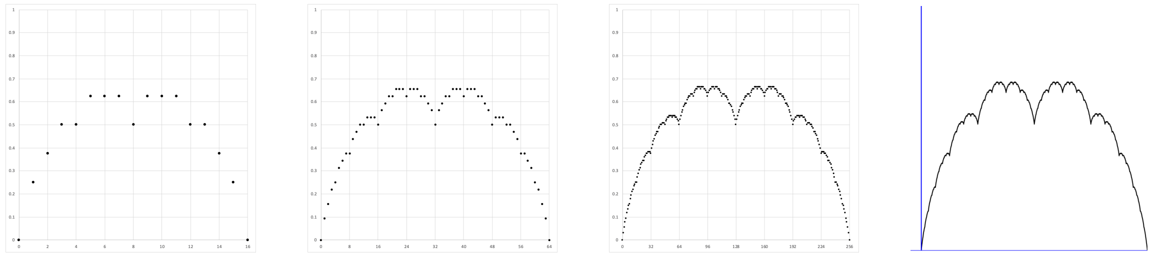
The binary tree generated is thus as balanced as it can be. At every level, the leaf descendants of an interior node are evenly or almost evenly divided between the left and the right branches. There is only one divide-and-conquer tree on n leaves, up to tree isomorphism.

Notation 7. The number of D-nodes in a divide-and-conquer tree is denoted by $\delta(n)$.

Theorem 8 is the same as Corollary 4 of Coronado et al. [4] but is stated here in terms of D-nodes and Takagi's original definition. It is also closely related to Baruchel's proposition 3.3 [2]. We show several examples of these dilations in Figure 1.

Theorem 8. *[4] Let $\delta(n)$ be the number of D-nodes in a divide-and-conquer tree on $n = 2^k + r$ leaves, with $0 \le r \le 2^k$. Let $\tau(x)$ be the Takagi function on x, with $0 \le x \le 1$. Then*

$$\tau\left(\frac{r}{2^k}\right) = \frac{\delta(2^k + r)}{2^k}.$$



(a) $y = \frac{\delta(16+x)}{16}$ (b) $y = \frac{\delta(64+x)}{64}$ (c) $y = \frac{\delta(256+x)}{256}$ (d) Takagi curve.

Figure 1: **Divide-and-conquer dilations of the Takagi function on the dyadic rationals.** Subfigures (a), (b) and (c) show examples of the dilations $y = \frac{\delta(2^k+x)}{2^k} = \tau\left(\frac{x}{2^k}\right)$ from Theorem 8, where $\delta(n)$ is the number of D-nodes on a divide-and-conquer tree on n leaves. Here, $k = 4, 6,$ and 8, and x is an integer with $0 \leq x \leq 2^k$. These may be visually compared to Subfigure (d), showing the continuous, self-similar, nowhere-differentiable Takagi (blancmange) curve on $[0, 1]$. (The blancmange curve image in Subfigure (d) is taken from Wiki Commons.)

4 D-nodes in divide-and-conquer trees

In this section, we develop several identities on $\delta(n)$, the number of D-nodes in a divide-and-conquer tree. The first subsection is general, and the second develops identities in terms of $s_1(n)$, the Hamming weight of n. We will then apply Theorem 8 to these in Section 5, for some quick identities on the Takagi function.

4.1 Counting D-nodes in a divide-and-conquer tree

We state here formulas for the number of D-nodes in a divide-and-conquer form with n leaves. The number of D-nodes corresponds to OEIS sequence A296062 [10], as noted by Coronado et al. [4].

Theorem 9. *[4] The number of D-nodes in a divide-and-conquer SD-tree with n leaf nodes is*

$$\delta(n) = \begin{cases} 2\delta(m), & \text{if } n = 2m; \\ \delta(m) + \delta(m+1) + 1, & \text{if } n = 2m+1; \\ 0, & \text{if } n = 1. \end{cases}$$

The following corollary applies to n when n is not a power of 2.

Corollary 10. *Let $n = 2^k + r$, where $0 < r < 2^k$, and let ρ_1 be the position of the smallest non-0 bit in r. The number of D-nodes in a divide-and-conquer SD-tree with n leaf nodes satisfies the equation*

$$\delta(n) = \frac{1}{2}(\delta(n-1) + \delta(n+1)) + 1 - \rho_1.$$

Proof. If $r = 2d + 1$ is odd, then $\rho_1 = 0$, and

$$\begin{aligned}
\delta(n) &= \delta(2(2^{k-1} + d) + 1) \\
&= \delta(2^{k-1} + d) + \delta(2^{k-1} + d + 1) + 1 && \text{(by Theorem 9)} \\
&= \frac{1}{2}(2 \cdot \delta(2^{k-1} + d) + 2 \cdot \delta(2^{k-1} + d + 1)) + 1 \\
&= \frac{1}{2}(\delta(2^k + 2d) + \delta(2^k + 2d + 2)) + 1 && \text{(by Theorem 9)} \\
&= \frac{1}{2}(\delta(n-1) + \delta(n+1)) + 1 - 0 \\
&= \frac{1}{2}(\delta(n-1) + \delta(n+1)) + 1 - \rho_1.
\end{aligned}$$

Now let $r = 2d$ be even. First, we use what has been shown for n odd:

$$\delta(n-1) = \frac{1}{2}(\delta(n-2) + \delta(n)) + 1 = \frac{1}{2}\delta(n-2) + \frac{1}{2}\delta(n) + 1,$$

and

$$\delta(n+1) = \frac{1}{2}(\delta(n) + \delta(n+2)) + 1 = \frac{1}{2}\delta(n+2) + \frac{1}{2}\delta(n) + 1.$$

Thus

$$\delta(n-1) + \delta(n+1) = \delta(n) + \frac{1}{2}(\delta(n-2) + \delta(n+2)) + 2,$$

and

$$\delta(n-1) + \delta(n+1) - 2 - \delta(n) = \frac{1}{2}(\delta(n-2) + \delta(n+2)). \tag{1}$$

Now we apply induction on k, noting that the smallest non-0 bit in $\frac{n}{2} = (2^{k-1} + d)$ is $(\rho_1 - 1)$:

$$\begin{aligned}
\delta(n) &= \delta(2(2^{k-1} + d)) \\
&= 2\delta(2^{k-1} + d) && \text{(by Theorem 9)} \\
&= 2 \cdot (\frac{1}{2}(\delta(2^{k-1} + d - 1) + \delta(2^{k-1} + d + 1)) + 1 - (\rho_1 - 1)) && \text{(by induction)} \\
&= \frac{1}{2}(2 \cdot \delta(2^{k-1} + d - 1) + 2 \cdot \delta(2^{k-1} + d + 1)) + 2 - (2 \cdot \rho_1 - 2) \\
&= \frac{1}{2}(\delta(2^k + 2d - 2) + \delta(2^k + 2d + 2)) + 2 - (2 \cdot \rho_1 - 2) && \text{(by Theorem 9)} \\
&= \frac{1}{2}(\delta(n-2) + \delta(n+2)) + 4 - 2 \cdot \rho_1 \\
&= \delta(n-1) + \delta(n+1) - 2 - \delta(n) + 4 - 2 \cdot \rho_1 && \text{(by Equation (1))} \\
&= \delta(n-1) + \delta(n+1) - \delta(n) + 2 - 2 \cdot \rho_1.
\end{aligned}$$

So

$$2 \cdot \delta(n) = \delta(n-1) + \delta(n+1) + 2 - 2 \cdot \rho_1,$$

and

$$\delta(n) = \frac{1}{2}(\delta(n-1) + \delta(n+1)) + 1 - \rho_1.$$

□

Theorem 11. *The number of D-nodes in a divide-and-conquer SD-tree with n leaf nodes is*

$$\delta(n) = \sum_{i=0}^{\lfloor \log_2(n) \rfloor - 1} \lambda_i(n), \text{ where } \lambda_i(n) = \begin{cases} (n \bmod 2^i), & \text{if } n_i = 0; \\ 2^i - (n \bmod 2^i), & \text{if } n_i = 1. \end{cases} \tag{2}$$

An explicit form is

$$\delta(n) = \sum_{i=0}^{\lfloor \log_2(n) \rfloor - 1} (n_i \cdot 2^i + (-1)^{n_i} \cdot (n \bmod 2^i)). \tag{3}$$

Proof. Because every SD-tree is a full complete binary tree, there are 2^i nodes at level i except for the bottom level. At each level, the descendant leaf nodes are almost evenly divided between the nodes: at the i^{th} level, $2^i - (n \bmod 2^i)$ nodes have $\left\lfloor \frac{n}{2^i} \right\rfloor$ leaf descendants, and $(n \bmod 2^i)$ of these nodes have $\left\lfloor \frac{n}{2^i} \right\rfloor + 1$ leaf descendants.

Consider the binary representation of the number n where the bits are ordered from least significant to most significant. A node in a divide-and-conquer tree is an D-node if and only if it has an odd number of leaf descendants. If n_i is 0, then $\left\lfloor \frac{n}{2^i} \right\rfloor$ is even and $(n \bmod 2^i)$ nodes have an odd number of descendants. If n_i is 1, then $\left\lfloor \frac{n}{2^i} \right\rfloor$ is odd, and $2^i - (n \bmod 2^i)$ nodes have an odd number of descendants. So the number of D-nodes at level i is

$$\beta_i(n) = \begin{cases} n \bmod 2^i, & \text{if } n_i \text{ is } 0; \\ 2^i - (n \bmod 2^i), & \text{if } n_i \text{ is } 1. \end{cases}$$

Equation (2) is obtained by summing across the levels.

So the number of D-nodes at level i is

$$n_i \cdot 2^i + (-1)^{n_i} \cdot (n \bmod 2^i).$$

Equation (3) is obtained by summing across the levels. □

An extensive analysis of solutions to divide-and-conquer recurrences is provided in [5]. Theorem 11 might be obtained from Theorem 9 using the methods in that paper. Instead we prove Theorem 11 directly by analyzing D-nodes.

4.2 Counting D-nodes on a divide-and-conquer tree in terms of Hamming weight

We begin this section with some simple lemmas on Hamming weight used throughout this section that appear in many papers. They can be seen by inspection.

Definition 12 (**Hamming weight**)**.** The *Hamming weight* of n, $\sum_{i=0}^{k} n_i$, is the number of 1s in the binary expansion of n, and is denoted in this paper by $s_1(n)$.

Lemma 13. *The Hamming weight $s_1(n) = s_1(r) + 1$.*

Lemma 14. *If n is even, then $s_1(n+1) = s_1(n) + 1$.*

Lemma 15. *If n is odd, then $s_1(n-1) = s_1(n) - 1$.*

Lemma 16. *If $r < 2^{k-1}$ is odd, then $s_1(2^k - r) = 1 + k - s_1(r)$.*

Proof. The binary representation of 2^k is 1 0...0, where there are k 0s following the leading 1. Let the binary representation of r be $0\ 0\ r_{k-2} \cdots\ r_1\ 1$, since $r < 2^{k-1}$ is odd. Subtracting r from 2^k gives $0\ 1\ \overline{r_{k-2}} \cdots\ \overline{r_1}\ 1$, where $\overline{r_i}$ is the negation of r_i. The lemma follows. □

We use the above lemmas to prove a series of assertions on the D-nodes of a tree with n leaves.

Lemma 17. *Let $n = 2^k + r$, with r odd and $0 < r < 2^k$. Let λ be as in Theorem 11, with $0 < i < k$. Then*

$$\lambda_i(n) - \lambda_i(n-1) = \begin{cases} 1, & \text{if } n_i = 0; \\ -1, & \text{if } n_i = 1. \end{cases}$$

Proof. For $0 < i < k$, all bits of r and $(r-1)$ besides the 0^{th} are the same, since r is odd. Thus

$$\lambda_i(n) - \lambda_i(n-1) = \begin{cases} (n \bmod 2^i) - ((n-1) \bmod 2^i) = 1, & \text{if } n_i = 0; \\ (2^i - (n \bmod 2^i)) - (2^i - ((n-1) \bmod 2^i)) = -1, & \text{if } n_i = 1. \end{cases}$$

□

The next lemma follows from Theorem 9 by induction.

Lemma 18. *Let $0 \le r \le 2^k$. Then $\delta(2^{k+1} - r) = \delta(2^k + r)$.*

Lemma 19. *Let $n > 0$ be even. Then*

$$\delta(n+1) = \delta(n) + \lfloor \log_2(n) \rfloor - 2 \cdot s_1(n) + 2.$$

Proof. Let $n = 2^k + r$, with $0 \le r < 2^k$, and let n have binary expansion $n_k n_{k-1} \cdots n_0$. Now n is even, so $\lfloor \log_2(n) \rfloor = \lfloor \log_2(n+1) \rfloor$.

$$\begin{aligned} \delta(n+1) - \delta(n) &= \sum_{i=0}^{k-1} \lambda_i(n+1) - \sum_{i=0}^{k-1} \lambda_i(n) && \text{(by Theorem 11)} \\ &= \sum_{i=0}^{k-1} (\lambda_i(n+1) - \lambda_i(n)) \end{aligned}$$

$$= \lambda_0(n+1) - \lambda_0(n) + \sum_{i=1}^{k-1} (\lambda_i(n+1) - \lambda_i(n)).$$

Since n is even, $n_0 = 0$, so $\lambda_0(n+1) - \lambda_0(n) = (2^0 - 0) - 0 = 1$.

There are $s_1(n) - 1 = (s_1(n+1) - 2)$ 1s in the 1^{st} through $(k-1)^{\text{th}}$ bits of $n+1$. Thus there are $((k-1) - (s_1(n+1) - 2))$ 0s in 1^{st} through $(k-1)^{\text{th}}$ bits of $(n+1)$. So

$$\begin{aligned}
\delta(n+1) - \delta(n) &= 1 - 1 \cdot (s_1(n+1) - 2) + 1 \cdot ((k-1) - (s_1(n+1) - 2)) && \text{(by Lemma 17)} \\
&= k - 2 \cdot s_1(n+1) + 4 \\
&= k - 2 \cdot (s_1(n) + 1) + 4 && \text{(by Lemma 14)} \\
&= k - 2 \cdot s_1(n) + 2 \\
&= \lfloor \log_2(n) \rfloor - 2 \cdot s_1(n) + 2.
\end{aligned}$$

□

Lemma 20. *Let $n = 2^k + r > 0$ be odd. Then*

$$\delta(n+1) = \delta(n) + \lfloor \log_2(n) \rfloor - 2 \cdot s_1(n) + 2.$$

Proof. Let $n = 2^k + r$, with $0 < r < 2^k$. We proceed by moving from n and $n+1$ to the symmetric $m = 2^{k+1} - (r+1)$ (which is even) and $m + 1 = 2^{k+1} - r$ (which is odd), by Lemma 18. We then apply Lemma 19, and move back to n and $n+1$, again by Lemma 18.

$$\begin{aligned}
\delta(n+1) &= \delta(m) && \text{(by Lemma 18)} \\
&= \delta(m+1) - (\lfloor \log_2(m) \rfloor - 2 \cdot s_1(m) + 2), && \text{(by Lemma 19)} \\
&= \delta(m+1) - (k - 2 \cdot s_1(m) + 2) \\
&= \delta(n) - (k - 2 \cdot s_1(m) + 2) && \text{(by Lemma 16)} \\
&= \delta(n) - (k - 2 \cdot (s_1(m+1) - 1) + 2) && \text{(by Lemma 14)} \\
&= \delta(n) - (k - 2 \cdot s_1(2^{k+1} - r) + 4) \\
&= \delta(n) - (k - 2 \cdot (2 + k - s_1(r)) + 4) && \text{(by Lemma 16)} \\
&= \delta(n) - (k - 2 \cdot (2 + k - (s_1(n) - 1)) + 4) && \text{(by Lemma 13)} \\
&= \delta(n) + k - 2 \cdot s_1(n) + 2 \\
&= \delta(n) + \lfloor \log_2(n) \rfloor - 2 \cdot s_1(n) + 2.
\end{aligned}$$

□

We then have a general recurrence relation for $\delta(n)$, following from Lemmas 19 and 20.

Theorem 21. *The number of D-nodes, $\delta(n+1)$, in a divide-and-conquer tree with $n+1$ leaf nodes satisfies the equation*

$$\delta(n+1) = \delta(n) + \lfloor \log_2(n) \rfloor - 2 \cdot s_1(n) + 2.$$

Repeated application of Theorem 9 (for even sub-sums) and Lemma 19 (for odd) gives rise to Theorem 22, another explicit formula for the number of D-nodes in a divide-and-conquer tree.

Theorem 22. *The number of D-nodes, $\delta(n)$, in a divide-and-conquer tree with $n+1$ leaf nodes satisfies the equation*

$$\delta(n) = \sum_{i=0}^{k-1} 2^i \cdot n_i \cdot \big(k - i - 2 \cdot \sum_{j=i}^{k} n_i + 4\big),$$

or alternatively

$$\delta(n) = \sum_{i=0}^{k-1} 2^i \cdot n_i \cdot \big(k - i - 2 \cdot s_1\big(\lfloor n/2^i \rfloor\big) + 4\big).$$

Proof. Let n have binary expansion $n_k n_{k-1} \cdots n_0$. We proceed by induction, noting that the theorem is true for $n = 1$ and $n = 2$.
If n is even, the quotient $n/2$ has binary expansion $n_k n_{k-1} \cdots n_1$, so

$$\begin{aligned}
\delta(n/2) &= \sum_{i=1}^{k-1} 2^{i-1} \cdot n_i \cdot \big(k - i - 2 \cdot \sum_{j=i}^{k} n_i + 4\big) && \text{(by induction)} \\
&= \frac{1}{2} \sum_{i=1}^{k-1} 2^i \cdot n_i \cdot \big(k - i - 2 \cdot \sum_{j=i}^{k} n_i + 4\big),
\end{aligned}$$

and

$$\begin{aligned}
\delta(n) &= 2\delta(n/2) && \text{(by Theorem 9)} \\
&= 2 \cdot \frac{1}{2} \sum_{i=1}^{k-1} 2^i \cdot n_i \cdot \big(k - i - 2 \cdot \sum_{j=i}^{k} n_i + 4\big) \\
&= \sum_{i=0}^{k-1} 2^i \cdot n_i \cdot \big(k - i - 2 \cdot \sum_{j=i}^{k} n_i + 4\big). && \text{(since } n_0 = 0\text{)}
\end{aligned}$$

If n is odd then $n_0 = 1$ and $n - 1$ has binary expansion $m_k m_{k-1} \cdots m_0$, where $m_i = n_i$ for $1 \le i \le k$, and $m_0 = 0$. Then

$$\begin{aligned}
\delta(n) &= \delta(n-1) + (\lfloor \log_2(n-1) \rfloor - 2 \cdot s_1(n-1) + 2) && \text{(by Theorem 21)} \\
&= \delta(n-1) + (k - 2 \cdot s_1(n-1) + 2) \\
&= \sum_{i=0}^{k-1} 2^i \cdot m_i \cdot \big(k - i - 2 \cdot \sum_{j=i}^{k} m_i + 4\big) + (k - 2 \cdot s_1(n-1) + 2) && \text{(by induction)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{k-1} 2^i \cdot m_i \cdot \big(k - i - 2 \cdot \sum_{j=i}^{k} m_i + 4\big) + (k - 2 \cdot s_1(n-1) + 2) \\
&= \sum_{i=1}^{k-1} 2^i \cdot n_i \cdot \big(k - i - 2 \cdot \sum_{j=i}^{k} n_i + 4\big) + (k - 2 \cdot s_1(n-1) + 2) \\
&= \sum_{i=1}^{k-1} 2^i \cdot n_i \cdot \big(k - i - 2 \cdot \sum_{j=i}^{k} n_i + 4\big) + (k - 2 \cdot (s_1(n) - 1) + 2) && \text{(by Lemma 15)} \\
&= \sum_{i=1}^{k-1} 2^i \cdot n_i \cdot \big(k - i - 2 \cdot \sum_{j=i}^{k} n_i + 4\big) + \big(k - 2 \cdot \sum_{j=0}^{k} n_i + 4\big) \\
&= \sum_{i=1}^{k-1} 2^i \cdot n_i \cdot \big(k - i - 2 \cdot \sum_{j=i}^{k} n_i + 4\big) + \big((k - 0) - 2 \cdot \sum_{j=0}^{k} n_i + 4\big) \\
&= \sum_{i=0}^{k-1} 2^i \cdot n_i \cdot \big(k - i - 2 \cdot \sum_{j=i}^{k} n_i + 4\big). && \text{(since } 2^0 \cdot n_0 = 1\text{)}
\end{aligned}
$$

□

Theorem 21 also gives a corollary on the Hamming weight of n.

Corollary 23. *The Hamming weight of n, $s_1(n)$, satisfies the equation*

$$s_1(n) = \frac{\delta(n) - \delta(n+1) + \lfloor \log_2(n) \rfloor}{2} + 1.$$

5 From D-nodes to the Takagi function

In this section, we take the identities on D-nodes on a divide-and-conquer tree from Section 4 and apply Theorem 8 to obtain identities on the Takagi function.

Theorem 24. *Let $1 \leq r \leq 2^k - 1$. Let ρ_1 be the position of the smallest non-0 bit in r. Then*

$$\tau\left(\frac{r}{2^k}\right) = \frac{1}{2}\left(\tau\left(\frac{r-1}{2^k}\right) + \tau\left(\frac{r+1}{2^k}\right)\right) + \frac{1-\rho_1}{2^k}. \tag{4}$$

Proof. Follows from Theorem 8 and Corollary 10. □

We observe that when $x = \dfrac{r-1}{2^k}$ and $y = \dfrac{r+1}{2^k}$, Theorem 24 gives rise to a special case of Boros' inequality [3]:

$$\tau\Big(\frac{x+y}{2}\Big) \leq \frac{1}{2}(\tau(x) + \tau(y)) + \frac{|x-y|}{2}.$$

For this x and y, Equation (4) in the theorem may be restated as follows:

$$\tau\left(\frac{x+y}{2}\right) = \frac{1}{2}(\tau(x)+\tau(y)) + \frac{|x-y|}{2} - \frac{\rho_1}{2^k},$$

so here, equality holds in Boros' inequality if and only if $\rho_1 = 0$, i.e., r is odd.

Theorem 25. *Let r be an integer. The Takagi function on $\frac{r}{2^k}$ satisfies the equation*

$$\tau\left(\frac{r}{2^k}\right) = \frac{1}{2^k} \cdot \sum_{i=0}^{k-1} \lambda_i(n), \text{ where } \lambda_i(n) = \begin{cases} (n \bmod 2^i), & \text{if } n_i = 0; \\ 2^i - (n \bmod 2^i), & \text{if } n_i = 1. \end{cases}$$

An explicit form is

$$\tau\left(\frac{r}{2^k}\right) = \frac{1}{2^k} \cdot \sum_{i=0}^{k-1} (n_i \cdot 2^i + (-1)^{n_i} \cdot (n \bmod 2^i)).$$

Proof. Follows from Theorem 11. □

Theorem 26. *Let r be an integer. The Takagi function on $\frac{r+1}{2^k}$ satisfies the equation*

$$\tau\left(\frac{r+1}{2^k}\right) = \tau\left(\frac{r}{2^k}\right) + \frac{1}{2^k} \cdot (k - 2 \cdot s_1(n) + 2).$$

Proof. Follows from Theorem 21. □

Theorem 26 may also be derived from Krüppel's Proposition 2.1 [7, Formula (2.2)] by substituting 1 for x, r for k, and k for ℓ in the statement of the formula.

Theorem 27. *Let r be an integer. The Takagi function on $\frac{r}{2^k}$ satisfies the equation*

$$\tau\left(\frac{r}{2^k}\right) = \frac{1}{2^k} \cdot \sum_{i=0}^{k-1} 2^i \cdot n_i \cdot \left(k - i - 2 \cdot \sum_{j=i}^{k} n_i + 4\right),$$

or alternatively

$$\tau\left(\frac{r}{2^k}\right) = \frac{1}{2^k} \cdot \sum_{i=0}^{k-1} 2^i \cdot n_i \cdot \left(k - i - 2 \cdot s_1\left(\left\lfloor \frac{n}{2^i} \right\rfloor\right) + 4\right).$$

Proof. Follows from Theorem 22. □

We end the paper with Theorem 28 on the Hamming weight of binary integers and its Corollary 30 on the cumulative sum of the weights of integers. Corollary 30 was shown by Trollope in 1968 [12], but also follows directly from the weight result, so we provide that proof.

Theorem 28. *The Hamming weight of n, $s_1(n)$, satisfies the equation*

$$s_1(n) = 2^{k-1}\left(\tau\left(\frac{r}{2^k}\right) - \tau\left(\frac{r+1}{2^k}\right)\right) + \frac{k+2}{2}.$$

Proof. Follows from Theorem 26. □

Lemma 29. *Let $S_1(n) = \sum_{i=0}^{n-1} s_1(i)$. If n is a power of 2, then $S_1(n) = \frac{1}{2} \cdot n \cdot \log_2(n)$.*

Proof. Exactly half of the bits in the integers between 1 and $n-1$ have value 1. □

Corollary 30. *[12] Let $S_1(n) = \sum_{i=0}^{n-1} s_1(i)$. Then*

$$S_1(n) = \frac{n \cdot \log_2(n)}{2} + 2^{k-1} \cdot (2x - \tau(x) - (1+x) \cdot \log_2(1+x)).$$

Proof.

$$\begin{aligned}
S_1(n) &= \sum_{i=0}^{2^k-1} s_1(i) + \sum_{i=2^k}^{2^k+r-1} s_1(i) \\
&= \frac{2^k \cdot k}{2} + \sum_{i=2^k}^{2^k+r-1} s_1(i) && \text{(by Lemma 29)} \\
&= \frac{2^k \cdot k}{2} + 2^{k-1} \cdot \left(\tau\left(\frac{0}{2^k}\right) - \tau\left(\frac{r}{2^k}\right)\right) + \sum_{i=2^k}^{2^k+r-1} \frac{k+2}{2} && \text{(by Theorem 28)} \\
&= \frac{2^k \cdot k}{2} - 2^{k-1} \cdot \tau\left(\frac{r}{2^k}\right) + r \cdot \frac{k+2}{2} \\
&= \frac{(2^k + r) \cdot k}{2} - 2^{k-1} \cdot \tau\left(\frac{r}{2^k}\right) + r \\
&= \frac{(2^k + r) \cdot k}{2} + 2^{k-1} \cdot \left(2 \cdot \frac{r}{2^k} - \tau\left(\frac{r}{2^k}\right)\right) \\
&= \frac{n \cdot k}{2} + 2^{k-1} \cdot (2x - \tau(x)) && \text{(by Lemma 2)} \\
& && (5) \\
&= \frac{n \cdot k}{2} + \frac{n}{2} \cdot \log_2\left(\frac{n}{2^k}\right) + 2^{k-1} \cdot \left(2x - \tau(x) - \frac{n}{2^k} \cdot \log_2\left(\frac{n}{2^k}\right)\right) \\
&= \frac{n \cdot k}{2} + \frac{n}{2} \cdot (\log_2 n - k) + 2^{k-1} \cdot \left(2x - \tau(x) - \frac{n}{2^k} \cdot \log_2\left(\frac{n}{2^k}\right)\right) \\
&= \frac{n \cdot \log_2 n}{2} + 2^{k-1} \cdot \left(2x - \tau(x) - \frac{n}{2^k} \cdot \log_2\left(\frac{n}{2^k}\right)\right) \\
&= \frac{n \cdot \log_2 n}{2} + 2^{k-1} \cdot (2x - \tau(x) - (1+x) \cdot \log_2(1+x)). && \text{(by Lemma 2)}
\end{aligned}$$

□

Finally, Corollary 31 gives three succinct formulas of the cumulative binary digit sum in terms of the Takagi function and in terms of the number of D-nodes.

Corollary 31. *Let n be as in Notation 1, and let $S_1(n) = \sum_{i=0}^{n-1} s_1(i)$. Then*

1. $S_1(n) = \frac{1}{2} \cdot (nk + 2^k \cdot (2x - \tau(x)))$,
2. $S_1(n) = \frac{1}{2} \cdot \left(nk + 2r - 2^k \cdot \tau\left(\frac{r}{2^k}\right)\right)$, *and*
3. $S_1(n) = \frac{1}{2} \cdot (nk + 2r - \delta(n))$.

Proof. Follows from Equation (5) in the proof of Corollary 30, and Theorem 8. □

The first two subcorollaries of Corollary 31 may also be derived from Krüppel's Proposition 2.1 (Formula (2.4) [7], by substituting r for n and ℓ for k in the statement of the formula and using Lemma 2 to obtain $S_1(n)$. The third subcorollary would then follow by Theorem 8.

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