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The Stochastic City*

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This paper analyzes a simple model of an urban area with growth and uncertainty. Household income, rents, and prices for land follow stochastic processes. Even though investors are risk neutral, uncertainty affects both land rents and land prices in equilibrium because the conversion of land from agricultural to urban use is irreversible. Growth, on the other hand, affects urban and agricultural land prices but not the level of rents. We show that uncertainty (i) delays the conversion of land from agricultural to urban use, (ii) imparts an option value to agricultural land, (iii) causes land at the boundary to sell for more than its opportunity cost in other uses, and (iv) reduces equilibrium city size. © 1990 Academic Press, Inc.

I. INTRODUCTION

This paper examines the effects of uncertainty on equilibrium land rents and prices in a simple model of a growing urban area. We apply the theory of continuous time stochastic processes to the timing of land conversion under uncertainty. Our analysis emphasizes the importance of a common, and seemingly innocuous, assumption in dynamic models of cities: the decision to convert land from agricultural to urban use is economically irreversible. When development is irreversible, uncertainty affects both land prices and land rents even when land owners are risk neutral. Growth, on the other hand, affects the value of urban and agricultural land but not the level of rents. We also show that uncertainty (i) delays the

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conversion of land from agricultural to urban use, (ii) imparts an option value to agricultural land, (iii) causes land at the boundary to sell for more than its opportunity cost in other uses, and (iv) reduces equilibrium city size.

The model developed in this paper is a synthesis of several strands of the economics literature. First, our analysis builds on existing models of urban growth, especially the durable capital, perfect foresight models of Arnott and Lewis [1], Arnott [2], and Wheaton [16]. Second, we draw on the literature dealing with optimal investment policy under uncertainty, for example, Bernanke [3], McDonald and Siegel [12], and Heaney and Jones [9]. Important papers by Mills [13] and Titman [15] examine how uncertainty affects the value and development of vacant urban land. More recently, Clarke and Reed [7] independently developed a model examining some of the issues discussed in this paper.

Including this introduction, the paper has six sections. Section II describes the model. In Section III the model is solved for the special case when future rents are known with certainty. Section IV describes the structure of equilibrium land rents and prices with uncertainty and risk neutrality. Section V outlines some of the implications of our analysis. Section VI summarizes and concludes the paper.

II. THE MODEL

This model is an extension of the simple dynamic model of a monocentric urban area developed in Capozza, *et al.* [4] and Capozza and Helsley [6]. We focus on the implications of uncertainty for the components of equilibrium land rents and prices. In the model household income, rents, prices, and city size follow stochastic processes.

Geography

A small, open urban area is located on a homogeneous plain. Employment and production are concentrated at the central business district (CBD), a point to which all households commute daily. Locations are indexed by their distance z from the CBD, where distance is measured so that the cost of commuting a unit of distance is \$1. The boundary of the urban area at time t, which is also the location of current development, is denoted by $z^*(t)$.

Households

Households are identical. They derive utility from land and a composite numeraire good x. Consumption of land is fixed at one unit per household. The budget constraint of a household living at location z is

$$y(t) = x + R(t, z) + z, \tag{1}$$

where $y(t)$ is exogenous household income at time t and $R(t, z)$ is land rent.

Household Equilibrium

Migration between urban areas is costless. Equilibrium for households is characterized by the single condition

$$V[R(t,z), y(t)] = \bar{V}, \tag{2}$$

where $V(\cdot)$ is the direct utility function and $\bar{V}$ is the national utility level. Since lot sizes are fixed, (2) implies that all households consume the same quantity of the numeraire good in equilibrium, $\bar{x}$, where

$$U(\bar{x}, 1) = \bar{V}, \tag{3}$$

and $U(\cdot)$ is the direct utility function. The bid rent function for land is

$$R(t,z) = y(t) - (\bar{x} + z). \tag{4}$$

Stochastic Assumptions

We assume that the urban area exports to an expanding world market. Shocks in world demand are transmitted through the labor market to household income and land rents as the export sector adjusts its output.[1] As a result household income rises over time in an uncertain fashion. More precisely, the sequence of random variables $\{y(t), t \geq 0\}$ is a Brownian motion process with drift $g > 0$ and variance σ^2. This implies that incomes follow (the infinitesimal analog of) a random walk that drifts upward by $\$g$ per unit of time. Formally, this assumption implies

(a) the stationary increments property, $y(t+s) - y(t) \sim N[gs, \sigma^2 s]$ for all t;

(b) the independent increments property, for all $t_0 < t_1 < \cdots < t_n$, $y(t_1) - y(t_0), y(t_2) - y(t_1), \ldots, y(t_n) - y(t_{n-1})$ are independent random variables with distributions given in (a);

(c) $y(0) = 0$; and

(d) $y(t) \equiv gt + \sigma B(t)$, where $\{B(t), t \geq 0\}$ is a standard Brownian motion process with drift 0 and variance 1.

From (4), bid rent is also a random variable. In fact, for every z, the sequence $\{R(t,z), t \geq 0\}$ has properties (a) and (b) above, with (c) replaced by

$$(c')\ R(0,z) = -(\bar{x} + z)$$

[1] Mills [13] describes more completely how shocks in export demand are transmitted to land rents in a simple general equilibrium model of a growing urban area.

and the equation in (d) replaced by

$$(d')\ R(t,z) \equiv R(0,z) + gt + \sigma B(t).^2$$

(a), (b), and (d') imply

$$R(t+s,z) \stackrel{\mathrm{d}}{=} R(t,z) + gs + \sigma B(s), \tag{5}$$

where the random variables in (5) are equal "in distribution."

The Value of Urban Land

In a competitive market the price of land equals the expected present value of future land rents. We assume that land owners are risk neutral and share a common discount rate r. We also assume that land owners know current rents, $R(t, z)$, and the distribution of future rents, given by g and σ^2. The price at time t of a unit of urban land at location z is

$$P^u(t,z) = E\left\{\int_t^\infty R(\tau,z)e^{-r(\tau-t)}d\tau \middle| R(t,z)\right\}, \qquad z \le z^*(t). \tag{6}$$

The expectation in (6) is conditional on the information available at time t. Setting $\tau = t + s$ in (5) and substituting the resulting expression into (6) yields

$$\begin{aligned} P^u(t,z) &= E\left\{\int_0^\infty [R(t,z) + gs + \sigma B(s)]e^{-rs}ds \middle| R(t,z)\right\} \\ &= \frac{1}{r}\left[R(t,z) + \frac{g}{r}\right]. \end{aligned} \tag{7}$$

The Value of Agricultural Land

Land in agriculture earns rent A. The capital cost of converting a unit of land to urban use is C. Thus, urban land is produced from agricultural land by adding a fixed amount of capital per unit.[3] Once converted land

[2]Urban rent may be negative under some realizations. One curiosity of dynamic models of urban areas is that land rents (but not building rents) can be negative in equilibrium (see [13]). The model can also be solved when rents follow a log-normal diffusion which rules out negative rents (see [5]).

[3]The model is best viewed as a model of the land conversion process and not as a model of the decision to add structural capital to land. We have purposely kept the production and consumption decisions extremely simple to highlight the effects of uncertainty on prices. The central assumption is stochastic land rents. See Mills [13] for a more detailed articulation of the production and consumption sectors.

receives urban rents forever: the decision to convert land to urban use is irreversible. The capital used to convert land from agricultural to urban use does not depreciate. The value at time t of a unit of agricultural land at location $z > z^*(t)$ is

$$P^a(t, s, z)$$
$$\equiv E\left\{\int_t^{t+s} Ae^{-r(\tau-t)}d\tau + \int_{t+s}^{\infty} R(\tau, z)e^{-r(\tau-t)}d\tau - Ce^{-rs}\middle|R(t, z)\right\}, \quad (8)$$

where $t + s$ is the date the land is converted from agricultural to urban use and s is a stopping time. The first term in the expectation in (8) is the present value of agricultural rent up to the date of conversion. The second term is the present value of urban rent from the date of conversion onward. The last term is the present value of the cost of conversion at $t + s$. Using (5), (7), and the strong Markov property, (8) can be written

$$P^a(t, s, z) = \frac{A}{r} + \frac{1}{r}E\left\{\left[R(t + s, z) + \frac{g}{r} - A - rC\right]e^{-rs}\middle|R(t, z)\right\}. \quad (9)$$

Landowners choose the conversion time to maximize expected profits or the value of land. Formally, landowners solve the program

$$\max_s P^a(t, s, z). \quad (10)$$

Let t^* be the optimal time to convert. Then the price of land is $P^a(t, t^*, z) \equiv P^a(t, z)$.

III. RENTS AND PRICES WHEN THE FUTURE IS CERTAIN

The model is easy to solve when the future is certain. As σ approaches 0, (5) becomes

$$R(t + s, z) = R(t, z) + gs. \quad (11)$$

The optimal conversion time from (10) and (8) satisfies

$$R(t^*, z) = A + rC. \quad (12)$$

Under certainty land is converted when its rent in urban use equals the agricultural rent foregone plus the opportunity cost of the capital needed to convert the land. (12) implicitly defines the boundary of the urban area at time t.

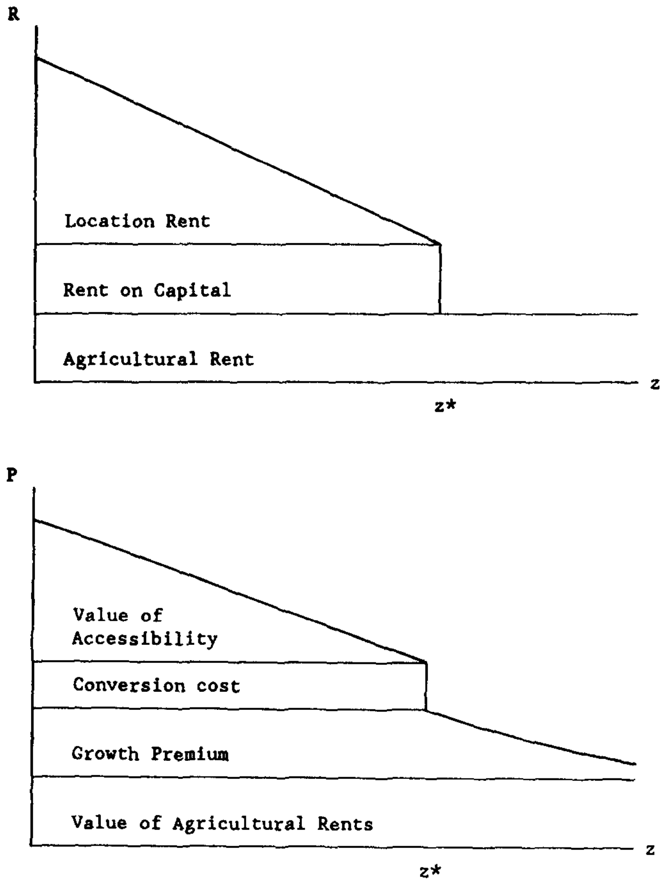

FIG. 1. Equilibrium land rents and prices under certainty.

From (4), the equilibrium rent function is

$$R(t,z) = \begin{cases} A + rC + z^*(t) - z & z \leq z^*(t) \\ A & z > z^*(t). \end{cases} \tag{13}$$

From (7), the price of urban land is

$$P^u(t,z) = \frac{A}{r} + C + \frac{g}{r^2} + \frac{1}{r}[z^*(t) - z], \qquad z \leq z^*(t). \tag{14}$$

Finally, the price of agricultural land from (10) is

$$P^a(t,z) = \frac{A}{r} + \frac{g}{r^2}e^{-r(t^*-t)}, \qquad z > z^*(t), \tag{15}$$

which may in turn be written[4]

$$P^a(t,z) = \frac{A}{r} + \frac{g}{r^2}e^{-r[z-z^*(t)]/g}, \qquad z > z^*(t). \tag{16}$$

Equilibrium land rents and prices have a simple structure in this model, especially when the future is certain. Figure 1 is a cross section of rents and prices inside and outside the city. Urban land rent consists of agricultural land rent, A, the opportunity cost of conversion capital, rC, and location rent, $z^*(t) - z$. Similarly, the price of urban land consists of the value of agricultural land rent, A/r, the cost of conversion, C, the value of accessibility, $(1/r)[z^*(t) - z]$, and a growth premium equal to the present value of anticipated increases in rents after development, $g/(r^2)$. The price of agricultural land equals the value of agricultural land rent, A/r, plus a growth premium that equals $g/(r^2)$ at the boundary of the urban area and decays as the distance from the boundary increases and the time of development moves further into the future.[5]

IV. RENTS AND PRICES WHEN THE FUTURE IS UNCERTAIN

First Hitting Time

To solve the model when the future is uncertain, we recast the landowner's problem into a hitting time problem. Let R^* represent the level of urban land rent at which it is optimal to convert land from rural to urban use. We refer to R^* as the reservation rent level. The time of conversion, t^*, known as the first hitting time, is defined by[6]

$$t^* = \min_s \{t + s \geq t | R(t+s, z) \geq R^*\}. \tag{17}$$

The first hitting time is illustrated in Fig. 2.

Let $F(R^*)$ represent the expected price of agricultural land conditioned on R^*. From (9),

$$\begin{aligned} F(R^*) &= E\{P^a(t, t^*, z) | R(t,z), R^*\} \\ &= \frac{A}{r} + \frac{1}{r}\left[R^* + \frac{g}{r} - A - rC\right] E\{e^{-r(t^*-t)} | R(t,z), R^*\}. \end{aligned} \tag{18}$$

[4](4) and (12) imply $z^*(t) = gt - (A + rC + \bar{x})$.

[5]The effects of expected future growth on the price of land in a deterministic model are examined in detail in Capozza and Helsley [6].

[6]Problems like the one analyzed in this section can be found in the literature on investment under uncertainty. See, for example, McDonald and Siegal [12] and Heaney and Jones [9].

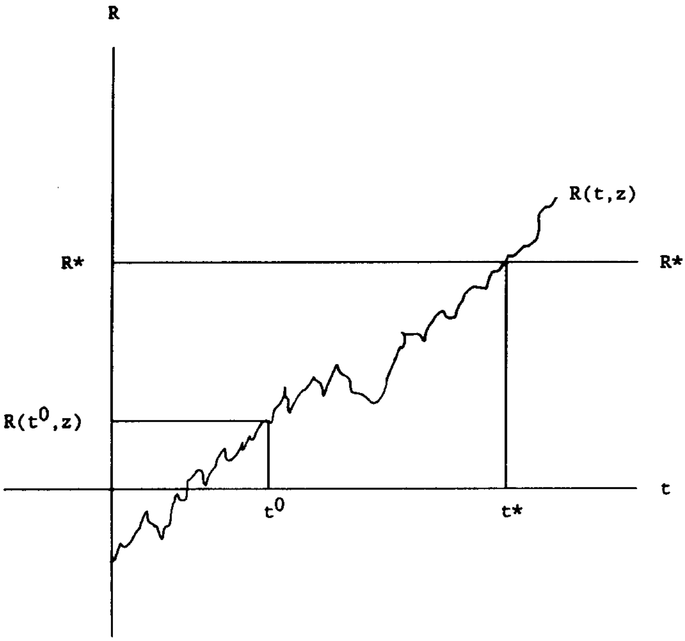

FIG. 2. The first hitting time.

Notice that the only random component of $F(R^*)$ is the first hitting time t^*.

The distribution of first hitting times under Brownian motion with drift is known. The moment generating function for t^* [11, p. 362] implies

$$E\left\{e^{-r(t^*-t)} \middle| R(t,z), R^*\right\} = e^{-\alpha[R^*-R(t,z)]}, \tag{19}$$

where

$$\alpha = \frac{\left(g^2 + 2\sigma^2 r\right)^{1/2} - g}{\sigma^2}. \tag{20}$$

Hence, the expected price of agricultural land can be written

$$F(R^*) = \frac{A}{r} + \frac{1}{r}\left[R^* + \frac{g}{r} - A - rC\right]e^{-\alpha[R^*-R(t,z)]}. \tag{21}$$

Reservation Rent and Reservation Price

The optimal conversion value, or reservation rent level, R^* maximizes (21), which implies

$$R^* = A + rC + \frac{r - \alpha g}{\alpha r}, \tag{22}$$

where $\alpha \leq r/g$. Comparing (12) and (22), we see that the reservation rent, the rent level that triggers the conversion of agricultural land to urban use, is higher when the future is uncertain.[7] From (7), the corresponding reservation price is

$$P^* = \frac{A}{r} + C + \frac{g}{r^2} + \frac{r - \alpha g}{\alpha r^2}. \tag{23}$$

The Structure of Land Rents and Prices

The equilibrium land rent function is

$$R(t, z) = \begin{cases} A + rC + \dfrac{r - \alpha g}{\alpha r} + z^*(t) - z & z \leq z^*(t) \\ A & z > z^*(t). \end{cases} \tag{24}$$

From (4) and (22), the boundary of the urban area at time t is given by

$$z^*(t) = y(t) - (R^* + \bar{x}). \tag{25}$$

The price of urban land, from (7), is

$$P^u(t, z) = \frac{A}{r} + C + \frac{g}{r^2} + \frac{r - \alpha g}{\alpha r^2} + \frac{1}{r}[z^*(t) - z], \qquad z \leq z^*(t). \tag{26}$$

Finally, the price of agricultural land, from (23), is

$$P^a(t, z) = \frac{A}{r} + \frac{g}{r^2}e^{-\alpha[z - z^*(t)]} + \frac{r - \alpha g}{\alpha r^2}e^{-\alpha[z - z^*(t)]} \qquad z > z^*(t). \tag{27}$$

The key pricing results of the model can now be summarized in two pictures, as illustrated in Fig. 3. In addition to the components outlined for the certainty case, urban land rent now includes an uncertainty or irreversibility term, $(r - \alpha g)/\alpha r$, which is precisely the difference between

[7]For $\sigma^2 > 0$, $\partial \alpha / \partial \sigma^2 < 0$.

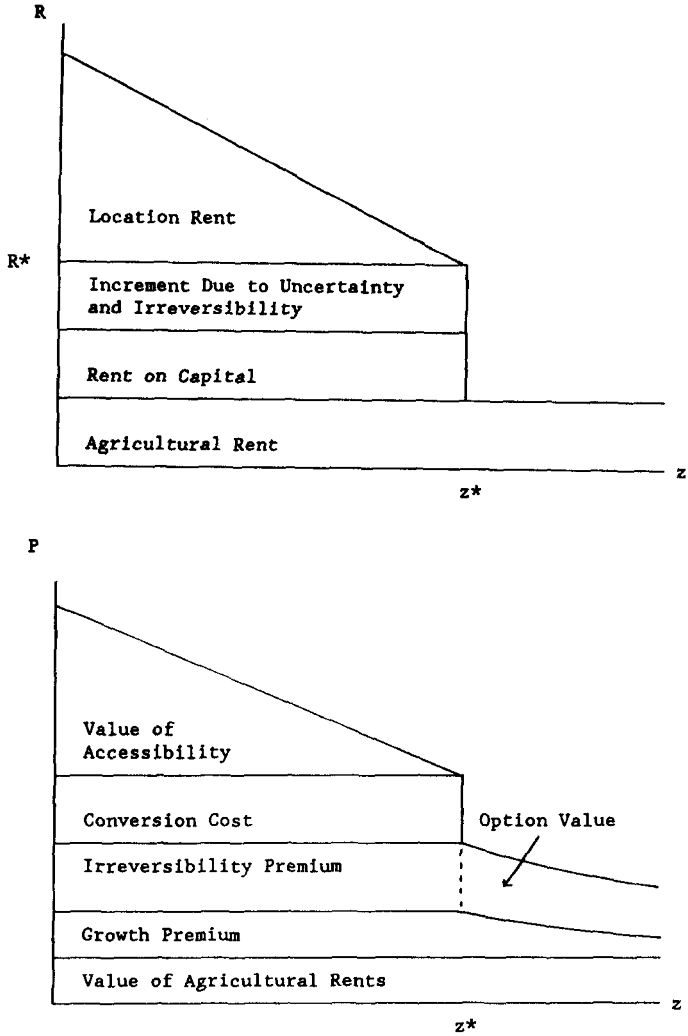

FIG. 3. Equilibrium land rents and prices under uncertainty.

the reservation rents in the uncertainty and certainty cases. This increment to rents is capitalized into the prices of urban and agricultural land. Thus, the price of urban land now consists of the value of agricultural land rent, A/r, the cost of conversion, C, the value of accessibility, $(1/r)(z^*(t) - z)$, the growth premium, $g/(r^2)$, and what we call an irreversibility premium, $(r - \alpha g)/(\alpha r^2)$. As before, the price of agricultural land includes a growth premium that decays as distance from the boundary increases.

However, the price of agricultural land now includes an option value as well, the counterpart of the irreversibility premium in the price of urban land.[8] This option value also grows smaller as the distance from the boundary of the urban area increases and the time of development moves further into the future.[9]

V. UNCERTAINTY, IRREVERSIBILITY, AND URBAN GROWTH

In this section we summarize the implications of our model for (a) development timing, (b) city size, (c) land prices and rents, (d) the gap between the price of land and the value of agricultural land rents at the urban boundary, and (e) the existence of vacant urban land in equilibrium.

Development Timing

In a deterministic model of a growing urban area, land is developed when rent in the urban use equals the opportunity cost of development—the sum of agricultural land rent and the opportunity cost of capital. How should the introduction of uncertainty about the future path of urban land rent affect this decision? Intuitively, we imagined that an optimal strategy would involve delaying development until urban land rent increased sufficiently to compensate investors for their aversion to risk. In other words, we imagined that development would occur when rent in the urban use equals the opportunity cost of development plus a risk premium.

While it is true in our model that conversion is delayed relative to the certainty case, the reasons for the delay differ from our intuitive expectations. The model implies that risk aversion is not the primitive cause of postponed development when future rents are uncertain. In fact, development is delayed even when investors are risk neutral. In the context of our model with risk neutral investors, development is postponed because the opportunity cost of development includes the option value of agricultural land—the value of the ability to avoid adverse outcomes in the risky urban land market while still retaining a claim on favorable outcomes. Development will not occur at the rent level that prompts conversion under certainty since investors know that rents may fall below this level in the immediate future. However, as conversion is postponed, and the level of urban land rent drifts upward, the probability that rents will fall below

[8]We have defined option value to be the net of the growth premium. This allows us to highlight the role of uncertainty. The finance literature usually includes the growth premium in option value.

[9]The growth premium in the price of agricultural land decays more quickly when the future is uncertain. Note that α approaches g/r as σ approaches 0, so the expressions for equilibrium rents and prices under certainty and uncertainty agree in the limit.

agricultural rents decreases. Thus, the present value of expected future returns is increased by delaying development, and even a risk neutral investor will adopt such a strategy.

With uncertainty and irreversibility, the rent level that triggers the conversion of land, the reservation rent, exceeds the sum of agricultural land rent and the opportunity cost of capital. Analogously, the reservation price of urban land exceeds the capitalized value of agricultural land rents plus the cost of conversion. Further, the reservation rent and the reservation price rise with the degree of uncertainty. From (22) and (23),

$$\frac{\partial R^*}{\partial \sigma^2} = -\frac{1}{\alpha^2}\frac{\partial \alpha}{\partial \sigma^2} > 0, \tag{28}$$

and

$$\frac{\partial P^*}{\partial \sigma^2} = -\frac{1}{r\alpha^2}\frac{\partial \alpha}{\partial \sigma^2} > 0. \tag{29}$$

City Size

In this open city model, uncertainty unambiguously reduces expected city size. Since rent at the boundary is higher the greater the degree of uncertainty about future land rents, at any point in time a greater fraction of income is devoted to land consumption and a smaller fraction to transportation in a "riskier" city. This can only be true if households at the boundary enjoy a shorter journey to work and the urban area with the greater variance of future rents is smaller. From (25), the expected boundary of the urban area at time t is

$$Ez^*(t) = gt - (R^* + \bar{x}), \tag{30}$$

where

$$\frac{\partial Ez^*(t)}{\partial \sigma^2} = -\frac{\partial R^*}{\partial \sigma^2} < 0. \tag{31}$$

Land Prices and Rents

The effects of uncertainty on land prices are closely related to its effects on city size and the timing of land development. A fundamental implication of our analysis is that equilibrium land prices in the agricultural area include an option value that increases with the variance of rents. This option value decreases as distance from the boundary of the urban area increases, implying a spatial gradient for agricultural land prices that is not directly related to the cost of transportation. Equilibrium land prices in the developed area include an irreversibility premium (equal to the option value at the boundary) that also rises with the variance of rents.

Our model implies that for a given city size, both urban and agricultural land prices should be higher in riskier urban areas. However, with the boundary of the urban area endogenously determined, the price of urban land is unaffected by uncertainty. From (26) and (27), the expected price of urban land can be written

$$EP^u(t,z) = \frac{g}{r^2} + \frac{1}{r}(gt - \bar{x} - z), \qquad z \le z^*(t), \tag{32}$$

which is independent of σ^2.

The expected price of agricultural land depends on the degree of uncertainty when city size is endogenously determined. From (27), the expected price of agricultural land may be written

$$EP^a(t,z) = \frac{A}{r} + \frac{1}{\alpha r} e^{-\alpha[z - Ez^*(t)]} e^{(\sigma^2/2)t}, \qquad z > z^*(t). \tag{33}$$

The effect of greater uncertainty on agricultural land prices is ambiguous.[10] In particular, when two cities with different variances of expected future incomes are compared, agricultural land prices may be lower in the riskier city for some locations near the urban boundary. However, it can be shown that agricultural land prices are eventually higher in the high variance urban area. Figure 4 illustrates the effect of the degree of uncertainty on land prices and rents when city size is endogenous.

The effect of growth and uncertainty on equilibrium land rents is perhaps more surprising. Since rents are payments for the current use of land one might not expect future growth or uncertainty to influence the current payment. But here, because the option value increases the reservation rent, city size and therefore average equilibrium urban rents are affected by both the drift and the variance of the stochastic process for bid rents. For a given city size both rents and prices are higher in an uncertain city.

The Price Gap

Should the price of land at the boundary of an urban area equal the value of agricultural land rents plus the cost of converting land to urban use? Except in the context of very simple, static models, the answer is no. We have identified several factors that contribute to such a gap. In a

[10]The expression for $\partial P^a(t,z)/\partial\sigma^2$ is messy and its sign is ambiguous. Details are available on request.

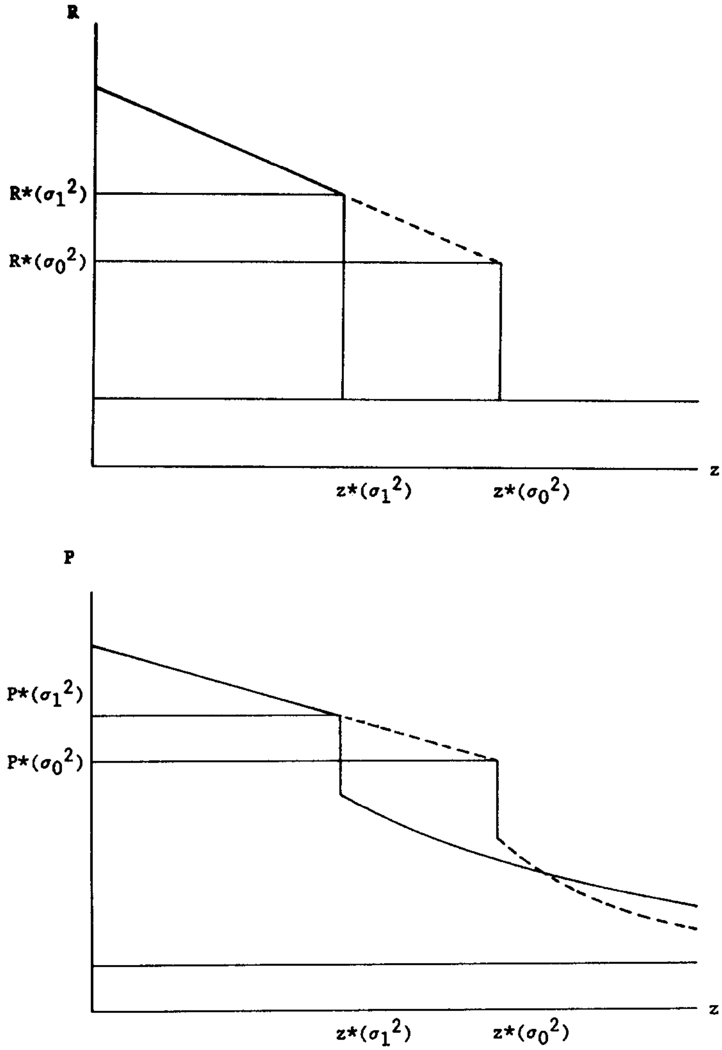

FIG. 4. Equilibrium land rents and prices and the degree of uncertainty ($\sigma_1^2 > \sigma_0^2$).

dynamic model where the future is certain, the price of land will include a growth premium equal to the value of expected future increases in urban land rents. When the future is uncertain and development is irreversible, the price of land will also include a premium caused by the option value in agricultural land. Economists sometimes appeal to government restrictions on land use and development to explain why the price of boundary land differs from the "static" opportunity cost of development. In fact, in all but the simplest of settings, such a differential is a characteristic of an efficient urban land market.

Leapfrogging

"Leapfrogging" is the discontinuous development of land in an urban area where vacant plots are surrounded by developed plots. There are two types of vacant plots—finished land that has been serviced and that is ready for development and raw land that has never been serviced.

Since a fixed amount of capital is applied to raw land, our model is best viewed as a model of the land conversion process, that is, as a model of the decision to convert raw land into finished land. The decision on the density of development is outside the scope of the model. As a result this model cannot address leapfrogging of the first kind. Vacant plots of finished land are not distinguishable from developed land.

However, the model can distinguish leapfrogging of the second kind—vacant raw land inside the urban boundary—and there is no leapfrogging of this kind in our model in equilibrium. There is vacant land awaiting conversion but it is all outside the urban boundary. Conversion takes place in a spatially continuous fashion.

Uncertainty about the future path of rents is neither necessary nor sufficient for the existence of vacant raw urban land in equilibrium. When the future is uncertain, it appears that an additional factor, such as variable densities [16, 15], competing land uses [8, 13], or heterogeneous expectations [13] is still required to generate vacant urban land in equilibrium.

VI. SUMMARY AND CONCLUSION

This paper analyzes a simple model of a growing urban area in which household income and land rent follow stochastic processes. Even though investors are risk neutral, uncertainty affects equilibrium land rents and prices, because the conversion of land from agricultural to urban use is irreversible.

In this model both rents and prices decompose into simple additive components with intuitive interpretations. Uncertainty affects the level of both rents and prices in the urban area. Growth affects the value of urban land and agricultural land but not the level of rents. We also show that uncertainty (i) delays the conversion of land from agricultural to urban use, (ii) reduces expected city size, and (iii) imparts an option value to agricultural land. The model suggests that land prices in two urban areas of equal size will be higher in a riskier urban area. Further, uncertainty and growth help explain why agricultural land near the boundary of an urban area sells for a large premium over the value of agricultural land rents.

The model developed in this paper is obviously highly simplified. To what degree do our results hinge on the assumptions we have made? Two of these assumptions seem especially important: (i) lot sizes are fixed, and (ii) migration is costless. Assuming fixed lot sizes simplifies the mathematics enormously, but, as noted above, it also predetermines the equilibrium spatial development pattern. We conjecture that uncertainty would encourage higher densities and discontinuous development in a more general model with variable lot sizes, but that prices and rents would no longer decompose in an additive fashion.

Assuming costless migration, or open cities, is common in long-run equilibrium models of city size. However, this is also an extreme assumption: in our model, it implies that unexpected growth in income is accompanied by unexpected growth in population to equalize utility levels. In a closed city, in which utility is endogenous, the effects of uncertain income growth might be very different. In fact, with population and hence the city boundary fixed in our model, income growth affects consumption of the composite good and utility in equilibrium, but it does not affect the level of land rents or prices. It would be more interesting and realistic to examine the effects of uncertain income growth in a partially closed model in which income changes in one period lead to migration in the next, and in which the adjustment to equilibrium is made explicit. This might illuminate precisely how our results depend on the open city assumption.

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