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ASSET PRICING WITH A FACTOR-ARCH COVARIANCE STRUCTURE
Empirical Estimates for Treasury Bills

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In this paper we suggest using the FACTOR-ARCH model as a parsimonious structure for the conditional covariance matrix of asset excess returns. This structure allows us to study the dynamic relationship between asset risk premia and volatilities in a multivariate system. One and two FACTOR-ARCH models are successfully applied to pricing of Treasury bills. The results show stability over time, pass a variety of diagnostic tests, and compare favorably with previous empirical findings.

1. Introduction

This paper is a contribution to the burgeoning literature which uses time series techniques to explain asset prices. Since almost all asset pricing theories rest on a specification of the way in which first moments (expected returns and risk premia) depend on second moments (variances and covariances), we focus on methods which allow us to explain the way in which second moments change. Our approach is explicitly multivariate; we characterize changes in the entire second moment matrix. We use the ARCH methodology originally proposed in Engle (1982); recent works which have used this approach to asset pricing include Bollerslev (1986, 1987), Bollerslev, Engle, and Wooldridge (1988), Chou (1988), Diebold and Nerlove (1988), Domowitz and Hakkio (1985), Engle and Bollerslev (1986), Engle, Lilien, and

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Robins (1987), French, Schwert, and Stambaugh (1986), McCurdy and Morgan (1988), and Milhoj (1987).

These studies typically use univariate time series models to represent asset returns. This seems inappropriate, as the major theme of static asset pricing theories is that an asset's risk premium depends as much on its covariance with other assets as on its own variance. Those studies which do specify a multivariate model consider at most two or three asset return series. This is also a drawback as empirical work on asset pricing has considered large numbers of assets. One leading theory, Ross's Arbitrage Pricing Theory, draws its theoretical sharpness from the assumption that the number of assets approaches infinity. See Chamberlain and Rothschild (1983).

A successful multivariate implementation of ARCH techniques is an exercise in parsimonious parameterization. An ARCH analysis of a univariate time series entails fitting something very like an ARIMA model to the (squared) disturbances of a time series model. The brute force generalization to a model with N assets would fit such a model to each of the $N \cdot (N+1)/2$ time series which characterizes the symmetric variance–covariance matrix. If each time series were generated by K parameters, then $K \cdot N \cdot (N+1)/2$ parameters must be estimated. Even though the number of parameters would grow as N^2, this kind of model would not capture the interdependence of the elements of the changing variance covariance matrix.

In this paper we suggest a particular parsimonious structure for the conditional covariance matrix of asset excess returns and apply it to the pricing of Treasury bills. Convenient specifications are necessarily restrictive; the usefulness of the model depends on its ability to fit real data. We believe our model emerges from its confrontation with the Treasury bill data in pretty good shape.

In the next section, we describe our covariance structure, discuss its attractive features, and its interpretation. Section 3 lays out the detailed statistical specification of our model and its relationship to Engle's (1987) FACTOR-ARCH. In section 4, we use our model to explain the pricing of Treasury bills. Section 5 concludes the paper.

2. Covariance structure

Let $y_t \in \mathbb{R}^N$ be a vector of asset excess returns[1] with conditional mean vector $\boldsymbol{\mu}_t$ and covariance matrix H_t (given all past information). Our purpose

[1]In this paper, the existence of a riskless asset is assumed. The excess return of an asset is defined as the difference between the 1-period holding return of the asset and the 1-period riskfree rate.

is to study the way in which H_t changes as t changes. We suppose that

$$H_t = \sum_{k=1}^{K} \boldsymbol{\beta}_k \boldsymbol{\beta}_k' \lambda_{kt} + \Omega, \tag{1}$$

where Ω is an $N \times N$ positive semi-definite matrix, the $\boldsymbol{\beta}_k$'s are linearly independent (nonscholastic) $N \times 1$ vectors, and the λ_{kt}'s are positive random variables. We require that $K < N$ but hope that it will be much less than N. This covariance structure has two natural interpretations that are related to the factor model proposed by Ross (1976) for Arbitrage Pricing Theory. A typical factor model for asset excess returns is

$$\boldsymbol{y}_t = \boldsymbol{\mu}_t + \sum_{k=1}^{K} \boldsymbol{g}_{kt} \cdot f_{kt} + \boldsymbol{v}_t, \tag{2}$$

where

$$\begin{aligned}
&\boldsymbol{\mu}_t\, \boldsymbol{g}_{kt} \in \mathscr{F}_{t-1}, && \forall k, t,\\
&\mathrm{E}_{t-1}(f_{kt}) = 0, && \forall k, t,\\
&\mathrm{E}_{t-1}(f_{kt} f_{jt}) = 0, && \forall j \neq k, \forall t,\\
&\mathrm{E}_{t-1}(\boldsymbol{v}_t) = \mathrm{E}_{t-1}(\boldsymbol{v}_t \,|\, f_{1t}, \dots, f_{Kt}) = 0, && \forall t,\\
&\mathrm{E}_{t-1}(\boldsymbol{v}_t \boldsymbol{v}_t') = \Omega \cdot
\end{aligned}$$

The f_{kt}'s are factors that affect the excess returns of *all* assets, $\boldsymbol{v}_t$ is a vector of idiosyncratic noises, and the $\boldsymbol{g}_{kt}$'s are time varying vectors of factor loadings.

The covariance structure specified in eq. (1) can be generated in this factor model setup under two different sets of additional assumptions. In the first case, $g_{kt} = \beta_k \cdot \lambda_{kt}^{1/2}$ and $V_{t-1}(f_{kt}) = 1$ $\forall k, t$; in the second case, $g_{kt} = \beta_k$ and $V_{t-1}(f_{kt}) = \lambda_{kt}$ $\forall k, t$. The first case is a time varying factor beta model in which the covariances (the betas) of different assets with a particular factor change proportionally. The response of each asset's risk premium to the risk (own-variance) of a particular factor is constant. Assets' risk premia change over time as the risk of particular factors changes. This model can also be interpreted as a dynamic factor model in which the factor loadings are constant over time but the factors themselves have time varying conditional second moments. Since the two interpretations lead to the same data generation process, they cannot be distinguished by the data.

The covariance structure has some attractive properties:

Property 1: The conditional covariance matrix of asset excess returns, H_t, is guaranteed to be positive semi-definite.

Property 2: Portfolios with excess return processes that have constant conditional variances can *always* be constructed.

To see this, let $\boldsymbol{w}$ be an N-vector in the (nonempty) null space of $[\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_K]'$, i.e., $\boldsymbol{w}'\boldsymbol{\beta}_k = 0\ \forall k$. The conditional variance of the excess return of a portfolio constructed with $\boldsymbol{w}$ as the vector of weights is $\boldsymbol{w}'H_t\boldsymbol{w} = \boldsymbol{w}'\Omega\boldsymbol{w}$ which is a constant.

Property 3: Portfolios with conditional excess return variances that can be used to replace the λ_{kt}'s can *always* be constructed. That is, H_t can always be rewritten in the form

$$H_t = \sum_{k=1}^{K} \boldsymbol{\beta}_k \boldsymbol{\beta}_k' \theta_{kt} + \Omega^*, \tag{3}$$

where the θ_{kt}'s are the conditional variances of some portfolios of the N assets and Ω^* is a time invariant $N \times N$ matrix.

To see this, choose $\boldsymbol{\alpha}_k$ such that $\boldsymbol{\alpha}_k$ is orthogonal to $\boldsymbol{\beta}_j$, $j \neq k$, and $\boldsymbol{\alpha}_k'\boldsymbol{\beta}_k = 1$. [Since the $\boldsymbol{\beta}_k$'s are linearly independent, such an $\boldsymbol{a}_k$ always exists. If the $\boldsymbol{\beta}_k$'s are orthogonal, then $\boldsymbol{\alpha}_k = \boldsymbol{\beta}_k/(\boldsymbol{\beta}_k'\boldsymbol{\beta}_k)$.] The conditional variance of the excess return of $P_{kt} \equiv \boldsymbol{\alpha}_k' y_t$, a portfolio constructed with $\boldsymbol{\alpha}_k$ as the vector of weights, is

$$\theta_{kt} \equiv \boldsymbol{\alpha}_k' H_t \boldsymbol{\alpha}_k = \lambda_{kt} + s_k, \tag{4}$$

where

$$s_k \equiv \boldsymbol{\alpha}_k' \Omega \boldsymbol{\alpha}_k.$$

Using eqs. (3) and (4), we can therefore write

$$H_t = \sum_{k=1}^{K} \boldsymbol{\beta}_k \boldsymbol{\beta}_k' \theta_{kt} + \Omega^*,$$

where

$$\Omega^* = \left[\Omega - \sum_{k=1}^{K} \boldsymbol{\beta}_k \boldsymbol{\beta}_k' s_k\right].$$

The portfolios constructed using the $\boldsymbol{\alpha}_k$'s as the weights are called 'factor-representing portfolios'. They are zero net investment portfolios since each of the y_t are excess returns. The portfolio weights can be either positive or negative. The conditional variance of each portfolio is perfectly correlated with its latent variable, λ_{kt}. Property 3 says that the information in the factor-representing portfolios is sufficient for predicting the variances and covariances of individual assets. In the terminology of Granger, Robins, and Engle (1984), there is *causality in variance* from the factor-representing portfolio to individual assets. This property allows us to study the dynamics of H_t by examining the dynamic behavior of the conditional variances of the factor-representing portfolios which is a much easier thing to do.

Property 4: The multiperiod forecasts of H_t can be obtained easily from the multiperiod forecasts of the θ_{kt}'s. In particular, the forecast for $H_{t+\tau}$ at time t is simply

$$\mathrm{E}_t(H_{t+\tau}) = \sum_{k=1}^{K} \boldsymbol{\beta}_k \boldsymbol{\beta}_k' \mathrm{E}_t(\theta_{kt+\tau}) + \Omega^*. \tag{5}$$

This property is particularly useful for the valuation of derivative assets written on more than one asset and for capital budgeting problems when the multiperiod forecast of the conditional covariance matrix of asset excess returns plays an important role.

Property 5: The 'persistence' of shocks to H_t is determined by the 'persistence' of the shocks to the θ_{kt}'s. Specifically, for $\mathrm{E}_t(H_{t+\tau}) = H_t$, it is necessary and sufficient that $\mathrm{E}_t(\theta_{kt+\tau}) = \theta_{kt}\ \forall t$.

This property is convenient for extending the works of Engle and Bollerslev (1986), Chou (1988), and Engle (1987) on integrated variance processes and cointegration in variance into a multivariate setting.

Let $\{P_{kt} \equiv \boldsymbol{\alpha}_k' y_t,\ k = 1, \ldots, K\}$ be the excess returns of a set of K factor-representing portfolios. Also let Π_{kt} be the risk premium of the kth factor-representing portfolio. The following asset pricing formula is considered:

$$\boldsymbol{\mu}_t = \sum_{k=1}^{K} \boldsymbol{\beta}_k \cdot \Pi_{kt}. \tag{6}$$

A simple derivation of (6) using the Consumption Beta model is provided in the appendix. Under an additional assumption about the constancy of preferences through time, Π_{kt} can be expressed as a linear function of the conditional variance of the excess returns of the kth factor-representing portfolio. That is,

$$\Pi_{kt} = c_k + \gamma_k \cdot \theta_{kt}, \qquad k = 1, \ldots, K. \tag{7}$$

The derivation for (7) is also given in the appendix. While these models are explicitly derived from a CCAPM framework, they are also close approximations to standard CAPM and APT models and we will continue to think of them from all three points of view. See, for example, Rothschild (1986).

3. Econometric specification

To complete the specification, an expression for θ_{kt}, or equivalently λ_{kt}, is needed. In principle, θ_{kt} can be any function of variables measurable with respect to the information set at time $t-1$. However, a very general representation is likely to be impractical. Although imposing the structure (1) [or equivalently (3)] can reduce the number of parameters to a large extent, a general formulation for the dynamics of the θ_{kt}'s still requires estimating the system of N assets as a whole. To simplify this problem, further restrictions on the dynamics of the θ_{kt}'s are sought.

The simplest but most restrictive assumption is the 'univariate portfolio representation assumption'. The set of K factor-representing portfolios is said to have a *univariate portfolio representation* if each of the excess return series (P_{kt}) of the factor-representing portfolios can be represented by a univariate time series process conditional on the full multivariate information set. An example of univariate portfolio representation is that the excess return of each of the K factor-representing portfolios follows an univariate ARCH-M model. Specifically, for all $k = 1, \ldots, K$,

$$P_{kt} = c_k + \gamma_k \cdot \theta_{kt} + u_{kt}, \qquad u_{kt} \mid \mathscr{F}_{t-1} \sim \mathrm{N}(0, \theta_{kt}), \tag{8}$$

$$\theta_{kt} = \omega_k + \phi_k \cdot u_{kt-1}^2 + \varphi_k \cdot \theta_{kt-1}.$$

Since $u_{kt} \equiv \boldsymbol{\alpha}_k' \boldsymbol{\varepsilon}_t$ and θ_{kt} and λ_{kt} are related by $\theta_{kt} = \lambda_{kt} + s_k$ [eq. (4)], the above specification requires that the dynamics of λ_{kt} satisfy

$$\lambda_{kt} = \left[\omega_k + s_k(\varphi_k - 1)\right] + \phi_k \cdot (\boldsymbol{\alpha}_k' \boldsymbol{\varepsilon}_{t-1})^2 + \varphi_k \cdot \lambda_{kt-1}. \tag{9}$$

Using eq. (8), the conditional covariance matrix of asset excess returns can be written as

$$H_t = C^* + \sum_{k=1}^{K} \left\{ \phi_k \boldsymbol{\beta}_k \boldsymbol{\beta}_k' (\boldsymbol{\alpha}_k' \boldsymbol{\varepsilon}_{t-1})^2 + \varphi_k \boldsymbol{\beta}_k \boldsymbol{\beta}_k' (\boldsymbol{\alpha}_k' H_{t-1} \boldsymbol{\alpha}_k) \right\}, \tag{10}$$

where

$$C^* \equiv \left[\sum_{k=1}^{K} \boldsymbol{\beta}_k \boldsymbol{\beta}_k' \omega_k + \Omega^*\right].$$

This is an example of the FACTOR-ARCH structure introduced by Engle (1987). It is also an example of the general positive definite structure proposed for multivariate ARCH models in Baba et al. (1987).

With the univariate portfolio representation, consistent estimates of the θ_{kt}'s can be obtained from maximum likelihood estimation of the univariate time series model for the P_{kt}'s. Using these consistent estimates of θ_{kt}'s as predetermined variables, consistent (though not efficient) estimates of the β_{ki}'s can be obtained from univariate time series models for individual assets.

A more general model can be achieved by relaxing the univariate portfolio representation assumption to a 'recursive portfolio representation' assumption. The set of K factor-representing portfolios is said to have a *recursive portfolio representation* if they can be rearranged such that the excess return of the kth portfolio depends only on information related to its own past and the past behavior of the excess returns of the first $k-1$ portfolios. An example of a recursive portfolio representation is

$$P_{kt} = c_k + \gamma_k \cdot \theta_{kt} + u_{kt}, \qquad u_{kt} \mid \mathscr{F}_{t-1} \sim \mathrm{N}(0, \theta_{kt}), \tag{11}$$

$$\theta_{kt} = \omega_k + \phi_{kk} \cdot u_{kt-1}^2 + \varphi_{kk} \cdot \theta_{kt-1} + \sum_{j=1}^{k-1} \left\{\phi_{kj} \cdot u_{jt-1}^2 + \varphi_{kj} \cdot \theta_{jt-1}\right\}.$$

If $\phi_{kj} = \varphi_{kj} = 0$ for $j \neq k$, then we are back to the univariate portfolio representation. If not, the information in one portfolio is useful in predicting the variance of another. In other words, there is 'causality in variance' from one factor-representing portfolio to another. Under (11), the conditional covariance matrix of asset excess returns can be written in the Baba et al. (1987) form as

$$H_t = C^* + \sum_{k=1}^{K} \{A_k \boldsymbol{\varepsilon}_{t-1} \boldsymbol{\varepsilon}_{t-1}' A_k' + G_k H_t G_k'\}, \tag{12}$$

where

$$C^* \equiv \left[\sum_{k=1}^{K} \boldsymbol{\beta}_k \boldsymbol{\beta}_k' \omega_k + \Omega^*\right], \quad A_k \equiv \sum_{j=1}^{k} \phi_{kj} \boldsymbol{\beta}_k \boldsymbol{\beta}_j', \quad G_k \equiv \sum_{j=1}^{k} \varphi_{kj} \boldsymbol{\beta}_k \boldsymbol{\beta}_j'.$$

This model is also a special case of the FACTOR-ARCH model introduced by Engle (1987). The recursive portfolio representation allows us to obtain consistent estimates of the θ_{kt}'s by sequential estimation of the single-equation models for the excess returns of the factor-representing portfolios.

An even more general specification is the 'general portfolio representation assumption'. The set of K factor-representing portfolios is said to follow a *general portfolio representation* if the excess return of any factor-representing portfolio depends only on information related to the past behavior of the excess returns of all K factor-representing portfolios. With a general portfolio representation we can get consistent estimates of the θ_{kt}'s by estimating a multivariate model for the system of K factor-representing portfolios which is a much smaller system than that for all N assets.

4. Application to the pricing of Treasury bills

To investigate whether the FACTOR-ARCH specification given in the last two sections is useful in modelling the dynamic behavior of asset excess returns, we apply it to the pricing of the short end of the term structure.

The data used in this paper consist of monthly percentage returns series on Treasury bills with maturities ranging from one to twelve months and the value-weighted index of NYSE & AMSE stocks. The Treasury bills data is obtained from the Fama Term Structure File in the 1985 CRSP Government

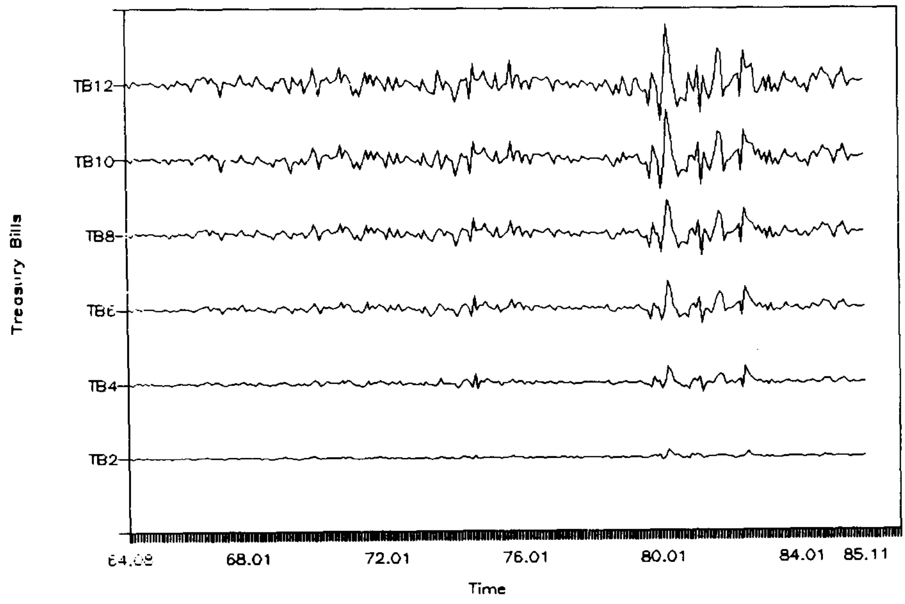

Fig. 1. Monthly excess returns.

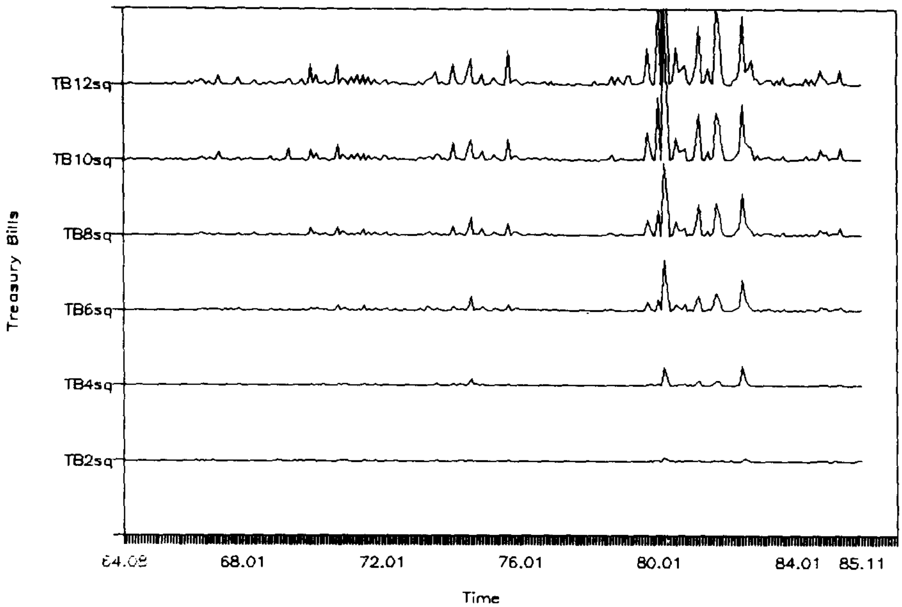

Fig. 2. Squared monthly excess returns.

Bond Tape. The stock index data is obtained from the 1985 CRSP Index Tape. The sample period is from August 1964 to November 1985.

From this dataset, the monthly excess returns of the 2- to 12-months Treasury bills ($TB2,TB3,\ldots,TB12$) and the stock market portfolio are constructed by subtracting from their monthly returns the 1-month T-bill rate under the assumption that it represents a riskless return. Fig. 1 presents the plots of excess returns for 2-, 4-, 6-, 8-, 10-, and 12-months maturities. Fig. 2 gives the plots for the squares of the excess returns of these assets. Fig. 3 shows the products of the excess returns of these assets with the excess return of the 3-months Treasury bill. Clearly, the excess returns of these T-bills have common periods of high volatility and therefore suggest the plausibility of the FACTOR-ARCH structure.

Summary statistics for the excess return series are given in table 1. The Ljung–Box statistics, $Q12$ and $QS12$, reported in the last two columns of the table show significant serial correlations for both the levels and the squares of the excess returns series. Results from principal component analysis on the sample unconditional covariance matrix of the excess returns are provided in table 2. The unconditional covariance matrix is nearly singular. The largest eigenvalue represents 92% of the total variance and the first two are 99.6% of the total variance. It is also interesting to observe that the eigenvector

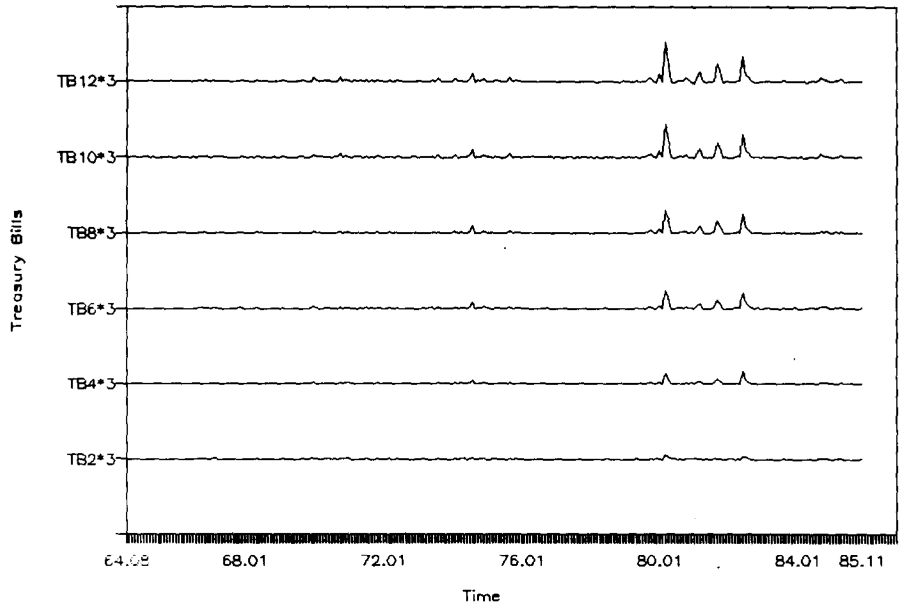

Fig. 3. Cross-product of monthly excess returns.

Table 1[a]

	Mean	Variance	*Skew*	*Kurto*	*Q12*	*QS12*
TB2	0.0307	0.0044	2.4956	16.414	45.765	42.246
TB3	0.0569	0.0144	2.0764	12.538	49.550	59.816
TB4	0.0613	0.0329	1.9510	13.825	33.512	43.268
TB5	0.0758	0.0593	1.5406	11.779	34.273	55.475
TB6	0.0755	0.0886	1.3491	11.040	45.973	64.932
TB7	0.0712	0.1164	0.8580	8.213	29.968	89.530
TB8	0.0908	0.1578	0.8278	8.973	31.078	87.773
TB9	0.0954	0.2201	1.0647	11.172	37.295	62.285
TB10	0.0670	0.2809	0.9447	10.468	31.567	68.361
TB11	0.0722	0.3319	0.8659	10.607	30.876	73.197
TB12	0.0672	0.4060	0.8290	9.819	34.089	72.393
VWS	0.2606	19.536	0.0518	3.867	11.906	21.792
EWB	0.0695	0.1161	1.1440	10.881	35.698	64.994

[a] *Skew* is the coefficient of skewness, *Kurto* is the coefficient of kurtosis, *Q12* and *QS12* are the Ljung–Box statistics for 12-order serial correlation in the levels and squares, respectively.

Table 2
Principal component analysis on the unconditional covariance matrix asset excess returns.

Rank in sizes	Eigenvalues	Trace percentage
1	19.67	92.23
2	1.59	7.45
3	0.03	0.14
4	0.02	0.08
⋮	⋮	⋮
	Eigenvector for the largest eigenvalue	Eigenvector for the 2nd largest eigenvalue
TB2	−0.0020	0.0124
TB3	−0.0044	0.0275
TB4	−0.0085	0.0430
TB5	−0.0121	0.0610
TB6	−0.0151	0.0758
TB7	−0.0173	0.0880
TB8	−0.0218	0.1029
TB9	−0.0262	0.1218
TB10	−0.0281	0.1370
TB11	−0.0276	0.1493
TB12	−0.0377	0.1620
VWS	1.2008	0.0195

corresponding to the largest eigenvalue loads primarily on the stock index and the eigenvector corresponding to the second largest eigenvalue loads primarily on the T-bills. Although there is no immediate relationship between the number of factors suggested by principal component analysis on the unconditional covariance matrix of asset excess returns and the number of 'dynamic' factors in our FACTOR-ARCH model, a 2-factor model seems to be a natural starting point.

In order to implement the FACTOR-ARCH model, it is necessary to identify the portfolio weights for the factor-representing portfolios. While it is in principle possible to estimate these weights and other parameters of the model jointly using a maximum likelihood procedure, this is an approach left for further research. In this paper, factor-representing portfolios with prespecified weights are constructed. Those selected for analysis are (i) a portfolio with equal weights on each of the bills and a zero on the stock index, which is labeled *EWB*, and (ii) a portfolio with zero weights on the bills and all weight on the stock index, which is labeled *VWS*.

We start with a special case of a univariate portfolio representation: the specification given in eq. (8) which is simply the GARCH-M model used by Domowitz and Hakkio (1984), Engle, Lilien, and Robins (1987), and French, Schwert, and Stambaugh (1987). Maximum likelihood estimation using the BHHH algorithm yields the following estimated processes (t-ratios in parentheses):

$$P_{EWBt} = \underset{(1.00)}{0.0164} + \underset{(2.85)}{0.4646} \cdot \theta_{EWBt} + u_{EWBt}, \tag{13}$$

$$\theta_{EWBt} = \underset{(2.05)}{0.0029} + \underset{(5.49)}{0.2756} \cdot u^2_{EWBt-1} + \underset{(19.2)}{0.7340} \cdot \theta_{EWBt-1},$$

$$P_{VWSt} = \underset{(-1.5)}{-3.376} + \underset{(1.58)}{0.1982} \cdot \theta_{VWSt} + u_{VWSt}, \tag{14}$$

$$\theta_{VWSt} = \underset{(1.68)}{1.9348} + \underset{(1.79)}{0.0518} \cdot u^2_{VWSt-1} + \underset{(12.6)}{0.8461} \cdot \theta_{VWSt-1}.$$

Next, we test for causality in variance from *EWB* to *VWS* by testing u^2_{EWBt-1} as an additional variable in the variance equation of *VWS*. The result is negative. The 1-degree-of-freedom LM test statistic takes a value of 0.4 which is insignificant at any reasonable level. We also test for causality in variance from *VWS* to *EWB* by testing u^2_{VWSt-1} as an additional variable in the variance equation of *EWB*. The 1-degree-of-freedom LM test statistic takes a value of 7.2 which is highly significant. These results suggest a one-sided causality in variance from the stock factor to the bill factor. The recursive portfolio representation seems to be more appropriate than the univariate portfolio representation.

The model for *EWP* is therefore re-estimated adding u^2_{VWSt-1} and θ_{VWSt-1} as explanatory variables in the variance equation of *EWB*. Maximum likelihood estimation gives the following results (t-ratios in parentheses):

$$P_{EWBt} = \underset{(0.28)}{0.0046} + \underset{(4.00)}{0.6965} \cdot \theta_{EWBt} + u_{EWBt}, \tag{15}$$

$$\theta_{EWBt} = \underset{(-2.4)}{-0.031} + \underset{(4.17)}{0.2997} \cdot u^2_{EWBt-1} + \underset{(8.98)}{0.5996} \cdot \theta_{EWBt-1}$$

$$+ \underset{(0.97)}{0.0002} \cdot u^2_{VWSt-1} + \underset{(2.37)}{0.0021} \cdot \theta_{VWSt-1}.$$

Eqs. (14) and (15) are used as our generating models for *VWS* and *EWB*, respectively. The GARCH effect is very strong for the excess returns of both *EWB* and *VWS*.

These models are subjected to a substantial battery of diagnostic checks. Tests for time-varying moments, for stability of the parameters across various regimes, for higher-order serial correlation or ARCH are all described below. Overall, the models perform surprisingly well.

The 3-degrees-of-freedom likelihood ratio test statistic for the null hypothesis that P_{VWSt} is generated by a normal model with constant mean and variance is 14.343 which is significant at the 5% level. Similarly the 5-degrees-of-freedom LR test statistics for the null hypothesis that P_{EWBt} is generated by a normal model with constant mean and variance is 148.63 which is highly significant. That is, (14) and (15) do much better than a standard normal model in fitting the data.

To test for parameter instability after the change in operating procedure of the Federal Reserve in 1979/9, we construct a dummy variable which takes a value of 1 after 1979/8 and 0 otherwise. The stability test is a standard LM test for nonzero interaction between the parameters and the dummy variable and is computed as an LM test for omitted variables as in Engle, Lilien, and Robins (1987). The 5-degrees-of-freedom LM test for *VWS* takes a value of 6.2235 which is insignificant at the 5% level. The 7-degrees-of-freedom LM test for *EWB* takes a value of 15.1665 which is only marginally significant at the 5% level. Redefining the dummy to be 1 between September 1979 and October 1982, the test statistics become 6.74 for the *VWS* and 19.01 for *EWB* showing a significant split for bills at the 1% level. When a recession dummy which takes the value of 1 during all the NBER defined recessions, is interacted with all the coefficients the statistics become 22.01 for *VWS* and 15.38 for *EWB*. Thus there is some evidence of instability of the bill equations around a carefully specified Fed operating procedures dummy and of the equity equations around business cycles. However, both of these dummy variables were defined with hindsight and, since they could not have been accurately predicted *ex ante*, may not be weakly exogenous to these markets.

We further test for own u_{t-2}^2 in the variance equation of *EWB* and *VWS*. The 1-degree-of-freedom LM test statistics are 0.2897 and 0.102, respectively. Both are insignificant at any reasonable level. Ljung–Box statistics for 12-order serial correlations are also computed for the levels and squares of the normalized residuals of both series. The statistics for the levels of the normalized residuals of *EWB* and *VWS* are 20.3036 and 11.637, respectively. The statistics for the squares of the normalized residuals of *EWB* and *VWS* are 15.5950 and 6.0837, respectively. They are all insignificant at the 5% level suggesting that there is little unexplained time dependence in the data. Based on the above evidence, (14) and (15) seem to provide a pretty good fit to the data. However, the coefficient of skewness and the coefficient of kurtosis of the normalized residuals [-0.11 and 3.57 for the *VWS* portfolio and -0.35

and 4.49 for the *EWB* portfolio, respectively] indicate slight evidence of misspecification of the conditional distribution, although in Engle and Gonzales (1989) the consequences do not appear to be very severe. See also Weiss (1982, 1984) for theoretical consistency results.

In the balance of the paper, these estimated conditional variances and risk premia for *EWS* and *VWS* are used as predetermined variables in the estimation of the conditional variances and risk premia of the Treasury bills based on (3) and (6). The model we estimate is

$$y_{it} = \Psi_i + \beta_{EWBi} \cdot \Pi_{EWBt} + \beta_{VWSi} \cdot \Pi_{VWSt} + \varepsilon_{it}, \tag{16}$$

$$\varepsilon_{it} \mid \mathscr{F}_{t-1} \sim \mathrm{N}(0, h_{it}),$$

$$h_{it} = \sigma_{ii} + \beta^2_{EWBi} \cdot \theta_{EWBt} + \beta^2_{VWSi} \cdot \theta_{VWSt}.$$

A constant term (Ψ_i) is included in the mean equation to capture the part of the individual asset risk premium that is possibly related to ignored 'static' factors which have time-invariant conditional variances, covariances, and risk premia. (In our time series model, 'static' factors cannot be identified.) The estimation results for bills with 2-, 4-, 6-, 8-, 10-, and 12-months maturities are reported in table 3. The odd maturities appeared in a previous draft of the paper and showed very similar behavior. The asymptotic t-ratios for the β_{VWSi}'s are very small for most of the series suggesting that the data-generating process might be a 1-factor model with *EWB* as the only factor whose volatility is partly driven by the volatility of *VWS*.

The models for individual bills are re-estimated as a 1-factor model as follows:

$$y_{it} = \Psi_i + \beta_{EWBi} \cdot \Pi_{EWBt} + \varepsilon_{it}, \qquad \varepsilon_{it} \mid \mathscr{F}_{t-1} \sim \mathrm{N}(0, h_{it}), \tag{17}$$

$$h_{it} = \sigma_{ii} + \beta^2_{EWBi} \cdot \theta_{EWBt}.$$

The results of maximum likelihood estimation using the BHHH algorithm are reported in table 4. Again, a battery of diagnostic tests is applied to each maturity bill and the results are given at the bottom of the table. These are very encouraging for the model.

The 1-degree-of-freedom LR test statistics for the null hypothesis that the excess return series are generated by normal models with constant means and variances are highly significant for all series as we expect.

Table 3

Two-factor model for individual assets (t-statistics in parentheses).[a]

$$y_{it} = \Psi_i + \beta_{EWBi} \cdot \Pi_{EWBt} + \beta_{VWSi} \cdot \Pi_{VWSt} + \varepsilon_{it}$$

$$h_{it} = \sigma_{ii} + \beta^2_{EWBi} \cdot \theta_{EWBt} + \beta^2_{VWSi} \cdot \theta_{VWSt}$$

	TB2	TB4	TB6	TB8	TB10	TB12
Ψ_i	0.0142	0.0185	0.0017	−0.002	−0.063	−0.053
	(5.468)	(2.955)	(0.152)	(−0.10)	(−2.63)	(−1.99)
σ_{ii}	−0.0002	−0.001	−0.007	−0.015	−0.019	0.0092
	(−0.373)	(−1.68)	(−0.48)	(−0.52)	(−0.37)	(0.514)
β^i_{EWB}	0.1805	0.5270	0.8816	1.1735	1.6140	1.9275
	(15.94)	(27.47)	(21.32)	(19.10)	(18.56)	(21.27)
β_{VWSi}	0.0042	0.0046	0.0184	0.0309	0.0360	−0.010
	(0.950)	(0.326)	(0.741)	(0.981)	(0.778)	(−0.17)
Fed79–	0.6112	0.7920	0.8455	0.9407	0.9172	0.8680
ARCH-M	0.9261	0.6133	0.5530	0.4156	0.8610	0.8021
Static	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Skew	0.6260	0.3343	−0.265	−0.420	−0.590	−0.406
Kurto	4.4988	5.0738	4.5294	4.1815	4.6749	4.2033
Q12	1.0000	0.8459	0.9715	0.8590	0.9765	0.9987
QS12	0.9082	0.7981	0.3134	0.7721	0.7574	0.9823

[a]*Fed79–* $= X^2_4$ point for LM test for structural change after 1979/9,
ARCH-M $= X^2_5$ point for LM test for adding ε^2_{it-j} $(j = 1,4)$ in variance equation and h_{it} in mean equation,
Static $= X^2_2$ point for LR test for H_0: $\beta_{EWBi} = \beta_{VWSi} = 0$,
Skew = coefficient of skewness for normalized residuals,
Kurto = coefficient of kurtosis for normalized residuals,
Q12 $= X^2_{12}$ point for Ljung–Box (12) for normalized residuals,
QS12 $= X^2_{12}$ point for Ljung–Box (12) for squares of normalized residuals.

Parameter instability is again investigated using a standard LM test for nonzero interaction between the parameters and the dummy variables for changes in operating procedures by the Federal Reserve and for recessions. Eighteen tests are computed and three are significant at the 5% level, none at the 1% level. The 79–82 *Fed* dummy is significant for maturities of 4 and 6 months and the *Recession* dummy is significant for the 6-months bills. In other words, after time-varying risk premia and time-varying conditional variances are accounted for by our model, parameter instability doesn't seem to be a big problem. The test for ARCH-M is a test for the importance of own residuals in predicting own variances and risk premia. It is not significant for any of the maturities.

We also tested the restriction that the beta in the variance equation is the square of the beta in the mean equation, as implied by eq. (6), and the risk

Table 4

One-factor model for individual assets (t-statistics in parentheses).[a]

$$y_{it} = \Psi_i + \beta_{EWBi} \cdot \Pi_{EWBt} + \varepsilon_{it}$$

$$h_{it} = \sigma_{ii} + \beta^2_{EWBi} \cdot \theta_{EWBt}$$

	TB2	*TB4*	*TB6*	*TB8*	*TB10*	*TB12*
Ψ_i	0.0139	0.0181	0.0008	−0.005	−0.063	−0.049
	(5.584)	(3.393)	(0.072)	(−0.28)	(−2.86)	(−1.89)
σ_{ii}	0.00005	−0.001	−0.001	−0.0003	−0.004	0.0077
	(0.446)	(−3.16)	(−0.92)	(−0.079)	(−0.84)	(0.854)
β_{EWBi}	0.1802	0.5249	0.8890	1.1657	1.6206	1.9212
	(20.64)	(30.16)	(24.60)	(24.16)	(25.83)	(23.00)
Fed79–	0.7045	0.8166	0.8757	0.8142	0.9045	0.8060
Fed79–82	0.3148	0.9544	0.9625	0.8437	0.8326	0.9126
Recess	0.1735	0.9592	0.7858	0.7206	0.6513	0.4901
ARCH-M	0.8972	0.5061	0.5094	0.1124	0.7719	0.7587
Static	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Skew	0.6121	0.3302	−0.277	−0.418	−0.608	−0.407
Kurto	4.5081	5.0162	4.5353	4.2020	4.6390	4.1926
D12	1.0000	0.8304	0.9587	0.7901	0.9541	0.9986
QS12	0.9357	0.7686	0.2740	0.7015	0.6360	0.9817

[a]*Fed79–* = X_3^2 point for LM test for structural change after 1979/9,
Fed79–82 = X_3^2 point for LM test for structural change from 1979/9 to 1982/10,
Recess = X_3^2 point for LM test for structural change during recessions,
ARCH-M = X_5^2 point for LM test for adding ε^2_{it-j} $(j = 1,4)$ in variance equation and h_{it} in mean equation,
Static = X_1^2 point for LR test for H_0: $\beta_{EWBi} = \beta_{VWSi} = 0$,
Skew = coefficient of skewness for normalized residuals,
Kurto = coefficient of kurtosis for normalized residuals,
Q12 = X_{12}^2 point for Ljung–Box (12) for normalized residuals,
QS12 = X_{12}^2 point for Ljung–Box (12) for squares of normalized residuals.

premium theory embodied in the FACTOR-ARCH-CCAPM model. To test this null hypothesis, we consider the following artificial model:

$$y_{it} = \Psi_i + (\beta_{EWBi} + d_{EWBi}) \cdot \Pi_{EWBt} + \varepsilon_{it}, \tag{18}$$

$$\varepsilon_{it} \mid \mathscr{F}_{t-1} \sim \mathrm{N}(0, h_{it}),$$

$$h_{it} = \sigma_{ii} + \beta^2_{EWBi} \cdot \theta_{EWBt}.$$

Under the null hypothesis that the restriction is valid, $d_{EWBi} = 0 \ \forall i$. The

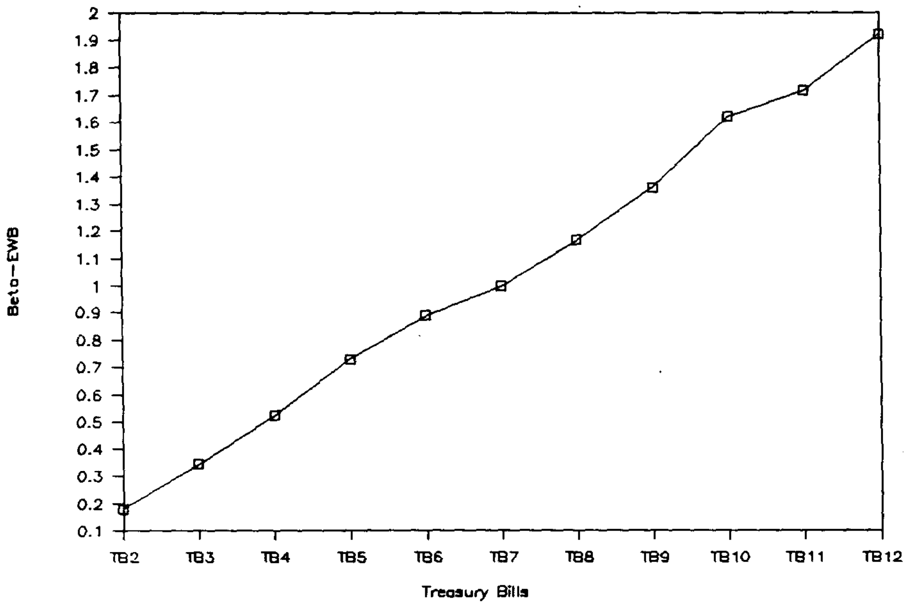

Fig. 4. Factor betas of Treasury bills.

1-degree-of-freedom LM test statistics are:

	TB2	*TB3*	*TB4*	*TB5*	*TB6*	*TB7*	*TB8*	*TB9*	*TB10*	*TB11*	*TB12*
LM1	0.31	1.1	0.11	0.19	0.08	0.25	0.16	0.34	0.56	1.1	2.5

They are insignificant for all excess returns series.

As further specification checks, the Ljung–Box statistics for 12-order serial correlations in the levels and squares of the normalized residuals from our 1-factor model are also reported in table 4. The normalized residuals for the excess returns of the 2-months and 12-months T-bills and the squares of the normalized residuals for the excess returns of the 12-months T-bills show significant autocorrelation. Moreover, the coefficient of skewness and the coefficient of kurtosis for the normalized residuals reported in the same tables do not support the conditional normality assumptions. Although these results indicate possible misspecification in the distributions of excess returns or the number of factors or the weights for the factor-representing portfolios, the bulk of the diagnostic tests give rather strong support to the model.

The plots for the estimates of the β_{EWBi}'s, as reported in table 4 and in Engle, Ng, and Rothschild (1989), are provided in fig. 4. The picture unambiguously reveals that the risk premia and volatilities of T-bills with longer

Table 5

Predicting term premia with Campbell's instruments (t-statistics, computed using White-heteroscadasticity-consistent standard errors, in parentheses).[a]

$$y_{it} = \Psi_i + \sum_{j=1}^{4} a_{ij} \cdot X_{jt} + e_{it}$$

	TB2	TB4	TB6	TB8	TB10	TB12
Ψ_i	0.019	−0.071	−0.124	−0.152	−0.273	−0.213
	(−1.25)	(−1.76)	(−1.84)	(−1.71)	(−2.29)	(−1.45)
a_{i1}	0.0054	0.0153	0.0215	0.0264	0.0346	0.0242
	(2.144)	(2.134)	(1.831)	(1.699)	(1.700)	(0.989)
a_{i2}	0.0165	−0.064	−0.182	−0.231	−0.310	−0.301
	(0.916)	(−1.23)	(−1.85)	(−1.64)	(−1.66)	(−1.28)
a_{i3}	0.0164	0.0809	0.1790	0.0842	0.3263	0.3479
	(1.378)	(2.487)	(2.954)	(2.661)	(2.970)	(2.485)
a_{i4}	−0.006	−0.022	−0.048	0.0405	−0.084	−0.101
	(−0.78)	(−1.16)	(−1.52)	(−1.57)	(−1.46)	(−1.46)

[a]X_1 = 1-month bill rate,
X_2 = 2-months less 1-month bill rate,
X_3 = 6-months less 1-month bill rate,
X_4 = 1-month lag of the 2-months bill excess return.

maturity are more sensitive to changes in the conditional variance of *EWB*. The superiority of the FACTOR-ARCH model over the GARCH-M model suggests that the conditional variances and covariances of the excess returns of the T-bills tend to move together. The predictability of the risk premia of the T-bills, as suggested by Campbell (1987), is also confirmed. Moreover, since the estimates for the models for the excess returns of the factor-representing portfolios, given in (14) and (15), reveal that the variance processes of *VWS* and *EWB* are near-integrated, the T-bills seem to have 'persistent variances and covariances' in the sense that the current information remains important for the forecast of the conditional covariance of the excess returns of the T-bills for all horizons.

Another way to assess the quality of our results is to see how well they explain Campbell's (1987) finding that he could predict the risk premia of interest rates with several yield and yield spread variables. They are: the 1-month bill rate (X_1), the 2-months less 1-month bill rate (X_2), the 6-months less 1-month bill rate (X_3), and the 1-month lag of the 2-months bill excess return (X_4). It would be convincing evidence for our model and for the proposition that risk premia change in response to changes in (conditional) second moments if the instruments which Campbell used to predict risk premia had no predictive power in a model with a FACTOR-ARCH covari-

Table 6

One-factor model for individual assets with Campbell's instrumental variables (t-statistics in parentheses).[a]

$$y_{it} = \Psi_i + \beta_{EWBi} \cdot \Pi_{EWBt} + \sum_{j=1}^{4} a_{ij} \cdot x_{jt} + \varepsilon_{it}$$

$$h_{it} = \sigma_{ii} + \beta^2_{EWBi} \cdot \theta_{EWBt}$$

	TB2	TB4	TB6	TB8	TB10	TB12
Ψ_i	0.0091	0.0098	−0.007	0.0025	−0.030	−0.0389
	(0.991)	(0.535)	(−0.21)	(0.057)	(−0.56)	(0.522)
σ_{ii}	0.00007	−0.0008	−0.001	−0.0017	−0.006	0.0084
	(0.584)	(−1.73)	(−0.74)	(−0.55)	(−1.50)	(0.891)
β_{EWBi}	0.1763	0.5088	0.8592	1.1675	1.5929	1.8650
	(19.09)	(28.91)	(23.99)	(23.70)	(25.65)	(21.89)
a_{i1}	−0.0004	0.0014	−0.006	−0.010	−0.018	−0.034
	(−0.23)	(−0.43)	(−1.21)	(−1.39)	(−1.91)	(−2.83)
a_{i2}	0.0185	0.0160	−0.010	0.0197	0.0372	−0.003
	(1.544)	(0.474)	(−0.18)	(0.261)	(0.356)	(−0.02)
a_{i3}	0.0072	0.0342	0.1032	0.1095	0.1638	0.2367
	(0.842)	(1.710)	(2.981)	(2.205)	(2.388)	(2.731)
a_{i4}	−0.004	−0.016	−0.037	−0.062	−0.090	−0.113
	(−0.85)	(−1.18)	(−1.62)	(−2.02)	(−2.18)	(−2.18)
FACTOR-ARCH	0.9682	0.9092	0.9945	0.9869	0.9956	0.9989

[a]FACTOR-ARCH = X_4^2 point for LR test for H_0: $a_{i1} = a_{i2} = a_{i3} = a_{i4} = 0$,
X_1 = 1-month bill rate,
X_2 = 2-months less 1-month bill rate,
X_3 = 6-months less 1-month bill rate,
X_4 = 1-month lag of the 2-months bill excess return.

ance structure. It would be negative evidence if the FACTOR-ARCH betas were no longer significant.

To study this, we examine the following two models for individual asset excess returns:

$$y_{it} = \Psi_i + \sum_{j=1}^{4} \cdot X_{jt} + e_{it}, \tag{19}$$

$$y_{it} = \Psi_i + \beta_{EWBi} \cdot \Pi_{EWBt} + \sum_{j=1}^{4} a_{ij} \cdot X_{jt} + \varepsilon_{it}, \tag{20}$$

$$h_{it} = \sigma_{ii} + \beta^2_{EWBi} \cdot \theta_{EWBt}.$$

The first model, as described by eq. (19), is a simple linear model with

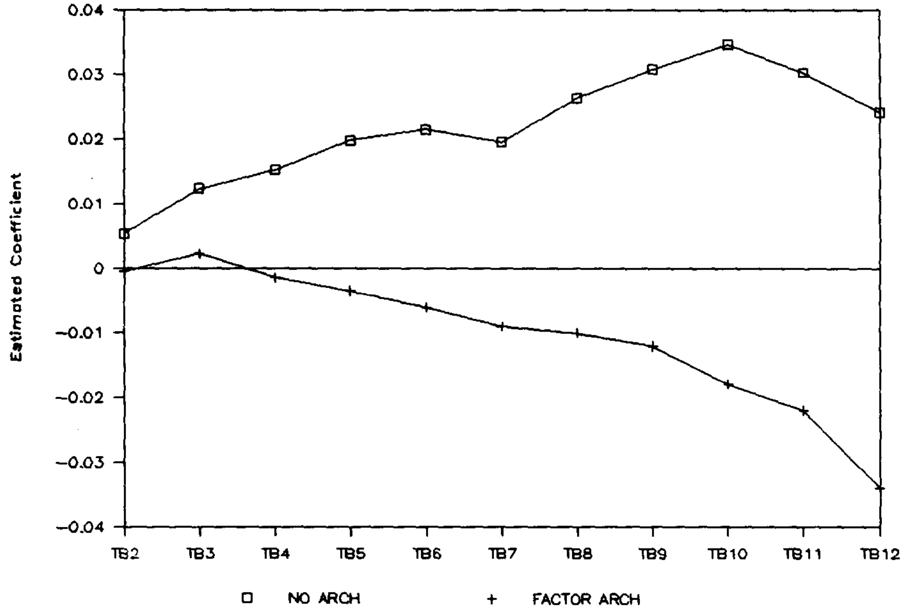

Fig. 5. One-month bill rate.

Campbell's instruments as explanatory variables. The second one, as described by eq. (20), is our FACTOR-ARCH model augmented by Campbell's instruments in the mean equation. The estimation results are reported in tables 5 and 6. The betas are very similar in size and significance to those in table 4. To examine the performance of the instruments, their estimated coefficients for the two models are plotted in figs. 5, 6, 7, and 8. The picture that conforms most to our anticipation is fig. 6. In that figure, the estimated coefficients corresponding to the 2-months less 1-month bill rate under both models are plotted. While the absolute value of the estimated coefficients are larger for bills with longer maturities under the simple linear model, the estimated coefficients are very close to zero and show no pattern of any kind under our augmented FACTOR-ARCH model. Figs. 5 and 7 also indicate that the coefficients corresponding to the 1-month bill rate and the 6-months yield spread are smaller under our FACTOR-ARCH model with only the exception of one coefficient for the 8-months Treasury bill. The worst picture is fig. 8 which is for the lagged 2-months excess return. The estimated coefficients under both models are very close except for the 8-months Treasury bill. Nevertheless, in general the figures do indicate that the predictive power of Campbell's instruments are smaller under our augmented FACTOR-ARCH model. The individual t-statistics for Campbell's instru-

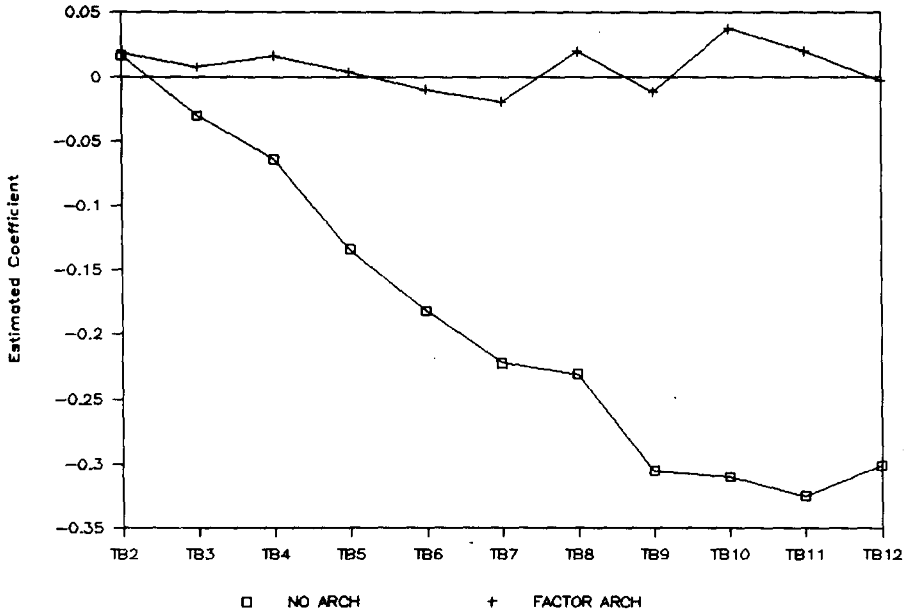

Fig. 6. Two-months less one-month bill rate.

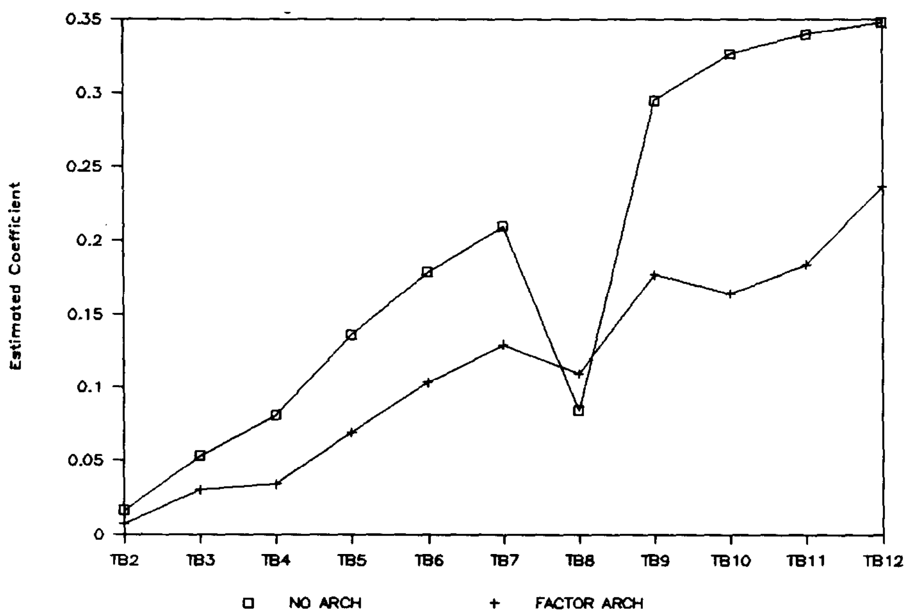

Fig. 7. Six-months less one-month bill rate.

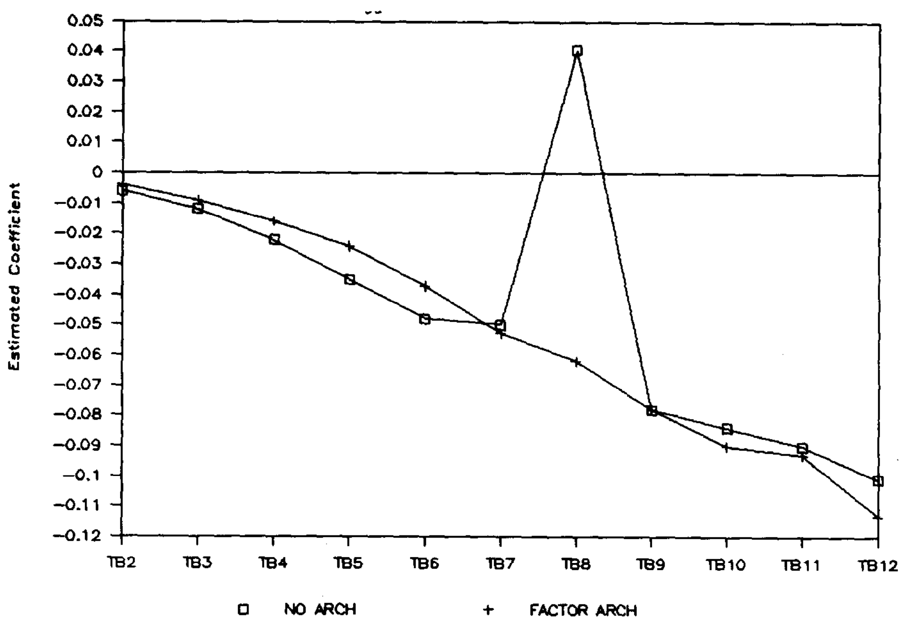

Fig. 8. Laggeed two-months excess return.

ments and the likelihood ratio statistics for the exclusion of all instruments reported in table 6 are, however, a little bothering. The t-statistics for the coefficients corresponding to the 6-months less 1-month bill rate and the lagged 2-months excess return are significant for bills with longer maturities. The highest t-statistic corresponding to the 6-months yield spread is the one for *TB6* and the highest t-statistic corresponding to the 2-months yield spread is the one for *TB2*. Therefore it is quite likely that at least part of the remaining predictive power of Campbell's instruments come from measurement error which is common to both excess return and its matching yield spread variable as in Stambaugh (1988) and McCulloch (1987). The fact that the likelihood ratio statistics for the exclusion of all instruments are significant for all but the 4-months Treasury bill, suggests the possibility of a second bill factor.

5. Conclusion

In this paper, we have proposed a FACTOR-ARCH structure for the conditional covariance matrix of asset excess returns coupled with a consumption beta, CAPM or APT theory of the corresponding time-varying risk premia. Empirical tests of this structure with Treasury bills supports the specification. An equally-weighted bill portfolio is effective in predicting both the volatility and the risk premia of individual maturities. There is little evidence that more accurate forecasts could be made using past information on the individual return histories. The *EWB* portfolio is called therefore a factor-representing portfolio which in this paper is taken to have prespecified weights. The volatility of this portfolio can itself be forecast by the volatility of the equity markets as summarized by the CRSP value-weighted index. Thus a volatility shock in equity markets leads to an increase in the volatility in the bill market and to an increase in the associated risk premia.

The model introduced in this paper can, in principle, be used for many standard exercises in finance including: forecasting the yield curve, valuing derivative assets written on more than one primary asset, selecting portfolios with particular dynamic properties, studying the predictability of asset risk premia, and measuring the persistence of asset return volatilities. However, it seems likely that future work will suggest more elegant and more convincing ways of estimating and assessing this model. Among the important econometric problems which we have left unsolved are devising methods for estimating (rather than imposing) both the number of factors and the weights of the factor-representing portfolios.

Appendix

Let R_t^c be the rate of change of the marginal utility of consumption at time t for a representative agent with time-separable von Neuman–Morgenstern utility. Assuming that the stochastic behavior of R_t^c and y_t (the vector of asset excess returns) are given by the following dynamic factor model:

$$y_t = \boldsymbol{\mu}_t + \sum_{k=1}^{K} \boldsymbol{\beta}_k \cdot f_{kt} + \boldsymbol{v}_t,$$

$$R_t^c = \mu_t^c + \sum_{k=1}^{K} b_k \cdot f_{kt} + u_t^c,$$

where

$$\begin{aligned}
&\boldsymbol{\mu}_t, \mu_t^c \in \mathscr{F}_{t-1}, && \forall t,\\
&\mathrm{E}_{t-1}(f_{kt}) = 0, && \forall k, t,\\
&\mathrm{E}_{t-1}(f_{kt} f_{jt}) = 0, && \forall j \neq k, \forall t,\\
&\mathrm{E}_{t-1}(\boldsymbol{v}_t) = \mathrm{E}_{t-1}(\boldsymbol{v}_{-t} \mid f_{1t}, \dots, f_{Kt}) = 0, && \forall t,\\
&\mathrm{E}_{t-1}(u_t^c) = \mathrm{E}_{t-1}(u_t^c \mid f_{1t}, \dots, f_{Kt}) = 0, && \forall t,\\
&V_{t-1}(f_{kt}) = \lambda_{kt}, && \forall k, t,\\
&\mathrm{E}_{t-1}(\boldsymbol{v}_t \boldsymbol{v}_t') = \Omega, \quad \mathrm{E}_{t-1}(\boldsymbol{v}_t \mid u_t^c) = 0, \quad V_{t-1}(u_t^c) = \sigma_c^2, && \forall t,
\end{aligned}$$

then

$$\begin{bmatrix} \boldsymbol{y}_t \\ R_t^c \end{bmatrix} \Bigg| \mathscr{F}_{t-1} \sim \mathrm{N}\left[\begin{bmatrix} \boldsymbol{\mu}_t \\ \mu_t^c \end{bmatrix}, \begin{bmatrix} \sum_{k=1}^{K} \boldsymbol{\beta}_k \boldsymbol{\beta}_k' \lambda_{kt} + \Omega & \sum_{k=1}^{K} \boldsymbol{\beta}_k b_k \lambda_{kt} \\ \sum_{k=1}^{K} \boldsymbol{\beta}_k' b_k \lambda_{kt} & \sum_{k=1}^{K} b_k^2 \lambda_{kt} + \sigma_c^2 \end{bmatrix} \right].$$

Under the Consumption Beta Model, as in for example Hansen and Singleton (1983), asset risk premia will satisfy the following pricing equation:

$$\boldsymbol{\mu}_t = \delta \cdot \mathrm{cov}_{t-1}(\boldsymbol{y}_t, R_t^c).$$

In our case, this becomes simply

$$\boldsymbol{\mu}_t = \sum_{k=1}^{K} \boldsymbol{\beta}_k \cdot (\delta_t \cdot b_k \lambda_{kt}), \tag{A.1}$$

where δ_t is a preference parameter.

The risk premium of the kth factor-representing portfolio is

$$\Pi_{kt} = \boldsymbol{a}_k' \boldsymbol{\mu}_t = \delta_t \cdot b_k \lambda_{kt}. \tag{A.2}$$

Using (A.2), eq. (A.1) can be rewritten as

$$\boldsymbol{\mu}_t = \sum_{k=1}^{K} \boldsymbol{\beta}_k \cdot \Pi_{kt}, \tag{A.3}$$

which is exactly eq. (6) in section 2.

Since θ_{kt} (the conditional variance of the kth factor-representing portfolio) is related to λ_{kt} by $\theta_{kt} = \lambda_{kt} + s_k$ [section 2, eq. (4)], under the additional assumption that $\delta_t = \delta \ \forall t$, eq. (A.2) can be rewritten as

$$\Pi_{kt} = c_k + \gamma \cdot \theta_{kt}, \tag{A.4}$$

where $c_k \equiv -\delta \cdot b_k s_k$ and $\gamma_k \equiv \delta \cdot b_k$. Eq. (A.4) is exactly eq. (7) in section 2.

The same results can in fact be obtained under still weaker assumptions. See for example Campbell (1987). The assumption that the conditional covariances between v_t, f_{kt}, and u_t^c are zero can be replaced by the assumption that they are time-invariant.

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