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COLLEGE OF ENGINEERING
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Technical Note

ON THE NUMBER OF CLASSES OF (n,k) SWITCHING NETWORKS

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SUMMARY

An (n,k) switching network is defined as an n-input,k-output network such that associated with each output is a Boolean transmission function of the n inputs. If we allow a group $\mathcal{G}$ on the inputs and a group $\mathcal{H}$ on the outputs, then the family of networks is decomposed into equivalence classes. In this paper the number of equivalence classes is derived for the important groups encountered in switching theory.

I. INTRODUCTION

Many writers have considered the problem of classifying Boolean functions under various groups (cf. 1, 4, 10, 14). In this paper we shall show how to extend the approach that this writer has taken in References 4, 5, 6, 7, in order to count the number of classes of sequences of k Boolean functions of n variables.

We shall be initially interested in classifying sequences of k Boolean functions of n variables under some transformation groups on the n variables. One may think of the functions in these sequences as the transmission functions of a switching circuit having k outputs. See Fig. 1 for a diagram of the generic network which we shall call an (n,k) network. Our (n,k) networks realize the (k,n) sequences of Povarov[12] as their behavior.

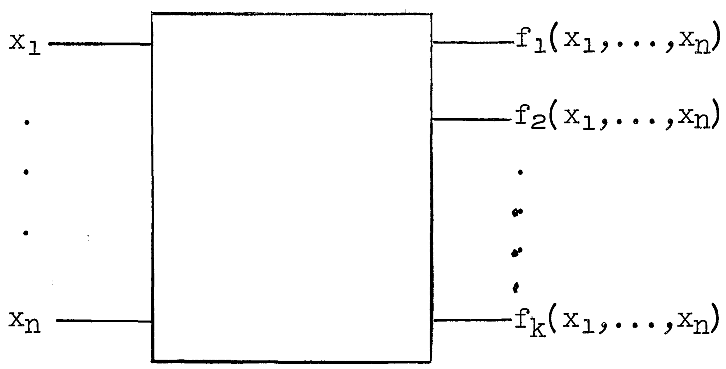

Fig. 1.

One may take either of two points of view in this investigation. Networks can be characterized by (1) their structure, i.e., the placement of certain components in arbitrary arrays or (2) by their behavior, i.e., the terminal relations. The former problem has been handled very neatly by Ninomiya[11] for the group of complementations and permutations. We shall not extend his results here, but we shall take the behavioral point of view. Some related results of a special nature have been obtained by Sagalovitch[13] who obtained the number of one input k output networks whose transmission functions are non-trivial Boolean functions of n variables. In this paper, the "behavioral" aspect of the problem is completely solved for all the groups commonly studied in switching theory.

Dramatis Personae

1. $\mathcal{C}_2^n$ is the group of all 2^n complementations of variables. This group was first studied as a group on Boolean functions by Ashenhurst.[1]
2. $\mathcal{S}_n$ denotes the symmetric group on the n variables. The order of $\mathcal{S}_n$ is n!; this group has been studied in Reference 4.
3. $\mathcal{G}_n$ is the smallest group containing $\mathcal{C}_2^n$ and $\mathcal{S}_n$. It has been studied by a great many people; References 1, 4, 10, 14 are a small subset of the entire class of papers.
4. $GL_n(Z_2)$ is the general linear group on the variables; the group has been studied by Slepian[14] and Harrison.[7]
5. $\mathcal{A}_n(Z_2)$ denotes the least group containing $\mathcal{C}_2^n$ and $GL_n(Z_2)$; this group is the affine group on the variables and has been studied by Nechiporuk[9] and Harrison.[7]

Before proceeding to our method, we shall briefly review the famous theorem of Pólya which will be used. We shall use a form of the theorem adopted from De Bruijn.[2]

Let F be the class of all functions from a finite set D to a finite set R. Suppose D has s elements and that $\mathfrak{G}$ is a permutation group of degree s and order g acting on D. Two functions $f_1, f_2 \in F$ are called equivalent if there exists a permutation $\alpha \in \mathfrak{G}$ such that $f_1(d) = f_2(\alpha(d))$ for all $d \in D$($\alpha(d)$ denotes the image of d under the permutation α). Consider R to be represented as the union of r disjoint subsets, i.e., $R = \bigcup_{i=1}^{r} R_i$ and $R_i \cap R_j = \phi$ if $i \neq j$. Let $k_1,\ldots,k_r$ be a partition of s. Pólya's theorem tells us the number of equivalence classes of functions from D to R such that for k_i values of $d \in D$, the image $f(d) \in R_i$ for $i = 1,\ldots,r$.

To every set R_i, we attach an indeterminate x_i and define ψ_i to be the number of elements in R_i for $i = 1,\ldots,r$. The figure counting series is defined as

$$\psi(x_1,\ldots,x_r) = \sum_{i=1}^{r} \psi_i x_i$$

Usually the convention is adopted of taking $x_1 = 1$. Let $P(x_1,\ldots,x_r)$ be the multi-variate generating function of the numbers that we are seeking, that is, the coefficient of $x_1 \ldots x_r$ is the number of classes of functions with the property that for k_i values of $d \in D$, $f(d) \in R_i$ where $i = 1,\ldots,r$. $P(x_1,\ldots,x_r)$ is often called the configuration counting series.

Before stating Pólya's theorem, we must develop the concept of the cycle index polynomial of $\mathfrak{G}$ (Zyklenzeiger), denoted by $Z_{\mathfrak{G}}$. Let $f_1,\ldots,f_s$ be s indeterminates, and let $g_{j_1,j_2,\ldots,j_s}$ be the number of permutations of $\mathfrak{G}$ having j_i cycles of length i for $i = 1,2,\ldots,s$, so that

$$\sum_{i=1}^{s} i j_i = s \tag{1}$$

Then we define

$$Z_{\mathfrak{G}} = \frac{1}{g} \sum_{(j)} g_{j_1,j_2,\ldots,j_s} f_1^{j_1} f_2^{j_2} \ldots f_s^{j_s}$$

where the sum is taken over all partitions of s which satisfy (1).

Now we can finally state Pólya's theorem which reduces the problem of determining the number of equivalence classes to the determination of the figure counting series and the cycle index polynomial.

Theorem 2.1 (Pólya). The configuration counting series is obtained by substituting the figure counting series into the cycle index polynomial of $\mathfrak{G}$. Symbolically

$$P(x_1,\ldots,x_r) = Z_{\mathfrak{G}}(\psi(x_1,\ldots,x_r), \psi(x_1^2,\ldots,x_r^2),\ldots,\psi(x_1^s,\ldots,x_r^s))$$

In our applications to single Boolean functions, Pólya's theorem takes an even simpler form. Since Boolean functions are mappings from $\{0,1\}^n$ into $\{0,1\}$, we find that $D = \{0,1\}^n$, $s = 2^n$, and $R = \{0,1\}$ with $r = 2$. Taking $R_1 = \{0\}$ and $R_2 = \{1\}$ and associating the indeterminate 1 with R_1 and indeterminate x with R_2, the figure counting series becomes

$$\psi(x) = 1 + x$$

Before proceeding to the theoretical results, we give an example of two networks to be considered equivalent under $\mathcal{S}^{\mu}_3$. Using the notation of Reference 4, let $\sigma = (2,3) \in \mathcal{S}^{\mu}_3$ be applied to the inputs of the top network of Fig. 2 to give the bottom network of the figure.

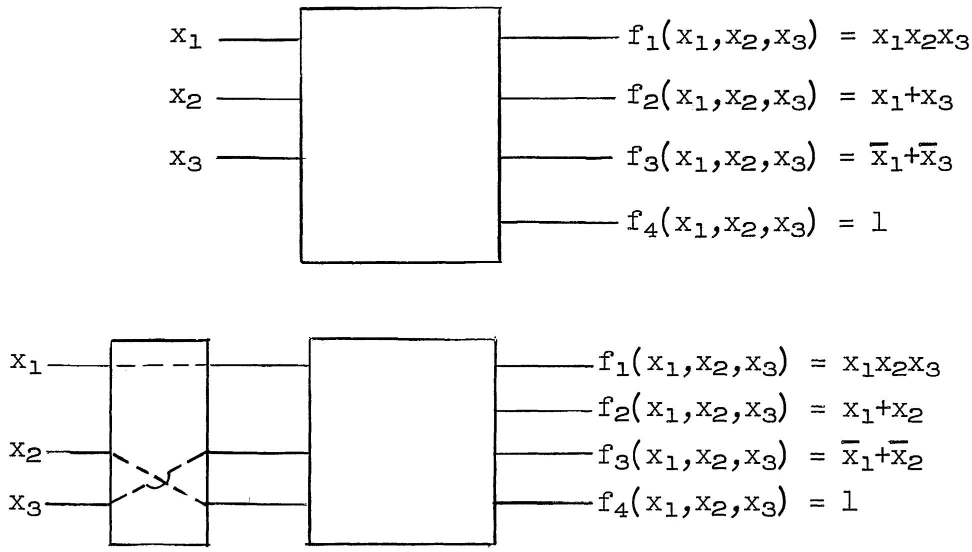

Fig. 2.

II. THE TOTAL NUMBER OF CLASSES

Our problem is now to count the number of classes of (n,k) networks under any group $\mathcal{G}$ acting on the domain of the functions, $\{0,1\}^n$. The total number of such networks is 2^{k2^n} since there are 2^{2^n} choices for each one of the k outputs.

In order to modify Pólya's formula for counting the number of classes of networks, we note that no change has been made in the way that $\mathcal{G}$ acts on $\{0,1\}^n$. The only change is in the range. We are now looking at mappings from $\{0,1\}^n$ into $\{0,1\}^k$ so that we need a new figure counting series. Obviously

$$\psi(x_1,\ldots,x_{2^k}) = \sum_{i=1}^{2^k} x_i$$

Usually x_i is chosen to be one, but this is of no consequence.

Theorem 1. The number of equivalence classes of (n,k) networks (sequences of k Boolean functions of n variables) under a group $\mathcal{G}$ is given by

$$Z_{\mathcal{G}}(2^k,\ldots,2^k)$$

Proof. To count the total number of classes, one takes $x_i = 1$ for $i = 1,\ldots,2^k$ in the figure counting series. Thus f_i is replaced by

$$\sum_{j=1}^{2^k} 1^i = 2^k \qquad \text{for} \qquad i = 1,\ldots,2^k$$

Corollary 2. The total number of classes of (n,k) networks under $\mathcal{C}_2^n$ is

$$\frac{1}{2^n}\left(2^{k2^n} + (2^n - 1)2^{k2^{n-1}}\right)$$

Proof. This follows from the fact that

$$Z_{\mathcal{C}_2^n} = \frac{1}{2^n}\left(f_1^{2^n} + (2^n - 1)f_2^{2^{n-1}}\right)$$

Cf. References 1 and 4.

It is interesting to note that the class of all mappings from $\{0,1\}^n$ into $\{0,1\}^k$ forms a Boolean algebra of 2^{k2^n} functions in the natural way. It appears that the generalization of many problems in switching theory to the multiple output case can be handled naturally from this point of view. This observation has already been verified for the multiple output minimization problem.

The calculations have been carried out for the five groups under discussion and the results are given below in Tables 1 through 5. Certain facts may be deduced from the tables. For example, the number of classes under $\mathcal{S}_1$ and $GL_1(Z_2)$ is 4^k since these groups consist of the identity alone.

III. SYMMETRIES IN THE RANGE

While the results of the previous section are of some interest, certain networks are considered non-equivalent which differ only in the order of the outputs. We wish to enlarge our definition of equivalence of networks by allowing permutations in the range. More precisely, let the symmetric group on k letters, $\mathcal{S}_k$, act on the range, and consider two networks equivalent if and only if there is an $\alpha \in \mathcal{G}$ and a permutation $\sigma \in \mathcal{S}_k$ such that $(f_1(d),\ldots,f_k(d)) = (g_{\sigma(1)}(\alpha d),\ldots,g_{\sigma(k)}(\alpha d))$ for every $d \in \{0,1\}^n$. This instance is a special case of the following theorem of De Bruijn.[2]

Theorem 3. (De Bruijn). *If* $\mathcal{G}$ *is a group on the domain D of a family of functions, and* $\mathcal{H}$ *is a group on the range R of the function* ($\bar{\bar{D}} = s$, *and* $\bar{\bar{R}} = r$), *then the number of equivalence classes of functions is given by*

$$Z_{\mathcal{G}}\left(\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_s}\right) Z_{\mathcal{H}}(h_1,\ldots,h_r)$$

evaluated at $z_1 = z_s = \ldots = z_s = 0$ *where*

$$h_t = \exp\left\{ t \sum_{k=1}^{\infty} z_{kt} \right\} \qquad \underline{\text{for}} \qquad t = 1,\ldots,r$$

In order to carry out calculations with this theorem, the following lemma was proved in Reference 5.

TABLE 1

THE NUMBER OF CLASSES UNDER Γ_2^n

n	k=1	k=2	k=3	k=4
1	3	10	36	136
2	7	76	1,072	16,576
3	46	8,416	2,100,736	536,928,256
4	4,336	268,496,896	17,592,201,773,056	1,152,921,508,633,378,816

TABLE 2

THE NUMBER OF CLASSES UNDER γ_n^{μ}

n	k=1	k=2	k=3	k=4
1	4	16	64	256
2	12	160	2,304	34,816
3	80	13,056	2,928,640	724,238,336
4	3,984	1,833,052,160	11,745,443,774,464	768,684,844,023,545,856

TABLE 3

THE NUMBER OF CLASSES UNDER $\mathcal{J}_n$

n	k=1	k=2	k=3	k=4
1	3	10	36	136
2	6	55	666	9,316
3	22	1,996	384,112	91,604,416
4	402	11,756,666	735,192,450,952	48,047,227,408,513,056

TABLE 4

THE NUMBER OF CLASSES UNDER $GL_n(Z_2)$

n	k=1	k=2	k=3	k=4
1	4	16	64	256
2	8	80	960	13,056
3	20	1,056	135,040	27,700,736
4	92	320,416	14,333,211,520	916,495,047,958,016

TABLE 5

THE NUMBER OF CLASSES UNDER $\mathcal{O}_n(Z_2)$

n	k=1	k=2	k=3	k=4
1	3	10	36	136
2	5	35	330	3,876
3	32	271	22,060	3,741,616
4	382	29,821	920,737,780	57,374,820,122,576

Lemma 4. <u>A term $h_1^{j_1} \dots h_r^{j_r}$ in $Z_{\mathcal{H}}$ gives rise to</u>

$$Z_{\mathcal{G}}\left(\sum_{t|1} t j_t, \dots, \sum_{t|s} t j_t\right)$$

The only information required before our calculations can begin is the determination of the cycle index of $\mathcal{S}_k$. Since there are 2^k elements of $\{0,1\}^k$, we need a representation of $\mathcal{S}_k$ of degree 2^k. Fortunately, such a representation was constructed in Reference 4. The result is

$$Z_{\mathcal{S}_k} = \frac{1}{k!} \sum_{(j)} \frac{k!}{\prod_{i=1}^{k} j_i!\, i^{j_i}} \mathop{\Large\times}_{i=1}^{k} \left(\prod_{d|i} f_d^{e(d)}\right)^{\times j_i}$$

where the sum is over all solutions of

$$\sum_{i=1}^{k} i j_i = k$$

and $e(d) = \frac{1}{d} \sum_{t|d} 2^t \mu(\frac{d}{t})$. The definition of the cross operation ($\times$) is given in Reference 4, and $\mu(a)$ denotes the Möbius function.

For the sake of reference, the results of De Bruijn's theorem are worked out below for $1 \leq k \leq 4$.

Theorem 5. <u>The number of classes of (n,k) networks with a group $\mathcal{G}$ on the domain and the symmetric group on the range is given below for $1 \leq k \leq 4$.</u>

$k = 1$ $\qquad Z_{\mathcal{G}}(\overline{2}, \dots)$

$k = 2$ $\qquad \frac{1}{2}\left(Z_{\mathcal{G}}(\overline{4}, \dots) + Z_{\mathcal{G}}(\overline{2,4}, \dots)\right)$

$k = 3$ $\qquad \frac{1}{6}\left(Z_{\mathcal{G}}(\overline{8}, \dots) + 3Z_{\mathcal{G}}(\overline{4,8}, \dots) + 2Z_{\mathcal{G}}(\overline{2,2,8}, \dots)\right)$

$k = 4$ $\qquad \frac{1}{24}\left(Z_{\mathcal{G}}(\overline{16}, \dots) + 6Z_{\mathcal{G}}(\overline{8,16}, \dots) + 3Z_{\mathcal{G}}(\overline{4,16}, \dots)\right.$
$\qquad\qquad \left. + 8Z_{\mathcal{G}}(\overline{4,4,16}, \dots) + 6Z_{\mathcal{G}}(\overline{2,4,2,16}, \dots)\right)$

where the notation $Z_{\mathcal{G}}(\overline{f_1,\ldots,f_m},\ldots)$ *means* $f_p = f_q$ *if and only if* $p \equiv q$ *mod m.* The calculations have been carried out and are given in Tables 6 through 10. The number of classes with (say) Σ_2^n on the domain and γ_k^μ on the range is denoted by $T(\Sigma_2^n, \gamma_k^\mu)$.

IV. COMPLEMENTATION IN THE RANGE

Suppose we allow ourselves to complement the k outputs of our networks and enlarge our definition of equivalence by considering two networks to be equivalent if there is an $\alpha \in \mathcal{G}$ and an $i = (i_1,\ldots,i_k) \in \Sigma_2^k$, such that $(f_1(d),\ldots,f_k(d)) = (f_1^{i_1}(\alpha(d)),\ldots,f_k^{i_k}(\alpha(d))$ for all $d \in \{0,1\}^n$. The notation $f_j^{i_j}(d)$ is defined as

$$f_j^{i_j}(d) = \begin{cases} f_j(d) \text{ if } i_j = 0 \\ \overline{f}_j(d) \text{ if } i_j = 1 \end{cases} \qquad \text{for } j = 1,\ldots,k$$

The number of such classes can again be determined from De Bruijn's theorem.

Theorem 6. *The number of classes of (n,k) networks with a group* $\mathcal{G}$ *on the domain and the complementing group* Σ_2^k *on the range is*

$$\frac{1}{2^k}\left(Z_{\mathcal{G}}(2^k,\ldots,2^k) + (2^k - 1)Z_{\mathcal{G}}(0,2^k,\ldots,0,2^k)\right)$$

Proof. The cycle index of Σ_2^k is known[1] to be

$$Z_{\Sigma_2^k} = \frac{1}{2^k}\left(f_1^{2^k} + (2^k - 1)f_2^{2^{k-1}}\right)$$

The result follows from Theorem 3.

TABLE 6

$T(\Gamma_2^n, \gamma^\mu_k)$

n	k=1	k=2	k=3	k=4
1	3	7	13	22
2	7	46	237	1,056
3	46	4,336	356,026	22,921,696
4	4,336	134,281,216	2,932,175,712,336	48,042,795,872,587,776

TABLE 7

$T(\gamma^{\mu}_{n}, \gamma^{\mu}_{k})$

n	k=1	k=2	k=3	k=4
1	4	10	20	35
2	12	88	484	2,180
3	80	6,616	497,760	31,017,356
4	3,984	916,539,904	1,957,701,217,238	32,031,538,353,966,080

TABLE 8

$T(\mathcal{O}_n, \gamma^{\mu}_k)$

n	k=1	k=2	k=3	k=4
1	3	7	13	22
2	6	32	158	652
3	22	1,056	66,336	3,945,992
4	402	5,884,954	122,543,247,874	2,002,161,498,159,934

TABLE 9

$T(GL_n(Z_2), \gamma^{\mu}_k)$

n	k=1	k=2	k=3	k=4
1	4	10	20	35
2	8	46	220	897
3	20	556	23,932	1,214,454
4	92	160,932	2,390,063,996	38,192,408,400,654

TABLE 10

$T(\mathfrak{A}_n(Z_2), \gamma^{\mu}_k)$

n	k=1	k=2	k=3	k=4
1	3	7	13	22
2	5	22	87	314
3	10	153	4,148	169,495
4	32	15,209	153,668,757	2,391,065,770,697

The special case when k = 1 has been previously investigated and is the topic of References 3, 5, and 10. The calculations are shown below in Tables 11 through 15, where $T(\mathcal{G}, \mathcal{C}_2^k)$ denotes the number of equivalence classes of (n,k) networks with a group $\mathcal{G}$ on the domain and $\mathcal{C}_2^k$ on the range.

It is somewhat artificial to consider complementation of the outputs only. One would prefer to consider both symmetries and complementations on the range. This suggests considering the least group containing $\mathcal{S}_k$ and $\mathcal{C}_2^k$. This group, $\mathcal{G}_k$, is very well known and its cycle index is known[4] to be

$$Z_{\mathcal{G}_k} = \frac{1}{k!2^k} \sum_{(j)} \frac{k!2^k}{\prod_{i=1}^{k} j_i!(2i)^{j_i}} \prod_{i=1}^{k} \left(\prod_{d|i} f_d^{e(d)} + \prod_{\substack{d \nmid i \\ d|2i}} f_d^{g(d)} \right)^{j_i}$$

where the sum is over all partitions of k, i.e., over all non-negative integer solutions $\sum_{i=1}^{k} i j_i = k$. $e(k) = \frac{1}{k} \sum_{d|k} 2^d \mu(\frac{k}{d})$ and

$$g(2k) = \frac{1}{2k} \sum_{\substack{d \nmid k \\ d|2k}} 2^{d/2} \mu(\frac{2k}{d})$$

where $\mu(a)$ is the Möbius function. Using Lemma 4, the polynomials to be evaluated are written below.

Theorem 7. <u>The number of classes of (n,k) networks with a group</u> $\mathcal{G}$ <u>on the domain and the group</u> $\mathcal{G}_k$ <u>of complementations and permutations on the range is given below for</u> $1 \leq k \leq 4$.

TABLE 11

$T(\lfloor^{n}_{2}, \lfloor^{k}_{2})$

n	k=1	k=2	k=3	k=4
1	2	4	8	16
2	5	22	176	1,216
3	30	2,128	265,728	33,611,776
4	2,288	67,127,296	2,199,025,221,832	72,057,598,064,459,776

TABLE 12

$T(\gamma^{\mu}_{n}, [^{k}_{2}])$

n	k=1	k=2	k=3	k=4
1	2	4	8	16
2	6	40	288	2,176
3	40	3,264	366,080	45,264,896
4	1,992	458,263,040	1,468,180,471,808	48,042,802,751,471,616

TABLE 13

$T(\mathcal{J}_n, [{k \atop 2})$

n	k=1	k=2	k=3	k=4
1	2	4	8	16
2	4	19	106	676
3	14	556	49,008	5,742,016
4	222	2,945,738	91,901,007,752	2,877,952,247,834,656

TABLE 14

$T\left(GL_n(Z_2), \left[{k \atop 2}\right)\right)$

n	k=1	k=2	k=3	k=4
1	2	4	8	16
2	4	20	120	816
3	10	264	16,880	1,731,296
4	46	80,104	1,791,651,440	51,030,940,434,876

TABLE 15

$$T\left(\mathcal{M}_n(Z_2), \left[{k \atop 2}\right)\right)$$

n	k=1	k=2	k=3	k=4
1	2	4	8	16
2	3	11	50	276
3	6	79	2,908	236,176
4	18	7,621	115,125,476	3,585,934,560,176

$k = 1 \qquad \frac{1}{2}\left(Z_{\mathcal{G}}(\overline{2},\ldots) + Z_{\mathcal{G}}(\overline{0,2},\ldots)\right)$

$k = 2 \qquad \frac{1}{8}\left(Z_{\mathcal{G}}(\overline{4},\ldots) + 3Z_{\mathcal{G}}(\overline{0,4},\ldots) + 2Z_{\mathcal{G}}(\overline{2,4},\ldots)\right.$
$\left. + 2Z_{\mathcal{G}}(\overline{0,0,0,4},\ldots)\right)$

$k = 3 \qquad \frac{1}{48}\left(Z_{\mathcal{G}}(\overline{8},\ldots) + 13Z_{\mathcal{G}}(\overline{0,8},\ldots) + 8Z_{\mathcal{G}}(\overline{2,2,8},\ldots)\right.$
$\left. + 8Z_{\mathcal{G}}(\overline{0,2,0,2,0,8},\ldots) + 6Z_{\mathcal{G}}(\overline{4,8},\ldots) + 12Z_{\mathcal{G}}(\overline{0,0,0,8},\ldots)\right)$

$k = 4 \qquad \frac{1}{384}\left(Z_{\mathcal{G}}(\overline{16},\ldots) + 51Z_{\mathcal{G}}(\overline{0,16},\ldots) + 48Z_{\mathcal{G}}(\overline{2,4,2,16},\ldots)\right.$
$+ 48Z_{\mathcal{G}}(\overline{0,0,0,0,0,0,0,16},\ldots) + 12Z_{\mathcal{G}}(\overline{8,16},\ldots)$
$+ 84Z_{\mathcal{G}}(\overline{0,0,0,16},\ldots) + 12Z_{\mathcal{G}}(\overline{4,16},\ldots)$
$\left. + 32Z_{\mathcal{G}}(\overline{4,4,16},\ldots) + 96Z_{\mathcal{G}}(\overline{0,4,0,4,0,16},\ldots)\right)$

Using this result, the number of classes has been computed and is tabulated below in Tables 16 through 20.

V. LINEAR GROUPS IN THE RANGE

Theorem 3 and Lemma 4 allow us to consider any group $\mathcal{G}$ on the domain and any group $\mathcal{H}$ on the range. We now complete our analysis of all remaining cases by allowing $GL_n(Z_2)$ and $\mathcal{O}_n(Z_2)$ on the range. Because of the length and complexity of the cycle indices of these groups, we shall not exhibit the explicit formulae to be used in the calculations; instead the reader is referred to Reference 7.

The results are given below in Tables 21 through 30.

TABLE 16

$T(\mathcal{L}_2^n, \mathcal{J}_k)$

n	k=1	k=2	k=3	k=4
1	2	3	4	5
2	5	18	51	122
3	30	1,200	45,777	1,476,032
4	2,288	33,601,536	366,543,984,720	3,002,400,587,673,600

TABLE 17

$T(\mathcal{U}_n, \mathcal{O}_k)$

n	k=1	k=2	k=3	k=4
1	2	3	4	5
2	6	24	98	194
3	40	1,676	63,440	1,992,430
4	1,992	235,384,592	244,644,311,176	2,002,158,848,764,301

TABLE 18

$T(\mathcal{T}_n, \mathcal{T}_k)$

n	k=1	k=2	k=3	k=4
1	2	3	4	5
2	4	13	34	640
3	14	308	8,906	257,658
4	222	1,476,218	15,320,103,918	125,147,156,711,032

TABLE 19

$T(GL_n(Z_2), \mathcal{T}_k)$

n	k=1	k=2	k=3	k=4
1	2	3	4	5
2	4	13	36	95
3	10	146	3,178	80,036
4	46	40,422	298,842,656	2,387,346,322,704

TABLE 20

$T(\mathcal{M}_n(Z_2), \mathcal{J}_k)$

n	k=1	k=2	k=3	k=4
1	2	3	4	5
2	3	8	19	41
3	6	49	632	11,917
4	18	3,963	19,245,637	149,471,180,139

TABLE 21

$T(\Gamma_2^n, GL_k(Z_2))$

n	k=1	k=2	k=3	k=4
1	3	4	4	4
2	7	21	27	28
3	46	1,531	14,056	39,839
4	4,336	44,782,251	104,751,025,086	57,280,291,273,345

TABLE 22

$T(\mathcal{H}_n, GL_k(Z_2))$

n	k=1	k=2	k=3	k=4
1	4	5	5	5
2	12	35	46	47
3	80	2,266	19,930	55,767
4	3,984	305,552,546	69,945,183,326	38,191,772,055,973

TABLE 23

$T(\mathfrak{A}_n, GL_k(Z_2))$

n	k=1	k=2	k=3	k=4
1	3	4	4	4
2	6	16	21	22
3	22	392	2,920	7,923
4	402	1,966,074	4,379,140,552	2,387,316,975,717

TABLE 24

$T(GL_n(Z_2), GL_k(Z_2))$

n	k=1	k=2	k=3	k=4
1	4	5	5	5
2	8	20	27	28
3	20	206	1,204	3,071
4	92	54,155	85,552,416	45,568,388,658

TABLE 25

$T(\mathfrak{A}_n(Z_2), GL_k(Z_2))$

n	k=1	k=2	k=3	k=4
1	3	4	4	4
2	5	11	15	16
3	10	64	276	653
4	32	5,276	5,534,421	178,471,391

TABLE 26

$T\left(\Gamma_2^n, \sigma_k(Z_2)\right)$

n	k=1	k=2	k=3	k=4
1	2	2	2	2
2	5	9	10	10
3	30	443	2,104	3,763
4	2,288	11,211,435	13,098,898,366	3,585,768,962,689

TABLE 27

$T(\gamma^{\mu}_n, \mathcal{M}_k(Z_2))$

n	k=1	k=2	k=3	k=4
1	2	2	2	2
2	6	11	12	12
3	40	590	2,828	5,066
4	1,992	76,384,114	8,747,130,342	2,390,901,609,645

TABLE 28

$T(\mathfrak{G}_n, \mathfrak{A}_k(Z_2))$

n	k=1	k=2	k=3	k=4
1	2	2	2	2
2	4	7	8	8
3	14	124	504	880
4	222	494,298	547,849,868	539,100,216

TABLE 29

$T(GL_n(Z_2), \mathcal{O}\!\!\mathcal{l}_k(Z_2))$

n	k=1	k=2	k=3	k=4
1	2	2	2	2
2	4	7	8	8
3	10	60	214	368
4	46	13,733	10,724,116	2,854,852,383

TABLE 30

$T(\mathfrak{A}_n(Z_2), \mathfrak{A}_k(Z_2))$

n	k=1	k=2	k=3	k=4
1	2	2	2	2
2	3	5	6	6
3	6	24	70	116
4	18	1,430	700,173	179,125,249

VI. MISCELLANEOUS RESULTS AND COMMENTS

One might be interested in comparing our results with some related results on invertible functions. In both cases we consider n-input,n-output networks with groups on the domain and the range, but in References 6 and 8 the functions are required to have inverses. Comparing the results of these references against $T(\mathcal{G}, \mathcal{H})$ for (n,n) networks, one finds that the ratio of the number of classes of invertible functions to $T(\mathcal{G}, \mathcal{H})$ approaches zero for large n. Thus the invertible classes are comparatively rare.

One degenerate group has not been discussed, namely the identity group on the n variables, $\mathcal{I}_n$. $\mathcal{I}_n$ has order 1, degree 2^n, and its cycle index is $f_1^{2^n}$. Our results with $\mathcal{I}_k$ on the range reduce to $Z_{\mathcal{G}}(2^k,\ldots,2^k)$ as one would expect. Separate calculations are required if $\mathcal{I}_n$ is applied to the domain and an arbitrary group $\mathcal{H}$ is used on the range. These calculations will not be performed here because of the limited interest in the results and the size of the numbers.

It is sometimes of interest to have lower bounds on the number of classes. The following theorem gives the desired bounds.

Theorem 8. <u>A lower bound on the number of classes of (n,k) networks with a group $\mathcal{G}$ of order g on the inputs and a group $\mathcal{H}$ of order h on the outputs is given by</u>

$$\frac{1}{h} Z_{\mathcal{G}}(2^k,\ldots,2^k) \geq \frac{1}{gh} 2^{k2^n}$$

Proof. The argument simply takes the largest terms in the polynomials given in Theorem 3. Note that for k = 1, and $\mathcal{H} = \mathcal{I}_1$, the bound reduces to $\frac{1}{g} 2^{2^n}$ which is a well known[7,8] lower bound for the number of classes of Boolean functions with a group of order g on the domain.

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