Optimal Fixed-Point Algorithms for Service Differentiation in Wireless Networks

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Abstract—**We study network utility maximization problems in wireless networks for service differentiation that optimize the Signal-to-Interference-plus-Noise Radio (SINR) and reliability under Rayleigh fading. Though seemingly nonconvex, we show that these problems can be solved using an optimization decomposition where each user calculates a payment for a given resource allocation, and the network uses the payment to optimize the performance of the user. We study three important examples of this utility maximization, namely the weighted sum logarithmic SINR maximization, the weighted sum inverse SINR minimization and the weighted sum logarithmic reliability maximization. These problems have hitherto been solved suboptimally in the literature. By exploiting the positivity, quasi-concavity and homogeneity properties in these problems using the nonlinear Perron-Frobenius theory, we propose fixed-point algorithms that converge geometrically fast to the globally optimal solution. Numerical evaluations show that our algorithms are stable (free of parameter configuration) and computationally fast.**

Index Terms—**Optimization, network utility maximization, resource allocation, nonlinear Perron-Frobenius theory.**

I. Introduction

Utility maximization in wireless networks is more complicated than its wired counterpart because of factors such as time-varying channel fading conditions, multiuser interference and adaptive resource allocations in the physical layer. It is also desired that algorithms for solving wireless resource allocation be simplified, for example with little parameter tunings or that base stations are oblivious to individual channel conditions or user utility functions. Utility maximization thus requires a joint optimization of resource allocation and interference management using algorithms with good properties. However, important and commonly-used performance metrics in the wireless utility maximization problems are often nonconvex and highly nonlinear, and thus are generally hard to solve. It is also challenging to design simple network algorithms that can solve them optimally.

Network utility maximization has its roots in the seminal work [1] by Kelly who established an optimization framework to decompose the original problem into separable smaller subproblems that are separately solved by individual users and a network controller. A novelty in this optimization framework is the use of proportional fairness as an intermediate mechanism for resource allocation, particularly to design rate control algorithms in [1]. The authors in [2], [3] used dual decomposition theory to tackle this utility maximization problem in wireless networks. The authors in [4] studied utility that is the sum of inverse SINR or nonlinear interference functions. To avoid the nonconvexity, the authors in [5] proposed a game-theoretic pricing mechanism to maximize a utility that is a weighted sum logarithmic reliability. The authors in [6] proposed a sum inverse SINR heuristic similarly to [4] to solve a utility maximization problem in [2]. In [7], the authors studied distributed Jacobi best-response algorithms that are obtained by a partial-linearization decomposition method to solve convexified version of the utility maximization problem.

In this paper, we study how to enable service differentiation in wireless networks by solving several of these nonconvex utility maximization problems, that have previously been solved suboptimally, e.g., in [5], [6] or partially in special cases, e.g., in [4]. By leveraging a decomposition and convex reformulation technique, which is in part inspired by [1], these utility maximization problems can be decomposed into multiple user subproblems and a network subproblem that iteratively weighs the link performance metric by the feedback from the users, so that the entire network converges to the global optimality. A novelty in our decomposition analysis is the use of the nonlinear Perron-Frobenius theory in [8] to systematically design fixed-point algorithms. These new fixed-point analysis characterizations overcome the limitations typically associated with the nonconvexity in these wireless network utility maximization problems.

The main contributions of this paper are summarized as follows.

- We propose a novel decomposition technique based on the nonlinear Perron-Frobenius theory to solve the nonconvex utility maximization problems that hitherto have been solved suboptimally or partially in [4]–[6].
- Our analysis demonstrates that four properties, namely positivity, monotonicity, homogeneity and quasi-concavity, that are inherent in these utility maximization problems, can be exploited to design fixed-point algorithms that converge geometrically fast.

The rest of this paper is organized as follows. We present the system model in Section II. In Section III, we formulate the utility maximization problems. In Section IV, we decompose

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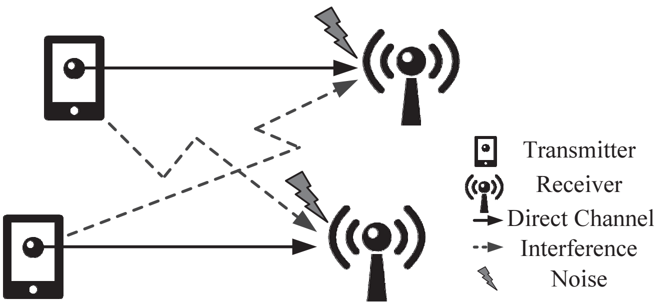

Fig. 1. The system model for the 2-user case. The received signal of the lth user is $G_{ll}p_l$ under frequency flat fading, and this received signal is $G_{ll}R_{ll}p_l$ under Rayleigh fading (the solid arrow line). The interference temperature received is the sum of interference signal power and additive noise power. The received interference temperature for the lth user is $\sum_{j\neq l} G_{lj}p_j + n_l$ under frequency flat fading, and this received interference temperature is $\sum_{j\neq l} G_{lj}R_{lj}p_j + n_l$ (the dashed arrow line).

the general utility maximization problem into multiple simpler subproblems. Next in Section V, by exploiting the optimality conditions and leveraging the nonlinear Perron-Frobenius theory, we propose fixed-point algorithms to solve the weighted sum logarithmic SINR maximization, the weighted sum inverse SINR minimization and the weighted sum logarithmic reliability maximization. We evaluate the performance of our algorithms numerically in Section VI. We conclude the paper in Section VII.

The following notation is used in our paper. Column vectors and matrices are denoted by boldfaced lowercase and uppercase respectively. Let $\rho(\mathbf{A})$ denote the Perron-Frobenius eigenvalue of a nonnegative matrix $\mathbf{A}$, and $\mathbf{x}(\mathbf{A})$ and $\mathbf{y}(\mathbf{A})$ denote the Perron right and left eigenvectors of $\mathbf{A}$ associated with $\rho(\mathbf{A})$ respectively. Furthermore, we denote $\mathbf{x} \circ \mathbf{y}$ as the Schur product of $\mathbf{x}$ and $\mathbf{y}$. We let $\mathbf{a}_l$ denote the lth column vector of matrix $\mathbf{A}$ and let $\mathbf{e}_l$ and $\mathbf{I}$ denote the lth unit coordinate vector and the identity matrix respectively. Let $\mathbf{1} = (1, \ldots, 1)^\top \in \mathbb{R}^L$ be an all-one vector. The super-script $(\cdot)^\top$ denotes transpose. For a given vector $\mathbf{x} = (x_1, \ldots, x_L)^\top$, $\text{diag}(\mathbf{x})$ is a diagonal matrix $\text{diag}(x_1, \ldots, x_L)$, $e^{\mathbf{x}}$ denotes $e^{\mathbf{x}} = (e^{x_1}, \ldots, e^{x_L})^\top$, and $\log \mathbf{x}$ denotes $\log \mathbf{x} = (\log x_1, \ldots, \log x_L)^\top$.

II. System Model

We consider a multiuser communication wireless network with L users (transmitter/receiver pairs) transmitting simultaneously on a shared spectrum. Let $G = [G_{lj}]_{l,j=1}^L > 0_{L\times L}$ represent the channel gain, where G_{lj} is the channel gain from the jth transmitter to the lth receiver, and $\mathbf{n} = (n_1, \ldots, n_L)^\top > \mathbf{0}$, where n_l is the noise power at the lth user. The vector $\mathbf{p} = (p_1, \ldots, p_L)^\top$ is the transmit power vector. Figure 1 shows the system model with the problem parameters for the 2-user case. In this paper, we consider two different scenarios of transmission using this channel model. These two scenarios are described below and are differentiated by the assumption whether there is a Rayleigh (fast) fading component on each individual signal path or not.

Let us describe the first transmission scenario. Supposed that there is no Rayleigh fading component on the signal path. The received SINR of the lth user can be given by:

$$\mathsf{SINR}_l(\mathbf{p}) = \frac{G_{ll}p_l}{\sum\limits_{j\neq l} G_{lj}p_j + n_l}. \tag{1}$$

Now, we define a nonnegative matrix $\mathbf{F}$ with entries:

$$F_{lj} = \begin{cases} 0, & \text{if } l = j \\ \dfrac{G_{lj}}{G_{ll}}, & \text{if } l \neq j \end{cases} \tag{2}$$

and the vector

$$\mathbf{v} = \left(\frac{n_1}{G_{11}}, \frac{n_2}{G_{22}}, \ldots, \frac{n_L}{G_{LL}}\right)^\top.$$

Moreover, we assume that $\mathbf{F}$ is irreducible, i.e., each link has at least one interferer. Using this notation, the SINR of the lth user can be represented compactly as: $\mathsf{SINR}_l(\mathbf{p}) = \frac{p_l}{(\mathbf{Fp}+\mathbf{v})_l}$.

Next, let us describe the second transmission scenario. Supposed that there is a Rayleigh fading component on each signal path. The received power from the jth transmitter at the lth receiver is given by $G_{lj}R_{lj}p_j$ where R_{lj} is a random variable due to the Rayleigh fading. In particular, R_{lj} are independent and exponentially distributed with unit mean, i.e., $E[G_{lj}R_{lj}p_j] = G_{lj}p_j$ [9]. The received SINR of the lth user under Rayleigh fading is thus given by:

$$\mathsf{SINR}_l(\mathbf{p}) = \frac{R_{ll}p_l}{\sum\limits_{j\neq l} F_{lj}R_{lj}p_j + v_l}. \tag{3}$$

Note that, unlike (1) which is a deterministic fractional function of $\mathbf{p}$, (3) is a random variable.

Now, an outage occurs when the received SINR of the lth user falls below β_l, a minimum SINR threshold for reliable communication. Assuming independent Rayleigh fading at all signals, the reliability function of the lth user is given as the complement of the outage probability [9]:

$$O_l \triangleq \text{Prob}(\mathsf{SINR}_l(\mathbf{p}) \geq \beta_l) = e^{\frac{-v_l\beta_l}{p_l}} \prod_{j=1}^{L} \left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right)^{-1}. \tag{4}$$

This means that when $\mathsf{SINR}_l(\mathbf{p}) \geq \beta_l$, the transmission from lth transmitter to its receiver is reliable; otherwise, the transmission fails.

In the following, we study the wireless network utility maximization problem subject to maximum power constraints under these two different scenarios. It should be clear from the context that whenever we use (4), the SINR function in (4) comes from (3) and not from (1).

III. Utility Maximization Problem Formulation

Let us denote the overall network utility in the wireless network by $U(f(\mathbf{p}))$. This is an objective function whose argument $f(\mathbf{p})$ is a vector function with entries $f_l(\mathbf{p})$ that is a bijective mapping from the transmit power $\mathbf{p}$ to a particular link metric of the lth user. In particular, we will study the link metrics $f_l(\mathbf{p})$ given by the SINR in (1) or the reliability function in (4) for the two different transmission scenarios introduced in the previous section. In addition, we assume that the overall network utility function is separable and continuous, i.e.,

$$U(\mathbf{p}) = \sum_{l=1}^{L} U_l(f_l(\mathbf{p})). \tag{5}$$

Each individual utility $U_l(f_l(\mathbf{p}))$ reflects the service obtained by the lth user based on the achievable link metric $f_l(\mathbf{p})$. Service differentiation in wireless networks can thus be realized by a suitable choice of the utility and its optimization.

As an example, $U(\cdot)$ can be given by the α-fairness utility [10]:

$$U(\mathbf{x}) = \begin{cases} \sum_{l=1}^{L} \log x_l, & \text{if } \alpha = 1, \\ \sum_{l=1}^{L} (1-\alpha)^{-1} x_l^{1-\alpha}, & \text{if } \alpha > 1. \end{cases} \tag{6}$$

Note that the special case for 1-fairness utility is also known as the *proportional fairness* in the literature [1], [10].

In addition, all the users are constrained by a weighted power constraint set $\mathcal{P}$ given by:

$$\mathcal{P} = \{\mathbf{p} \,|\, \mathbf{a}_l^\top \mathbf{p} \leq \bar{p}_l, \;\; l = 1, \ldots, L\}, \tag{7}$$

where $\bar{\mathbf{p}}$ is the upper bound for the weighted power constraints, and $\mathbf{a}_l$ can be any positive vector. Since $\mathbf{a}_l$ is the lth column vector of a nonnegative matrix $\mathbf{A}$, (7) can also be expressed as $\mathcal{P} = \{\mathbf{p} \,|\, \mathbf{A}^\top \mathbf{p} \leq \bar{\mathbf{p}}\}$. As special cases, when $\mathbf{a}_l = \mathbf{e}_l$ (i.e., $\mathbf{A} = \mathbf{I}$), we have the individual power constraints $\mathcal{P} = \{\mathbf{p} \,|\, \mathbf{p} \leq \bar{\mathbf{p}}\}$, and when $\mathbf{a}_l = \mathbf{1}$ (i.e., $\mathbf{A}$ is an all-one matrix), we have a total power constraint $\mathcal{P} = \{\mathbf{p} \,|\, \mathbf{1}^\top \mathbf{p} \leq \min_{l=1,\ldots,L} \bar{p}_l\}$.

The problem of maximizing the overall network utility subject to power constraints is given as:

$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} U_l(f_l(\mathbf{p})) \\ \text{subject to} & \mathbf{p} \in \mathcal{P}, \\ \text{variables:} & \mathbf{p}. \end{array} \tag{8}$$

In general, depending on the choice of the utility function in (5), (8) can be nonconvex, and thus it may be generally difficult to solve (8) optimally.

Next, using a logarithmic mapping of variable, for $\mathbf{p} = (p_1, \ldots, p_L)^\top > 0$, let $\tilde{p}_l = \log p_l$ for all l, i.e., $\mathbf{p} = e^{\tilde{\mathbf{p}}}$, (8) can be transformed to the following equivalent optimization problem:

$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} U_l(f_l(e^{\tilde{\mathbf{p}}})) \\ \text{subject to} & e^{\tilde{\mathbf{p}}} \in \mathcal{P}, \\ \text{variables:} & \tilde{\mathbf{p}}. \end{array} \tag{9}$$

In general, (9) may still be nonconvex. In the following, we study important special cases of (9) that can be solved optimally (and thereby solving (8) optimally). These special cases have been hitherto tackled suboptimally in the literature when they are viewed as the nonconvex problems in (8). In the following, we decompose (9) in Section IV and then apply the nonlinear Perron-Frobenius theory to link (8) and (9) in Section V. As a consequence, fixed-point algorithms with geometric convergence rate can be systematically obtained to solve (8) optimally.

IV. Service Differentiation by Proportional Fairness Decomposition

To overcome the hurdle of nonconvexity of (8), we decompose (9) and then exploit the optimality conditions of (9). Our approach is motivated by the Kelly's decomposition in [1], in which time is divided into discrete slots and we iteratively solve one of two different sets of subproblems in each slot: 1) The user subproblem: each user computes a positive payment associated with the current resource allocated by a centralized network controller (for example, a base station). This payment can be viewed as the willingness to pay for the allocated resource. 2) The network subproblem: based on the payments computed by all the users, the network controller maximizes a weighted sum of *proportional fairness* associated with the link metrics for all users, where the payment received from users acts as weight. Then, the controller returns the feedback of the updated resource allocation to all the users. The above subproblems are solved iteratively until the network reaches an equilibrium. At this equilibrium, we say that the resource allocation is proportionally fair to the equilibrium payments.

In the following, we describe this Kelly's decomposition applied to (9). By introducing an auxiliary variable $\boldsymbol{\alpha}$, we rewrite (9) as:

$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} U_l(\alpha_l) \\ \text{subject to} & \alpha_l \leq f_l(e^{\tilde{\mathbf{p}}}), \;\; l = 1, \ldots, L, \\ & e^{\tilde{\mathbf{p}}} \in \mathcal{P}, \\ \text{variables:} & \tilde{\mathbf{p}}, \boldsymbol{\alpha}. \end{array} \tag{10}$$

Note that (10) is nonconvex in general. However, by using a logarithmic transformation change of variable, i.e., $\boldsymbol{\alpha} = e^{\tilde{\boldsymbol{\alpha}}}$, (10) is equivalent to:

$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} U_l(e^{\tilde{\alpha}_l}) \\ \text{subject to} & \tilde{\alpha}_l \leq \log f_l(e^{\tilde{\mathbf{p}}}), \;\; l = 1, \ldots, L, \\ & e^{\tilde{\mathbf{p}}} \in \mathcal{P}, \\ \text{variables:} & \tilde{\mathbf{p}}, \tilde{\boldsymbol{\alpha}}. \end{array} \tag{11}$$

In this paper, we study utility functions that satisfy the following assumption.

Assumption 1: The following are sufficient for (11) to be convex: 1) $\sum_{l=1}^{L} U_l(e^{\tilde{\alpha}_l})$ is concave in $\tilde{\alpha}_l$, 2) $\log f_l(e^{\tilde{\mathbf{p}}})$ is concave in $\tilde{\mathbf{p}}$ for all l.

Lemma 1: Both $f(e^{\tilde{\mathbf{p}}})$, i.e., $\mathsf{SINR}(e^{\tilde{\mathbf{p}}})$ in (1) and $O_l(e^{\tilde{\mathbf{p}}})$ in (4), satisfy the second condition in Assumption 1.

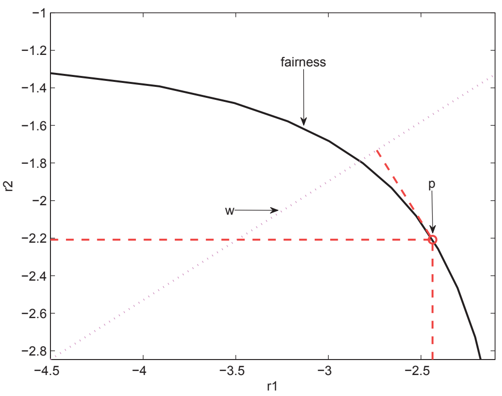

Fig. 2. Illustration of the proportional fairness in the log SINR domain. We use a two-user example, and the objective utility function is the α-fairness utility function where α is equal to 3. The channel gains are given by $G_{11} = 0.75$, $G_{12} = 0.12$, $G_{21} = 0.13$, $G_{22} = 0.70$ and the weight for the power constraints are $\mathbf{a}_1 = [4.50,\ 5.20]^\top$ and $\mathbf{a}_2 = [2.80,\ 2.40]^\top$ respectively. The upper bound for the weighted power constraints is $\bar{\mathbf{p}} = [1.20,\ 1.00]^\top$ W, and the noise power for both users are 1 W. We solve the utility maximization problem to obtain the optimal power $\mathbf{p}^\star$, and use this $\mathbf{p}^\star$ to calculate the proportional fairness which is $w_l = \gamma_l^{-2}$ in this case. Its perpendicular intersects the optimal solution of the utility maximization problem at the boundary of the feasible region.

Next, we form the partial Lagrangian of (11) by introducing the dual variable $\boldsymbol{\mu} \in \mathbb{R}_+^L$:

$$\mathcal{L}(\tilde{\mathbf{p}}, \tilde{\boldsymbol{\alpha}}, \boldsymbol{\mu}) = \sum_{l=1}^{L} U_l(e^{\tilde{\alpha}_l}) + \sum_{l=1}^{L} \mu_l \big(\log f_l(e^{\tilde{\mathbf{p}}}) - \tilde{\alpha}_l \big). \quad (12)$$

By taking the partial derivative of (12) with respect to $\tilde{\alpha}_l$ and using Lagrange duality, we can obtain the following result.

Lemma 2: The optimal solution $\tilde{\boldsymbol{\alpha}}^\star$ and $\tilde{\mathbf{p}}^\star$ of (11) satisfy $e^{\tilde{\alpha}_l^\star} = f_l(e^{\tilde{\mathbf{p}}^\star})$ for all l. In other words, the optimal solution $\boldsymbol{\alpha}^\star$ and $\mathbf{p}^\star$ to (10) satisfy $\alpha_l^\star = f_l(\mathbf{p}^\star)$ for all l. We also have that $\mu_l^\star = \alpha_l^\star \nabla_{\alpha_l} U_l(\alpha_l^\star)$.

According to the results stated in Lemma 2, we can decompose (9) as follows. For any feasible power vector $\mathbf{p}$, each user calculates a payment as follows:

$$w_l = f_l(\mathbf{p}) \nabla_{f_l} U_l(f_l(\mathbf{p})), \quad (13)$$

where $\mathbf{w}$ denotes the payment made by the lth user to the network controller, and $\nabla_{f_l} U_l(f_l(\mathbf{p}))$ is the first order derivative with respect to $f_l(\mathbf{p})$. Suppose the network controller receives the payment w_l from the lth user, it solves a problem that maximizes the weighted sum of the logarithm of the link metric $f_l(\mathbf{p})$ (i.e., proportional fairness), where the payment w_l acts as a weight. In particular, the network controller solves the following problem:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{l=1}^{L} w_l \log f_l(\mathbf{p}) \\ \text{subject to} & \mathbf{p} \in \mathcal{P}, \\ \text{variables:} & \mathbf{p}. \end{array} \quad (14)$$

We say that $\mathbf{w}$ is proportionally fair if $w_l = f_l(\mathbf{p}^\star) \nabla_{f_l} U_l(f_l(\mathbf{p}^\star))$, where $\mathbf{p}^\star$ is the optimal solution of (8). Interestingly, at optimality, both (8) and (14) have the same optimal solution. Figure 2 gives a geometric interpretation of the proportional fairness: a hyperplane intersecting the optimal point is perpendicular to $\mathbf{w}$ when $\mathbf{w}$ is proportionally fair. By leveraging this relationship, we propose the following algorithm that computes the optimal solution of (9).

Algorithm 1 (Utility Maximization):

1) Each lth user updates the proportional fairness:

$$w_l(k+1) = f_l(\mathbf{p}(k)) \nabla_{f_l} U_l(f_l(\mathbf{p}(k))).$$

2) The controller solves the following problem:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{l=1}^{L} w_l(k+1) \log f_l(\mathbf{p}) \\ \text{subject to} & \mathbf{p} \in \mathcal{P}, \\ \text{variables:} & \mathbf{p}, \end{array}$$

whose optimal solution is denoted as $\mathbf{p}(k+1)$.

Theorem 1: Algorithm 1 converges to the optimal solution of (8) from any initial point $\mathbf{p}(0)$.

Remark 1: We make the following remarks on Algorithm 1. At Step 1, the computation of payment can be made distributed by each user. The optimization problem in Step 2 is solved optimally, and further elaborated in Section V.

V. Fixed-Point Algorithm for Proportional Fairness

In this section, we study how to solve (14) for the weighted sum logarithmic SINR maximization and the weighted sum inverse SINR minimization in the first transmission scenario and the weighted sum reliability maximization in the second transmission scenario. It turns out that the stationarity of the Lagrangians for these three problems share an interesting property: they can be expressed as fixed-point problems involving concave self-mapping functions for the optimal power $\mathbf{p}^\star$. This enables us to propose tuning-free fixed-point algorithms that converge geometrically fast by leveraging the nonlinear Perron-Frobenius theory in [8] (also see Appendix).

A. Weighted Sum Logarithmic SINR Maximization

Consider the first transmission scenario. We let $f_l(\mathbf{p})$ be $\mathsf{SINR}_l(\mathbf{p})$, thus (14) is the weighted sum logarithmic SINR maximization problem given by:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{l=1}^{L} w_l \log \mathsf{SINR}_l(\mathbf{p}) \\ \text{subject to} & \mathbf{p} \in \mathcal{P}, \\ \text{variables:} & \mathbf{p}. \end{array} \quad (15)$$

In this subsection, let $\mathbf{p}^\star$ denote the optimal solution to (15).

Although (15) is nonconvex, it is equivalent to a convex optimization problem by making a change of variable $\tilde{p}_l = \log p_l$ for all l, and the equivalence is given by:

$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} w_l \log \mathsf{SINR}_l(e^{\tilde{\mathbf{p}}}) \\ \text{subject to} & \log\left(\frac{1}{\bar{p}_l}\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\right) \le 0, \quad l = 1,\dots,L, \\ \text{variables:} & \tilde{\mathbf{p}}. \end{array} \tag{16}$$

The Lagrangian associated with (16) is given by:

$$\mathcal{L}(\tilde{\mathbf{p}}, \boldsymbol{\lambda}) = \log \prod_{l=1}^{L} \left((\mathbf{F}e^{\tilde{\mathbf{p}}} + \mathbf{v})_l e^{-\tilde{p}_l}\right)^{w_l} + \sum_{l=1}^{L} \lambda_l \log \frac{1}{\bar{p}_l}\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}, \tag{17}$$

where $\boldsymbol{\lambda}_l \in \mathbb{R}_+$ is the dual variable vector for $\log\left((1/\bar{p}_l)\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\right) \le 0$. By taking first order derivative of (17) with respect to $\tilde{p}_l$, setting it to zero and substituting $\mathbf{p} = e^{\tilde{\mathbf{p}}}$ back, we have the following equations satisfied by $\mathbf{p}^\star$ in (15) and the optimal dual variable $\boldsymbol{\lambda}^\star$ in (16):

$$p_l^\star = \frac{w_l}{\sum_{j=1}^{L} \frac{w_j F_{jl}}{(\mathbf{F}\mathbf{p}^\star + \mathbf{v})_j} + \sum_{j=1}^{L} \frac{\lambda_j^\star A_{lj}}{\mathbf{a}_j^\top \mathbf{p}^\star}}, \quad l = 1,\dots,L, \tag{18}$$

$$\sum_{j=1}^{L} \frac{A_{lj} p_l^\star}{\mathbf{a}_j^\top \mathbf{p}^\star} \lambda_j^\star = w_l - p_l^\star \sum_{j \ne l} \frac{w_j F_{jl}}{(\mathbf{F}\mathbf{p}^\star + \mathbf{v})_j}, \quad l = 1,\dots,L. \tag{19}$$

Note that if $\boldsymbol{\lambda}^\star$ is the only unknown variable in (19), $\boldsymbol{\lambda}^\star$ can be easily computed by solving a system of equations (keeping $\mathbf{p}^\star$ fixed). Moreover, as the right-hand side of (19) is positive, (19) is in fact a positive system of equations in $\mathbf{p}^\star$ and $\boldsymbol{\lambda}^\star$.

Next, let us define the following nonnegative matrix:

$$\mathbf{B}(\mathbf{p}) = \mathbf{F} + \sum_{l=1}^{L} \frac{\hat{\lambda}_l^\star}{\bar{p}_l} \mathbf{v}\mathbf{a}_l^\top, \tag{20}$$

where $\hat{\boldsymbol{\lambda}}^\star \in \mathbb{R}_+^L$ is the normalized dual variable vector (i.e., $\hat{\boldsymbol{\lambda}}^\star = \boldsymbol{\lambda}^\star / \mathbf{1}^\top \boldsymbol{\lambda}^\star$). Note that $\mathbf{B}$ defined in (20) is a function of $\mathbf{p}$, since $\boldsymbol{\lambda}^\star$ in (19) depends on $\mathbf{p}$.

According to complementary slackness, since $\hat{\lambda}_l^\star > 0 \Rightarrow (1/\bar{p}_l)\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}} = 1$ or $\hat{\lambda}_l^\star = 0 \Rightarrow (1/\bar{p}_l)\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}} < 1$, only those tight constraints at optimality participate in forming $\mathbf{B}$. This leads to $\sum_{l=1}^{L} (\hat{\lambda}_l / \bar{p}_l)\mathbf{v}\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}} = 1$, i.e., $\mathbf{F}e^{\tilde{\mathbf{p}}} + \mathbf{v} = \mathbf{B}e^{\tilde{\mathbf{p}}}$.

Furthermore, using $\mathbf{B}$ in (20) and the complementary slackness condition, (16) can then be rewritten as:

$$\begin{array}{ll} \text{minimize} & \prod_{l=1}^{L} \left(\frac{(\mathbf{B}(e^{\tilde{\mathbf{p}}})e^{\tilde{\mathbf{p}}})_l}{e^{\tilde{p}_l}}\right)^{w_l} \\ \text{subject to} & \log\left(\frac{1}{\bar{p}_l}\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\right) \le 0, \quad l = 1,\dots,L, \\ \text{variables:} & \tilde{\mathbf{p}}. \end{array} \tag{21}$$

Theorem 2: The optimal power $\mathbf{p}^\star$ of (15) satisfies

$$p_l^\star = \frac{w_l}{\sum_{j=1}^{L} w_j B_{jl}(\mathbf{p}^\star) / (\mathbf{B}(\mathbf{p}^\star)\mathbf{p}^\star)_j}, \quad l = 1,\dots,L. \tag{22}$$

Moreover, $\tilde{p}_l^\star = \log p_l^\star$ solves (21) for all l.

Observe that the righthand-side of (22) is positive, homogeneous of degree one and quasi-concave. Therefore, it is a concave function. We can exploit this fact together with the nonlinear Perron-Frobenius theory in [8] (also see Appendix) to propose the following fixed-point algorithm that computes the optimal solution of (15) in Theorem 2.

Algorithm 2 (Weighted Sum Log SINR Maximization):

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1) Compute $\boldsymbol{\lambda}(k+1)$ by solving the following equations for a given $\mathbf{p}(k)$:
$$\sum_{j=1}^{L} \frac{A_{lj} p_l(k)}{\mathbf{a}_j^\top \mathbf{p}(k)} \lambda_j(k+1) = w_l - p_l(k) \sum_{j=1}^{L} \frac{w_j F_{jl}}{(\mathbf{F}\mathbf{p}(k) + \mathbf{v})_j}, \quad l = 1,\dots,L.$$
2) Normalize $\boldsymbol{\lambda}(k+1)$ to $\hat{\boldsymbol{\lambda}}(k+1)$ and update the nonnegative matrix $\mathbf{B}(k+1)$:
$$\hat{\boldsymbol{\lambda}}(k+1) = \boldsymbol{\lambda}(k+1) / \mathbf{1}^\top \boldsymbol{\lambda}(k+1),$$
$$\mathbf{B}(k+1) = \mathbf{F} + \sum_{l=1}^{L} \frac{\hat{\lambda}_l(k+1)}{\bar{p}_l} \mathbf{v}\mathbf{a}_l^\top.$$
3) Update the power $p_l(k+1)$ for the lth user by:
$$p_l(k+1) = \frac{w_l}{\sum_{j=1}^{L} w_j B_{jl}(k+1) / (\mathbf{B}(k+1)\mathbf{p}(k))_j}.$$
4) Normalize $\mathbf{p}(k+1)$:
$$\mathbf{p}(k+1) \leftarrow \mathbf{p}(k+1) / \max_{l=1,\dots,L} \{\mathbf{a}_l^\top \mathbf{p}(k+1) / \bar{p}_l\}.$$

Corollary 1: Algorithm 2 converges geometrically fast to the fixed point $\mathbf{p}^\star$ in Theorem 2 from any initial point $\mathbf{p}(0)$.

We study the weighted sum inverse SINR minimization problem in Section V-B. This problem is intimately related to (15) (through the arithmetic-geometric mean inequality [11]).

B. Weighted Sum Inverse SINR Minimization

Consider the first transmission scenario. Let us study the weighted sum inverse SINR minimization problem given by:

$$\begin{array}{ll} \text{minimize} & \sum_{l=1}^{L} w_l \frac{1}{\mathsf{SINR}_l(\mathbf{p})} \\ \text{subject to} & \mathbf{p} \in \mathcal{P}, \\ \text{variables:} & \mathbf{p}. \end{array} \tag{23}$$

In this subsection, we denote the optimal solution to (23) by $\mathbf{p}^\star$. Although (23) has a different objective from (14), it is closely related to (15), which can be viewed as an alternative for solving (15) [12], [13]. Furthermore, (23) provides a useful upper bound to (15), which is discussed later in this section.

Similar to the convexification technique used on (15) to get

(16), (23) is equivalent to:

$$\begin{array}{ll} \text{minimize} & \sum_{l=1}^{L} w_l \frac{1}{\mathsf{SINR}_l(e^{\tilde{\mathbf{p}}})} \\ \text{subject to} & \log\left(\frac{1}{\bar{p}_l}\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\right) \le 0, \quad l = 1,\ldots,L, \\ \text{variables:} & \tilde{\mathbf{p}}. \end{array} \tag{24}$$

The optimization problem in (24) is convex. By introducing the dual variable $\boldsymbol{\lambda} \in \mathbb{R}_+^L$, we form the Lagrangian for (24), given by:

$$\mathcal{L}(\tilde{\mathbf{p}}, \boldsymbol{\lambda}) = \sum_{l=1}^{L} w_l (\mathbf{F}e^{\tilde{\mathbf{p}}} + \mathbf{v})_l e^{-\tilde{p}_l} + \sum_{l=1}^{L} \lambda_l \log \frac{1}{\bar{p}_l}\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}. \tag{25}$$

Taking the partial derivative of (25) with respect to $\tilde{p}_l$, setting it to zero and substituting $\mathbf{p} = e^{\tilde{\mathbf{p}}}$ back, we have the following equations satisfied by the optimal power $\mathbf{p}^\star$ in (23) and the optimal dual variable $\boldsymbol{\lambda}^\star$ in (24):

$$p_l^\star = \sqrt{\frac{w_l(\mathbf{F}\mathbf{p}^\star + \mathbf{v})_l}{\sum_{j \ne l} w_j F_{jl}/p_j^\star + \sum_{j=1}^{L} \lambda_j^\star A_{lj}/\mathbf{a}_j^\top \mathbf{p}^\star}}, \quad l = 1,\ldots,L, \tag{26}$$

$$\sum_{j=1}^{L} \frac{A_{lj} p_l^\star}{\mathbf{a}_j^\top \mathbf{p}^\star}\lambda_j^\star = \frac{w_l(\mathbf{F}\mathbf{p}^\star + \mathbf{v})_l}{p_l^\star} - p_l^\star \sum_{j \ne l} \frac{w_j F_{jl}}{p_j^\star}, \quad l = 1,\ldots,L. \tag{27}$$

Note that (27) is a system of equations in $\mathbf{p}^\star$ and $\boldsymbol{\lambda}^\star$ with a positive right-hand side, and hence a positive system.

Recall the same definition of $\mathbf{B}$ in (20). In this case, $\hat{\boldsymbol{\lambda}}^\star \in \mathbb{R}_+^L$ is the normalized dual variable for (24). Notice that (24) can be further rewritten as:

$$\begin{array}{ll} \text{minimize} & \sum_{l=1}^{L} w_l \frac{(\mathbf{B}(e^{\tilde{\mathbf{p}}})e^{\tilde{\mathbf{p}}})_l}{e^{\tilde{p}_l}} \\ \text{subject to} & \log\left(\frac{1}{\bar{p}_l}\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\right) \le 0, \quad l = 1,\ldots,L, \\ \text{variables:} & \tilde{\mathbf{p}}. \end{array} \tag{28}$$

Theorem 3: The optimal power $\mathbf{p}^\star$ of (23) satisfies

$$p_l^\star = \sqrt{\frac{w_l \sum_{j \ne l} B_{lj}(\mathbf{p}^\star) p_j^\star}{\sum_{j \ne l} w_j B_{jl}(\mathbf{p}^\star)/p_j^\star}}, \quad l = 1,\ldots,L. \tag{29}$$

As in the previous, the following algorithm computes the optimal solution of (23) by using the nonlinear Perron-Frobenius theory in [8].

Algorithm 3 (Weighted Sum Inverse SINR Minimization):

---

1) Compute $\boldsymbol{\lambda}(k+1)$ by solving the following equations for a given $\mathbf{p}(k)$:
$$\sum_{j=1}^{L} \frac{A_{lj} p_l(k)}{\mathbf{a}_j^\top \mathbf{p}(k)} \lambda_j(k+1) = \frac{w_l(\mathbf{F}\mathbf{p}(k) + \mathbf{v})_l}{p_l(k)} - p_l(k) \sum_{j \ne l} \frac{w_j F_{jl}}{p_j(k)}, \quad l = 1,\ldots,L.$$
2) Normalize $\boldsymbol{\lambda}(k+1)$ to $\hat{\boldsymbol{\lambda}}(k+1)$ and update the nonnegative matrix $\mathbf{B}(k+1)$:
$$\hat{\boldsymbol{\lambda}}(k+1) = \boldsymbol{\lambda}(k+1)/\mathbf{1}^\top \boldsymbol{\lambda}(k+1),$$
$$\mathbf{B}(k+1) = \mathbf{F} + \sum_{l=1}^{L} \frac{\hat{\lambda}_l(k+1)}{\bar{p}_l} \mathbf{v}\mathbf{a}_l^\top.$$
3) Update the power $p_l(k+1)$ for the lth user by:
$$p_l(k+1) = \sqrt{\frac{w_l \sum_{j \ne l} B_{lj}(k+1) p_j(k)}{\sum_{j \ne l} w_j B_{jl}(k+1)/p_j(k)}}.$$
4) Normalize $\mathbf{p}(k+1)$:
$$\mathbf{p}(k+1) \leftarrow \mathbf{p}(k+1)/\max_{l=1,\ldots,L}\{\mathbf{a}_l^\top \mathbf{p}(k+1)/\bar{p}_l\}.$$

Corollary 2: Algorithm 3 converges geometrically fast to the fixed point $\mathbf{p}^\star$ in Theorem 3 from any initial point $\mathbf{p}(0)$.

Remark 2: We connect the weighted sum logarithmic SINR maximization in Section V-A with (23). In particular, by applying the arithmetic-geometric mean inequality and the Friedland-Karlin inequality in [14], we have

$$\sum_{l=1}^{L} w_l \frac{(\mathbf{B}(\mathbf{p})\mathbf{p})_l}{p_l} \ge \prod_{l=1}^{L} \left(\frac{(\mathbf{B}(\mathbf{p})\mathbf{p})_l}{p_l}\right)^{w_l} \ge \rho(\mathbf{B}(\mathbf{p})). \tag{30}$$

The equality holds for both the first and the second inequalities in (30) if and only if $(\mathbf{B}(\mathbf{p})\mathbf{p})_1/p_1 = \cdots = (\mathbf{B}(\mathbf{p})\mathbf{p})_L/p_L$, i.e., $\mathsf{SINR}_1(\mathbf{p}) = \cdots = \mathsf{SINR}_L(\mathbf{p})$.

Interestingly, the weighted sum inverse SINR minimization in (23) has the same optimal solution $\mathbf{p}^\star$ as the weighted sum logarithmic SINR maximization in (15) when

$$\mathbf{w} = \mathbf{x}(\mathbf{B}(\mathbf{p}^\star)) \circ \mathbf{y}(\mathbf{B}(\mathbf{p}^\star)), \tag{31}$$

where $\mathbf{x}(\mathbf{B}(\mathbf{p}^\star))$ and $\mathbf{y}(\mathbf{B}(\mathbf{p}^\star))$ are respectively the Perron right and left eigenvectors of $\mathbf{B}(\mathbf{p}^\star)$, and $\mathbf{x}(\mathbf{B}(\mathbf{p}^\star)) \circ \mathbf{y}(\mathbf{B}(\mathbf{p}^\star))$ is a probability vector.

Finally, the same technique as in Section V-A and V-B can be used to solve the weighted sum logarithmic reliability maximization problem in Section V-C, i.e., finding a concave self-mapping, and then leveraging the nonlinear Perron-Frobenius theory (also see Appendix) to propose a fixed-point algorithm.

C. Weighted Sum Logarithmic Reliability Maximization

Consider the second transmission scenario. Let $f_l(\mathbf{p})$ be the reliability $R_l(\mathbf{p})$. Then the weighted sum logarithmic reliability maximization problem is given by:

$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} w_l \log O_l(\mathbf{p}) \\ \text{subject to} & \mathbf{p} \in \mathcal{P}, \\ \text{variables:} & \mathbf{p}. \end{array} \tag{32}$$

In this subsection, let $\mathbf{p}^\star$ denote the optimal solution to (32).

Through a logarithmic transformation change of variable on (32), we obtain:

$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} w_l \log O_l(e^{\tilde{\mathbf{p}}}) \\ \text{subject to} & \log\left(\frac{1}{\bar{p}_l}\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\right) \le 0, \quad l = 1,\ldots,L, \\ \text{variables:} & \tilde{\mathbf{p}}. \end{array} \tag{33}$$

Next, let us define the nonnegative matrix $\mathbf{C}$ with the entries (that are functions of $\mathbf{p}$):

$$C_{lj}(\mathbf{p}) = \begin{cases} 0, & \text{if } l = j, \\ \dfrac{p_l}{\beta_l p_j} \log\Big(1 + \dfrac{\beta_l F_{lj} p_j}{p_l}\Big), & \text{if } l \neq j. \end{cases} \quad (34)$$

We then form the Lagrangian for (33) by introducing the dual variable $\boldsymbol{\lambda} \in \mathbb{R}_+^L$ for the L inequality constraints in (33):

$$\begin{aligned} \mathcal{L}(\tilde{\mathbf{p}}, \boldsymbol{\lambda}) = & \sum_{l=1}^{L} w_l \Big(v_l \beta_l e^{-\tilde{p}_l} + \sum_{j=1}^{L} \log\big(1 + \beta_l F_{lj} e^{\tilde{p}_j - \tilde{p}_l}\big)\Big) \\ & + \sum_{l=1}^{L} \lambda_l \log\Big(\frac{1}{\bar{p}_l} \mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\Big). \end{aligned} \quad (35)$$

Taking the partial derivative of (35) with respect to $\tilde{p}_l$, setting it to zero and then substituting $\mathbf{p} = e^{\tilde{\mathbf{p}}}$ back, we have the following equations for the optimal power $\mathbf{p}^\star$:

$$\begin{aligned} p_l^{\star 2} \sum_{j=1}^{L} \frac{A_{lj}}{\mathbf{a}_j^\top \mathbf{p}^\star} \lambda_j = & \; w_l v_l \beta_l + \textstyle\sum_{j\neq l} \dfrac{w_l \beta_l F_{lj} p_j^\star p_l^\star}{p_l^\star + \beta_l F_{lj} p_j^\star} \\ & - p_l^{\star 2} \textstyle\sum_{j \neq l} \dfrac{w_j \beta_j F_{jl}}{p_j^\star + \beta_j F_{jl} p_l^\star}, \quad l = 1, \ldots, L. \end{aligned} \quad (36)$$

As in the previous, (36) is a positive system of equations in $\mathbf{p}^\star$ in (32) and $\boldsymbol{\lambda}^\star$ in (33) that characterizes the optimality conditions of (32).

In Section V-A, dual variables are used to construct a nonnegative matrix $\mathbf{B}$ in (20). Likewise, we define a nonnegative matrix $\mathbf{D}$ given by:

$$\mathbf{D}(\mathbf{p}) = \mathbf{C}(\mathbf{p}) + \sum_{l=1}^{L} \frac{\hat{\lambda}_l^\star}{\bar{p}_l} \mathbf{v} \mathbf{a}_l^\top, \quad (37)$$

where $\hat{\boldsymbol{\lambda}}^\star \in \mathbb{R}_+^L$ are the normalized dual variables of (33) such that $\hat{\boldsymbol{\lambda}}^\star = \boldsymbol{\lambda}^\star / \mathbf{1}^\top \boldsymbol{\lambda}^\star$. We can then rewrite (33) as

$$\begin{aligned} \text{minimize} \quad & \sum_{l=1}^{L} w_l \frac{(\operatorname{diag}(\boldsymbol{\beta}) \mathbf{D}(e^{\tilde{\mathbf{p}}}) e^{\tilde{\mathbf{p}}})_l}{e^{\tilde{p}_l}} \\ \text{subject to} \quad & \log\Big(\frac{1}{\bar{p}_l} \mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\Big) \leq 0, \quad l = 1, \ldots, L, \\ \text{variables:} \quad & \tilde{\mathbf{p}}. \end{aligned} \quad (38)$$

Theorem 4: The optimal power $\mathbf{p}^\star$ of (32) satisfies

$$p_l^\star = \sqrt{\frac{w_l \sum_{j \neq l} \Big(v_l \beta_l \hat{a}_j(\mathbf{p}^\star) p_j^\star + \dfrac{\beta_l F_{lj} p_j^\star p_l^\star}{p_l^\star + \beta_l F_{lj} p_j^\star}\Big)}{\sum_{j\neq l} w_j \Big(v_j \beta_j \hat{a}_l(\mathbf{p}^\star)/p_j^\star + \dfrac{\beta_j F_{jl}}{p_j^\star + \beta_j F_{jl} p_l^\star}\Big)}} \quad (39)$$

for all l, where $\hat{\mathbf{a}}$ is given by $\hat{\mathbf{a}} = \sum_{l=1}^{L} (\hat{\lambda}_l^\star / \bar{p}_l) \mathbf{a}_l$.

Similar to (20), the function of $\hat{\mathbf{a}}$ involving $\hat{\lambda}_l^\star$ is also a function of $\mathbf{p}^\star$. Moreover, $\tilde{p}_l^\star = \log p_l^\star$ solves (38) for all l.

It can be verified that the righthand-side of (39) is positive, quasi-concave and homogeneous of degree one, and hence it is concave. We thus leverage the nonlinear Perron-Frobenius theory in [8] to propose the following algorithm that computes $\mathbf{p}^\star$ in Theorem 4.

Algorithm 4 (Weighted Sum Log Reliability Maximization):

---

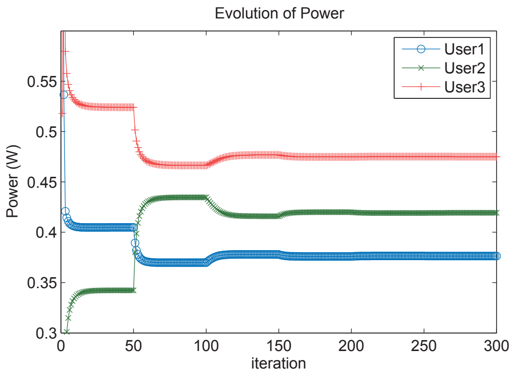
1) Compute $\boldsymbol{\lambda}(k+1)$ by solving the following equations for a given $\mathbf{p}(k)$:
$$\begin{aligned} & \sum_{j=1}^{L} \frac{A_{lj} p_l(k)}{\mathbf{a}_j^\top \mathbf{p}(k)} \lambda_j(k+1) = w_l v_l \beta_l / p_l(k) \\ & + \sum_{j \neq l} \frac{w_l \beta_l F_{lj} p_j(k)}{p_l(k) + \beta_l F_{lj} p_j(k)} - \sum_{j\neq l} \frac{w_j \beta_j F_{jl} p_l(k)}{p_j(k) + \beta_j F_{jl} p_l(k)}, \\ & \qquad l = 1, \ldots, L. \end{aligned}$$
2) Normalize $\boldsymbol{\lambda}(k+1)$ to $\hat{\boldsymbol{\lambda}}(k+1)$ and update the vector $\hat{\mathbf{a}}(k+1)$:
$$\hat{\boldsymbol{\lambda}}(k+1) = \boldsymbol{\lambda}(k+1) / \mathbf{1}^\top \boldsymbol{\lambda}(k+1),$$
$$\hat{\mathbf{a}}(k+1) = \sum_{l=1}^{L} \frac{\hat{\lambda}_l(k+1)}{\bar{p}_l} \mathbf{a}_l.$$
3) Update the power $p_l(k+1)$ for the lth user by:
$$p_l(k+1) = \sqrt{\frac{w_l \sum_{j\neq l} \Big(v_l \beta_l \hat{a}_j(k+1) p_j(k) + \dfrac{\beta_l F_{lj} p_j(k) p_l(k)}{p_l(k) + \beta_l F_{lj} p_j(k)}\Big)}{\sum_{j\neq l} w_j \Big(v_j \beta_j \hat{a}_l(k+1)/p_j(k) + \dfrac{\beta_j F_{jl}}{p_j(k) + \beta_j F_{jl} p_l(k)}\Big)}}.$$
4) Normalize $\mathbf{p}(k+1)$:
$\mathbf{p}(k+1) \leftarrow \mathbf{p}(k+1) / \max_{l=1,\ldots,L} \{\mathbf{a}_l^\top \mathbf{p}(k+1) / \bar{p}_l\}$.

Corollary 3: Algorithm 4 converges geometrically fast to the fixed point $\mathbf{p}^\star$ in Theorem 4 from any initial point $\mathbf{p}(0)$.

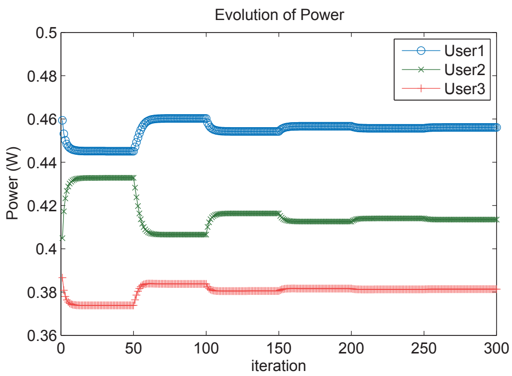
In summary, the three problems studied in this section exhibit the following features: 1) They can be reformulated as optimization problems involving a suitably constructed nonnegative matrix $\mathbf{B}$ or $\mathbf{D}$; 2) Leveraging the nonlinear Perron-Frobenius theory (also see Appendix), concave self-mappings can be associated with them to design fixed-point algorithms to solve these problems optimally.

VI. Numerical Examples

In this section, we evaluate the performance of Algorithm 1 numerically to solve (8) for $U(\cdot)$ being the 1-fairness utility function in (6) by replacing Step 2 of Algorithm 1 with the fixed-point algorithms in Section V. Since we connect (15) and (23) (cf. (31)), we compare the convergence of these two different problems that have the same optimal solution. We consider a three-user case, using the following channel gain matrix: $G_{11} = 0.71$, $G_{12} = 0.13$, $G_{13} = 0.12$, $G_{21} = 0.11$, $G_{22} = 0.73$, $G_{23} = 0.14$, $G_{31} = 0.15$, $G_{32} = 0.16$ and $G_{33} = 0.69$, and the following weights for the power constraints: $\mathbf{a}_1 = (0.93\ \ 0.72\ \ 0.74)^\top$, $\mathbf{a}_2 = (0.63\ \ 0.86\ \ 0.93)^\top$



(a)



(b)

Fig. 3. Illustration of the convergence of Algorithm 1 with two different network utility functions. We plot the power evolution for the three users that run Algorithm 2 in (a) and Algorithm 4 in (b) respectively for 50 iterations in the inner loop. We can observe that both Algorithm 2 and Algorithm 4 converge in each inner loop, and Algorithm 1 converges in the outer loop.

and $\mathbf{a}_3 = (0.98\ \ 0.86\ \ 0.78)^\top$. We set $\bar{\mathbf{p}} = (1.50\ 1.00\ 1.10)^\top$ W, and the noise power of each user is 1 W.

Figure 3 plots the evolution of the power for three users that run Algorithm 1 with only 6 outer loops. Note that the network utility used in (a) and (b) of Figure 3 are, respectively, in terms of $f_l(\mathbf{p})$ as given by (1) and (4). In Figure 3(a), we set the initial power vector to $\mathbf{p}(0) = [0.45\ 0.80\ 0.54]^\top$, and run Algorithm 2 for 50 iterations as an inner loop of Algorithm 1. The optimal power $\mathbf{p}^\star$ is $[0.38\ 0.42\ 0.48]^\top$. In Figure 3(b), we set the initial power vector to $\mathbf{p}(0) = [0.50\ 0.38\ 0.42]^\top$ W, and run Algorithm 4 for 50 iterations as an inner loop of Algorithm 1. The optimal power $\mathbf{p}^\star$ is $[0.46\ 0.41\ 0.38]^\top$ W.

VII. Conclusion

We studied network utility maximization problems for service differentiation in wireless networks. These problems are nonconvex, and thus are generally hard to solve. Using the Kelly's decomposition and a logarithmic change-of-variable technique, we decomposed the utility maximization problem into separable user subproblems and the network subproblem. Maximizing the weighted sum logarithmic SINR, minimizing the weighted sum inverse SINR and maximizing the weighted sum logarithmic reliability are three special cases solved optimally by our technique. These three special cases have hitherto been solved suboptimally in the literature. We exploited a shared optimality feature in these three special cases, which are positive, homogeneous and quasi-concave self-mappings related to the stationarity of the Lagrangian. The nonlinear Perron-frobenius theory is applied to these concave self-mappings to design fixed-point iterative algorithms that converge geometrically fast. Numerical examples verified that these fixed-point algorithms are computationally fast and have good convergence property.

Appendix

A. Nonlinear Perron-Frobenius Theory [8]

Let $\|\cdot\|$ be a monotone norm on $\mathbb{R}^L$. For a concave mapping $f : \mathbb{R}^L_+ \rightarrow \mathbb{R}^L_+$ with $f(\mathbf{z}) > 0$ for $\mathbf{z} \geq \mathbf{0}$, the following statements hold. The conditional eigenvalue problem $f(\mathbf{z}) = \lambda\mathbf{z},\ \lambda \in \mathbb{R},\ \mathbf{z} \geq \mathbf{0},\ \|\mathbf{z}\| = 1$ has a unique solution $(\lambda^*, \mathbf{z}^*)$, where $\lambda^* > 0,\ \mathbf{z}^* > \mathbf{0}$. Furthermore, $\lim_{k\rightarrow\infty} \tilde{f}(\mathbf{z}(k))$ converges geometrically fast to $\mathbf{z}^*$, where $\tilde{f}(\mathbf{z}) = f(\mathbf{z})/\|f(\mathbf{z})\|$.

For all the results in this paper, we can define a monotone norm on $\mathbb{R}^L$ as $\max\limits_{l=1,\ldots,L} \{\mathbf{a}_l^\top \mathbf{p}/\bar{p}_l\}$ and suitably identify a nonnegative concave mapping $f(\mathbf{p})$ in order to use the result in [8].

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