The value of observations in predicting transmission success in wireless networks under slotted Aloha

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***Abstract*—We consider a wireless network of static nodes where each transmitter–receiver pair employs slotted Aloha, electing to transmit with a common contention probability p, and we further assume the Rayleigh fading varies with each time slot. The random point processes of actual transmitters in a given time slot (along with their random fades) determine the interference seen by a reference receiver, and the interference across time slots is dependent due to common dependence on the underlying set of potential interferers. It follows that observations of the set of transmitters over several time slots (or summary statistics of this process) may be leveraged to yield improved estimates of the probability of success of the transmission attempted at the reference receiver. In this paper we study the value of several different forms of such observations in improving this estimated success probability. Specifically, we consider five cases: the observer has $i)$ zero knowledge, $ii)$ full knowledge of the point process of potential transmitters, $iii)$ knowledge of the number of "nearby" potential transmitters, $iv)$ N binary observations under the "physical" model at the reference receiver, and $v)$ N binary observations under the "protocol" model at the reference receiver.**

***Index Terms*—slotted Aloha; wireless networks; Poisson networks; protocol model; physical model.**

I. Introduction

The focus of this paper is on the value of observations in estimating the probability of success of an attempted transmission between a particular (reference) transmitter–receiver (TX-RX) pair in a static wireless network. Suppose the wireless network is formed by placing the locations of the radios using the Poisson bipolar model, wherein the set of potential TX is a Poisson point process, and each potential TX is matched with an associated RX positioned at a random location a fixed distance away (so all TX-RX pairs are separated by a common distance). The spatial point process of potential TX and RX, denoted Φ, is random but is fixed in time. Suppose nodes in this network employ the slotted Aloha protocol where each TX elects to transmit in each time slot independently with a common contention probability p. As is well understood (e.g., [1], and the extensive subsequent literature) the point processes of nodes that elect to transmit in time slots k, l, say, are dependent on account of their shared dependence on Φ, and the positive correlation (of, say, the interferences seen at the origin at times k and l) increases in p. Suppose further, as is common in the literature, that the network is subject to Rayleigh fading, with the random fade being independent across both nodes and time slots. There are three sources of randomness in the network: the (fixed in time) random locations of the potential TX and RX, the random contention decisions by the potential TX in each time slot, and the random fading coefficients in each time slot. As Φ is fixed in time, partial or full knowledge of Φ and/or network observations (of varying forms discussed below) in different time slots may be used to predict future network performance. Specifically, in this paper we consider the goal of predicting whether or not an attempted transmission by the reference TX-RX pair will be successful under the "physical" reception model, meaning the transmission attempt is successful if the signal to interference ratio (SIR) at the reference RX is above a specified threshold.

In particular, we consider five different forms of knowledge about the potential TX-RX point process Φ:

- Zero knowledge of Φ, aside from distribution parameters;
- Full knowledge of Φ, i.e., knowledge of the position of each potential TX;
- Knowledge of the number, M, of potential interferers within an *observation radius* r_{O} of the reference RX;
- N binary observations of the success or failure of the attempted reference transmission;
- N binary observations of whether or not an interferer was in the observation disk around the reference RX.

The first scenario is the "baseline", the second scenario represents a "best case", and the remaining three represent various forms of partial information about Φ, of the kind that might feasibly be gathered in a real network. The questions to be asked are: $i)$ how much better can we estimate the success probability at the reference RX given full knowledge of Φ relative to no knowledge of Φ?, and $ii)$ how useful for estimation are the various forms of partial knowledge of Φ?

A. Related work

Prior work by the author on a related subject was presented at WiOpt 2016 [2] with extension [3]; various results from [3] are leveraged in this paper. We leverage the "diversity polynomial" of [4], and the "meta-distribution" of [5]. This work (like [3]) is part of the literature evaluating the connections between the protocol and physical reception models, including [6], [7], and bears some similarities to the contention probability estimation work pioneered in [8]. Several relevant references are not discussed due to space constraints.

B. Outline and summary of contributions

The key contributions of this paper are: $i)$ we frame the question of how valuable are observations of a wireless network in a rigorous manner; $ii)$ we leverage results in stochastic geometry to obtain estimates of the success probability of

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TABLE I
NOTATION

Symbol	Meaning
n	ambient network dimension ($n \in \{1, 2, 3\}$)
c_n	volume of a unit ball in $\mathbb{R}^n$
Φ	homogeneous PPP with potential TX, RX locations
λ	(spatial) intensity of Φ
x_i	RV location in $\mathbb{R}^n$ of TX i
z_i	RV location in $\mathbb{R}^n$ of RX i relative to TX i
r_T	constant TX-RX separation distance
$(\mathsf{x}_i, \mathsf{y}_i)$	TX i at RV x_i and RX i at RV $\mathsf{y}_i = \mathsf{x}_i + \mathsf{z}_i$
$(\mathsf{x}_o, \mathsf{y}_o)$	RV location of reference TX and RX, with y_o at origin
p	slotted Aloha transmission attempt probability
$\mathsf{T}_{i,k}$	RV transmission attempt indicator from TX i at time k
$\mathcal{T}_k$	Random set of transmitters (with $\mathsf{T}_{i,k} = 1$) at time k
Φ_k	homogeneous PPP of active (TX, RX) pairs at time k
$\mathsf{F}_{i,k}$	RV for Rayleigh fade from TX i to reference RX at time k
$\mathsf{P}_{i,k}$	RV for received power from TX i at reference RX at time k
$l(r)$	large-scale pathloss function, $l(r) \equiv r^{-\alpha}$
α	large-scale pathloss constant, $\alpha > n$
δ	characteristic exponent n/α
β	SIR threshold
$\Sigma_{o,k}$	RV for SIR at reference RX at time k
$\mathsf{I}_{o,k}$	RV for sum interference at reference RX at time k
H_k	RV for ref. TX physical model success/failure at time k
κ_δ	convenience parameter $\Gamma(1+\delta)\Gamma(1-\delta)$
σ	convenience parameter $\beta r_\mathrm{T}^\alpha$
μ_h	convenience function $\lambda c_n \kappa_\delta \sigma^\delta$
$I(u,\delta)$	convenience function $\delta \int_0^u \frac{t^\delta}{1+t} \mathrm{d}t$
r_O	protocol model guard zone / observation radius
D_k	RV for ref. TX protocol model success/failure at time k
μ_d	convenience parameter $\lambda c_n r_\mathrm{O}^n$
χ	convenience parameter $r_\mathrm{O}^\alpha/\sigma = (r_\mathrm{O}/r_\mathrm{T})^\alpha/\beta$
ξ	convenience parameter $\frac{p}{\bar{p}}\chi^{-\delta} I(\chi,\delta)$
N	number of protocol / physical model observations
$\mathsf{K}_h^{(N)}$	RV for number of physical model successes
$\mathsf{K}_d^{(N)}$	RV for number of protocol model successes

the reference transmission under the five assumed forms of knowledge of Φ; and iii) we present numerical results that quantify the value of the various types of observations.

The paper is organized as follows. The mathematical model is presented in §II, followed by analysis of each of the four non-trivial assumptions about Φ: full knowledge in §III, knowledge of the number of nearby potential TX in §IV, multiple physical model observations in §V, and multiple protocol model observations in §VI. Numerical results addressing the two questions above are found in §VII, and a short conclusion is given in §VIII. The appendix holds the proof of Cor. 1.

II. MODEL

A. General notation

Tab. I lists key notation. Euclidean distance of a point $x \in \mathbb{R}^n$ from the origin o is denoted $\|x\|$, and the ball in $\mathbb{R}^n$ centered at the origin of radius r is denoted $b(o,r)$, with $c_n \equiv |b(o,1)|$ the volume of the unit ball. Natural and real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. All logs are natural. Denote $\{1, \ldots, N\}$ by $[N]$, for $N \in \mathbb{N}$. The indicator $\mathbf{1}_A$ or $\mathbf{1}\{A\}$, for any statement A, equals 1 (0) if A is true (false). The notation $A \equiv B$ means $A = B$ by definition. Random variables (RVs) are given a sans-serif font, e.g., x, m. We write IID for independent and identically distributed. Probability is written $\mathbb{P}(\cdot)$ and expectation is written $\mathbb{E}[\cdot]$. A bar denotes complement: $\bar{p}(\cdot) \equiv 1-p(\cdot)$. We will write $\mathrm{Ber}(p)$, $\mathrm{bin}(N,p)$, $\mathrm{Uni}(\mathcal{A})$, $\mathrm{Exp}(\nu)$, and $\mathrm{Po}(\nu)$ to denote Bernoulli, binomial, uniform, exponential, and Poisson distributions.

B. Transmitters and receivers, Aloha, and Rayleigh fading

Let $n \in \{1,2,3\}$ denote the network dimension (i.e., linear, planar, or 3D). Fix a reference TX-RX pair at $(\mathsf{x}_o, \mathsf{y}_o)$, with the reference RX at the origin, $\mathsf{y}_o = o$. Let $\Phi \equiv \{(\mathsf{x}_i, \mathsf{z}_i)\} \subset \mathbb{R}^n$ be a homogeneous bipolar PPP of intensity $\lambda > 0$ representing the random (but fixed in time) locations of *potential* TX and RX, with TXs at $\{\mathsf{x}_i\} \subset \mathbb{R}^n$ and RXs at $\{\mathsf{y}_i\} \subset \mathbb{R}^n$, where $\mathsf{y}_i = \mathsf{x}_i + \mathsf{z}_i$. Let $\{\mathsf{z}_i\}$ (and $\mathsf{z}_o = \mathsf{x}_o$) be IID on the n-dimensional sphere centered at o with radius r_T, the TX-RX separation distance. A realization of Φ is denoted ϕ.

Let time be slotted and indexed by $k \in \mathbb{N}$. We consider multiple time instants under the (slotted) Aloha protocol with contention parameter $p \in (0,1)$: each potential TX attempts transmission at each time with probability p, independently of other nodes and independent across time slots. Let $\bar{p} \equiv 1 - p$.

Define RVs $\mathsf{T} \equiv (\mathsf{T}_{i,k}, (i,k) \in \mathbb{N}^2)$ with $\mathsf{T}_{i,k} \sim \mathrm{Ber}(p)$, and $\mathsf{T}_{i,k} = 1$ denoting TX i attempts transmission at time k. Under Aloha T is IID in nodes i and times k. Let $\mathcal{T}_k \equiv \{i \in \mathbb{N} : \mathsf{T}_{i,k} = 1\}$ be the random set of TXs (not including the reference TX at x_o) attempting transmission at time k.

We further assume the time slot durations and fading coherence times are matched and synchronized, with the idealization that the RVs $\mathsf{F} \equiv (\mathsf{F}_{i,k}, (i,k) \in \mathbb{N}^2)$, with $\mathsf{F}_{i,k} \sim \mathrm{Exp}(1)$ the random fade from TX i to the reference receiver at o at time k, are likewise IID across both nodes i and times k.

The process Φ generates a sequence of identically distributed PPPs $(\Phi_k, k \in \mathbb{N})$, with $\Phi_k \subseteq \Phi$ the PPP of attempted TX at time k, with intensity λp, and $\mathsf{T}_{i,k} = \mathbf{1}_{\mathsf{x}_i \in \Phi_k}$. Equivalently, we view $\Phi_k = \{(\mathsf{x}_i, (\mathsf{z}_i, \mathsf{T}_{i,k}, \mathsf{F}_{i,k})\}$ as the process Φ augmented with IID marks $(\mathsf{T}_{i,k}, \mathsf{F}_{i,k})$ for each $i \in \mathbb{N}$. The elements of $\{\Phi_k\}$ are dependent due to their shared connection with Φ, but are conditionally independent given Φ, due to the independent transmission attempts and fades.

C. Physical reception model

We assume a (standard) signal propagation model for large-scale, distance-based pathloss with Rayleigh fading, and unit transmission power. The signal power at RX o from TX i at time k is $\mathsf{P}_{i,k} \equiv \mathsf{F}_{i,k} l(\|\mathsf{x}_i\|)$, for $\mathsf{F}_{i,k}$ as above, with $l(r) \equiv r^{-\alpha}$ the pathloss function with exponent $\alpha > n$, and $\|\mathsf{x}_i\|$ the (random) distance from TX i to RX o (note: $\|\mathsf{x}_o\| = r_\mathrm{T}$, by assumption). Call $\delta \equiv n/\alpha < 1$ the characteristic exponent.

A transmission between the reference TX-RX pair o is considered successful under the physical interference model if the (random) SIR[1] at the reference RX, denoted $\Sigma_{o,k}$, exceeds an SIR threshold $\beta > 0$, with $\Sigma_{o,k} \equiv \mathsf{P}_{o,k}/\mathsf{I}_{o,k}$, and $\mathsf{I}_{o,k} \equiv \sum_{i \in \mathcal{T}_k} \mathsf{P}_{i,k}$ the (random) sum interference power at RX o at time k. The Bernoulli RV $\mathsf{H}_k \equiv \mathbf{1}\{\Sigma_{o,k} \geq \beta\}$ represents

[1]To simplify the presentation we assume throughout there is no noise at the receiver, i.e., the SINR equals the SIR. However, this assumption is inessential and the results are easily extensible to include noise.

physical model success or failure of the reference transmission o in Φ_k at time k. We also employ the convenience functions

$$\begin{array}{rclrcl} \kappa_\delta & \equiv & \Gamma(1+\delta)\Gamma(1-\delta), & \sigma & \equiv & \beta r_{\mathrm{T}}^{\alpha} \\ \mu_h & \equiv & \lambda c_n \kappa_\delta \sigma^\delta, & I(u,\delta) & \equiv & \delta \int_0^u \frac{t^\delta}{1+t}\mathrm{d}t \end{array} \quad (1)$$

The probability of the reference transmission attempt being successful under the physical model in some time slot k, say, is denoted $p_{\mathsf{H}}(1) = \mathbb{P}(\mathsf{H}_k = 1)$, and is given by

$$p_{\mathsf{H}}(1) = \exp\left(-\lambda p c_n \kappa_\delta \sigma^\delta\right) = \mathrm{e}^{-\mu_h p}. \quad (2)$$

This follows from standard stochastic geometry arguments on the outage probability of the power law pathloss function with Rayleigh fading for a PPP [9, p. 104] (c.f., [3, Lem. 1]).

D. Protocol reception model

We also employ a (standard) protocol interference model, characterized by a guard zone distance r_{O}. A transmission between TX-RX pair o is considered successful under the protocol interference model iff there are no interfering TX's within distance r_{O} of the reference RX at o. The Bernoulli RV $\mathsf{D}_k \equiv \mathbf{1}\{\|\mathsf{x}_i\| \geq r_{\mathrm{O}}, \forall i \in \mathcal{T}_k\}$ represents the success or failure under the protocol model of the reference transmission o in Φ_k at time k. We also employ the convenience functions

$$\mu_d \equiv \lambda c_n r_{\mathrm{O}}^n,\ \chi \equiv \frac{r_{\mathrm{O}}^\alpha}{\sigma} = \frac{1}{\beta}\left(\frac{r_{\mathrm{O}}}{r_{\mathrm{T}}}\right)^\alpha,\ \xi(p) \equiv \frac{p}{\bar{p}}\chi^{-\delta} I(\chi,\delta). \quad (3)$$

The probability of the reference transmission attempt being successful under the protocol model in some time slot k, say, is denoted $p_{\mathsf{D}}(1) = \mathbb{P}(\mathsf{D}_k = 1)$, and is given by

$$p_{\mathsf{D}}(1) = \exp\left(-\lambda p c_n r_{\mathrm{O}}^n\right) = \mathrm{e}^{-\mu_d p}. \quad (4)$$

E. Observations

Let $N \in \mathbb{N}$ be the number of prior observations, in each of which the reference transmission has been attempted. We consider separately the cases when these observations indicate success or failure under the physical model (denoted $h^{(N)} = (h_1, \ldots, h_N) \in \{0,1\}^N$), and the protocol model (denoted $d^{(N)} = (d_1, \ldots, d_N) \in \{0,1\}^N$). In particular, $h_k = 1$ (0), indicates that the reference transmission attempt was (not) successful under the physical model at time $k \in [N]$, and $d_k = 1$ (0) indicates that the reference transmission attempt was (not) successful under the protocol model at time $k \in [N]$. Observe $K_h^{(N)} \equiv \sum_{k \in [N]} h_k$ and $K_d^{(N)} \equiv \sum_{k\in[N]} d_k$ are sufficient statistics for $h^{(N)}$ and $d^{(N)}$, respectively. Given either $K_h^{(N)}$ or $K_d^{(N)}$, the observer is asked to predict the outcome under the physical model $h_{N+1} \in \{0,1\}$. The corresponding RVs are denoted $\mathsf{H}_{N+1}, \mathsf{K}_d^{(N)}, \mathsf{K}_h^{(N)}$.

III. Full knowledge of potential transmitters

Given full knowledge of the locations of the potential transmitters, i.e., $\Phi = \phi$ (i.e., knowledge of the realization $\mathsf{x}_i = x_i$ for $i \in \mathbb{N}$), the event that the reference TX attempt is successful under the physical model in a given time slot, say k, is a function of the remaining RVs in the model, namely, the transmission decisions T_k and the fading coefficients F_k.

Proposition 1. *The physical model feasibility RV H_k given $\Phi = \phi$ (i.e., $\mathsf{x}_i = x_i$ for $i \in \mathbb{N}$) has distribution*

$$p_{\mathsf{H}|\Phi}(1|\phi) \equiv \mathbb{P}(\mathsf{H}_k = 1 | \Phi = \phi) = \prod_{i\in\mathbb{N}} \frac{1 + \bar{p}\sigma|x_i|^{-\alpha}}{1+\sigma|x_i|^{-\alpha}}. \quad (5)$$

Proof: The definitions in §II-C give

$$\begin{aligned} p_{\mathsf{H}|\Phi}(1|\phi) &= \mathbb{P}(\Sigma_{o,k} > \beta | \Phi = \phi) \\ &= \mathbb{P}(r_{\mathrm{T}}^{-\alpha}\mathsf{F}_{o,k} > \beta \mathsf{I}_{o,k} | \Phi = \phi) \\ &= \mathbb{P}(\mathsf{F}_{o,k} > \sigma \mathsf{I}_{o,k} | \Phi = \phi) \\ &= \mathbb{E}[\mathbb{P}(\mathsf{F}_{o,k} > \sigma \mathsf{I}_{o,k} | \Phi = \phi, \mathsf{I}_{o,k}) | \Phi = \phi] \\ &= \mathbb{E}[\mathrm{e}^{-\sigma \mathsf{I}_{o,k}} | \Phi = \phi] \end{aligned} \quad (6)$$

Note $\mathsf{I}_{o,k}$, given $\Phi = \phi$, equals $\sum_{i\in\mathbb{N}} |x_i|^{-\alpha}\mathsf{T}_{i,k}\mathsf{F}_{i,k}$, where the TX locations (x_i) are known, while the TX decisions T_k and fades F_k are random. Thus

$$\begin{aligned} p_{\mathsf{H}|\Phi}(1|\phi) &= \mathbb{E}\left[\exp\left(-\sigma \sum_{i\in\mathbb{N}} |x_i|^{-\alpha}\mathsf{T}_{i,k}\mathsf{F}_{i,k}\right)\middle| \Phi = \phi\right] \\ &= \mathbb{E}\left[\prod_{i\in\mathbb{N}} \exp\left(-\sigma|x_i|^{-\alpha}\mathsf{T}_{i,k}\mathsf{F}_{i,k}\right)\middle| \Phi = \phi\right] \\ &= \prod_{i\in\mathbb{N}} \mathbb{E}\left[\exp\left(-\sigma|x_i|^{-\alpha}\mathsf{T}_{i,k}\mathsf{F}_{i,k}\right)\middle| \Phi = \phi\right] \end{aligned} \quad (7)$$

It is simple to compute, for $\mathsf{T} \sim \mathrm{Ber}(p)$ and $\mathsf{F} \sim \mathrm{Exp}(1)$, that

$$\mathbb{E}\left[\mathrm{e}^{-s\mathsf{TF}}\right] = \frac{1+\bar{p}s}{1+s}. \quad (8)$$

Substitution of (8) into (7) yields (5). ■

The product in (5) has an infinite number of terms, over $i \in \mathbb{N}$, yet very accurate approximations of $p_{\mathsf{H}|\Phi}(1|\phi)$ are obtainable by computing the product for only the nearest R (say) interferers, i.e., over points $\{1, \ldots, R\}$ when the points in Φ are labeled such that $|\mathsf{x}_1| < |\mathsf{x}_2| < \cdots$. This approximation is employed in §VII-C where $\mathsf{R} = \Phi(\mathcal{A}_l^n)$ is the (random) number of points from Φ lying in the (bounded) arena $\mathcal{A}_l^n$.

The protocol model RV D_k given $\Phi = \phi$ has distribution

$$p_{\mathsf{D}|\Phi}(1|\phi) \equiv \mathbb{P}(\mathsf{D}_k = 1 | \Phi = \phi) = \bar{p}^M, \quad (9)$$

where $M = M(\phi) = \phi(b(o, r_{\mathrm{O}}))$ is the number of potential interferers within the observation disk $b(o, r_{\mathrm{O}})$.

IV. Knowledge of number of nearby potential transmitters

Instead of full knowledge of the locations of the potential transmitters, i.e., $\Phi = \phi$, suppose instead we are given only knowledge of the number of potential transmitters within the observation radius, r_{O}, i.e., $\mathsf{M} = \Phi(b(o, r_{\mathrm{O}}))$. Define $p_{\mathsf{H}|\mathsf{M}}(1|M) \equiv \mathbb{P}(\mathsf{H}_k = 1 | \mathsf{M} = M)$ as the conditional distribution on H_k given $\mathsf{M} = M$. This quantity is given in Prop. 2, from recent prior work by the author, [3, Thm. 2].

Proposition 2 (Thm. 2 in [3]). *The physical model feasibility RV H_k given $\mathsf{M} = M$ has distribution*

$$p_{\mathsf{H}|\mathsf{M}}(1|M) = \mathrm{e}^{\mu_d p (1 - \chi^{-\delta}(\kappa_\delta + I(\chi,\delta)))} (1+\xi)^M \bar{p}^M. \quad (10)$$

The proof is found in Appendix G (proof of Thm. 2) in [3].

There is an intuitive tradeoff in the selection of the observation radius r_{O}. First, for small r_{O} the observation ball $b(o, r_{\mathrm{O}})$ will be empty with high probability, in which case the typical observation $\mathsf{M} = 0$ of Φ carries little information about the success probability of the reference transmission under the physical model, H_k. But, if the ball *is* nonempty ($\mathsf{M} > 0$) it will signal the presence of one or more strong (nearby) interferers, and in turn will indicate the likely (at least for large p) failure of the reference transmission under the physical model, $\mathsf{H}_k = 0$. Second, for large r_{O} the ball $b(o, r_{\mathrm{O}})$ will be nonempty with high probability, in which case the typical observation $\mathsf{M} > 0$ carries little information about the success probability of the reference transmission under the physical model, H_k. But, if the ball *is* empty, i.e., $\mathsf{M} = 0$, this will indicate the likely success of the reference transmission under the physical model, $\mathsf{H}_k = 1$. The correlation between physical and protocol model observations is addressed in [3, §4], and we leverage the main result, Prop. 2 in that work, which gives the correlation $\rho_{\mathsf{H},\mathsf{D}}(\chi)$ as a function of $\chi = \chi(r_{\mathrm{O}}) = r_{\mathrm{O}}^{\alpha}/\sigma$, a reparameterization of the observation radius r_{O}. The radius r_{O} will be selected to maximize this correlation in §VII-D.

V. Multiple physical model observations

Suppose that the observer sees K *physical* model successes out of N observations, i.e., sees $h_1, \ldots, h_N$ corresponding to the realizations $\phi_1, \ldots, \phi_N$ (recall §II-E). Define $p_{\mathsf{H}|\mathsf{K}_h}(1|K) \equiv \mathbb{P}(\mathsf{H}_{N+1} = 1|\mathsf{K}_h^{(N)} = K)$ as the conditional distribution on H_{N+1} given the observation of K out of N physical model successes. The distribution $p_{\mathsf{H}|\mathsf{K}_h}(1|K)$ is given in Cor. 1, a simple extension of [4, Thm. 1], which gives the probability of N successes under the Aloha protocol in terms of the *diversity polynomial* $D_h(k, p, \delta)$ in (12) below. Whereas [4] gives $i)$ the probability of N successes (§III-A), $ii)$ the conditional probability of success/failure given a single previous success/failure, $iii)$ the probability of one or more successful transmissions out of N, and $iv)$ the conditional probability of failure given N previous failures (§III-D), our formulation below gives the (slightly) more general probability of success conditioned on K successes in N attempts.

Corollary 1 (of Thm. 1 in [4])**.** *The distribution of the physical model feasibility RV H_{N+1} given $\mathsf{K}_h^{(N)} = K$ out of N physical model successes were observed is*

$$p_{\mathsf{H}|\mathsf{K}_h}(1|K) = \frac{\sum_{j=0}^{N-K} \binom{N-K}{j} (-1)^j \mathrm{e}^{-\mu_h p D_h(K+j+1,p,\delta)}}{\sum_{j=0}^{N-K} \binom{N-K}{j} (-1)^j \mathrm{e}^{-\mu_h p D_h(K+j,p,\delta)}} \tag{11}$$

for $\mu_h \equiv \lambda c_n \kappa_\delta \sigma^\delta$, and D_h [4, Def. 1], defined, for $k \in \mathbb{N}$, $0 < p < 1$, and $0 < \delta < 1$, as:

$$D_h(k, p, \delta) \equiv \sum_{l=1}^{k} \binom{k}{l} \binom{\delta - 1}{l - 1} p^l. \tag{12}$$

The proof is given in App. A. For a given realization $\Phi = \phi$ of the potential transmitters, we compute $p_{\mathsf{H}|\Phi}(1|\phi)$ in (5), denoted here by the shorthand $p_h(\phi)$. We may then use it to compute $\mathbb{E}[p_{\mathsf{H}|\mathsf{K}_h}(1|\mathsf{K}_h)|\Phi = \phi]$, where, conditioned on $\Phi = \phi$, the RV K_h has a binomial distribution with parameters N and $p_h(\phi)$, i.e., $\mathsf{K}_h(\phi) \sim \mathrm{Bin}(N, p_h(\phi))$, yielding:

$$\mathbb{E}[p_{\mathsf{H}|\mathsf{K}_h}(1|\mathsf{K}_h)|\Phi = \phi] = \sum_{K=0}^{N} \binom{N}{K} p_h(\phi)^K \bar{p}_h(\phi)^{N-K} p_{\mathsf{H}|\mathsf{K}_h}(1|K). \tag{13}$$

This quantity is, for a given realization ϕ, the expected estimate of the success probability of the reference transmission under the physical model, where the expectation is with respect to the random number of observed successes, $\mathsf{K}_h(\phi)$.

VI. Multiple protocol model observations

Suppose now instead that the observer sees K *protocol* model successes out of N observations. Define $p_{\mathsf{H}|\mathsf{K}_d}(1|K) \equiv \mathbb{P}(\mathsf{H}_{N+1} = 1|\mathsf{K}_d^{(N)} = K)$ as the conditional distribution on H_{N+1} given K out of N protocol model successes. The distribution $p_{\mathsf{H}|\mathsf{K}_d}(1|K)$ is given in Prop. 3 below, taken from recent prior work by the author, [3, Prop. 10].

Proposition 3 (Prop. 10 in [3])**.** *The distribution of the physical model feasibility RV H_{N+1} given $\mathsf{K}_d^{(N)} = K$ out of N protocol model successes were observed is: $p_{\mathsf{H}|\mathsf{K}_d}(1|K)$*

$$= \mathrm{e}^{\mu_d p (1 - \chi^{-\delta} \kappa_\delta + \xi(p))} \frac{D_d(\mu_d(1 + \xi(p)), \bar{p}, K + 1, N - K)}{D_d(\mu_d, \bar{p}, K, N - K)}. \tag{14}$$

for $\chi \equiv r_{\mathrm{O}}^{\alpha}/\sigma$, $\xi(p) \equiv \frac{p}{\bar{p}} \chi^{-\delta} I(\chi, \delta)$, $\mu_d \equiv \lambda c_n r_{\mathrm{O}}^n$, and[2]

$$D_d(\nu, a; k, l) \equiv \sum_{j=0}^{l} \binom{l}{j} (-1)^j \mathrm{e}^{-\nu(1 - a^{k+j})}. \tag{15}$$

The proof of Prop. 3 is in Appendix H in [3].

For a given realization $\Phi = \phi$ of the potential transmitters, we compute $p_{\mathsf{D}|\Phi}(1|\phi)$ in (9), denoted here by the shorthand $p_d(\phi)$. We may then use it to compute $\mathbb{E}[p_{\mathsf{H}|\mathsf{K}_d}(1|\mathsf{K}_d)|\Phi = \phi]$, where, conditioned on $\Phi = \phi$, the RV $\mathsf{K}_d(\phi)$ has a binomial distribution with parameters N and $p_d(\phi)$, i.e.,

$$\mathbb{E}[p_{\mathsf{H}|\mathsf{K}_d}(1|\mathsf{K}_d)|\Phi = \phi] = \sum_{K=0}^{N} \binom{N}{K} p_d(\phi)^K \bar{p}_d(\phi)^{N-K} p_{\mathsf{H}|\mathsf{K}_d}(1|K). \tag{16}$$

This quantity is, for a given realization ϕ, the expected estimate of the success probability of the reference transmission, where the expectation is with respect to the random number of observed successes, $\mathsf{K}_d(\phi)$, under the protocol model.

A. Estimating M from K successes in N protocol observations

Protocol model observations provide estimates of the success probability of the reference transmission under the physical model by giving improved estimates of the number of potential transmitters $M = \phi(b(o, r_{\mathrm{O}}))$ within the observation ball. An improved estimate of M improves the estimated

[2] D_d in (15) is denoted by f_d in our prior work [3] (c.f., Eq. (32) in that work). The change is to emphasize that D_d plays a role in Prop. 3 similar to that of D_h in Cor. 1, while preserving notation for D_h used in [4].

success probability relative to not having the protocol observations. Prop. 4 gives the expected value of M conditioned on observing K successes out of N protocol model observations.

Define a variant on D_d in (15):

$$\tilde{D}_d(\nu, a; k, l) \equiv \sum_{j=0}^{l} \binom{l}{j} (-1)^j \nu a^{k+j} \mathrm{e}^{-\nu(1-a^{k+j})}. \quad (17)$$

Proposition 4. *Given K protocol successes in N trials of the reference TX, the conditional expectation of M is*

$$\mathbb{E}[\mathsf{M}|\mathsf{K}_d = K] = \frac{\tilde{D}_d(\mu_d, \bar{p}; K, N-K)}{D_d(\mu_d, \bar{p}; K, N-K)}. \quad (18)$$

Proof: Lemma 8 in [3] gives

$$\frac{\mathbb{P}(\mathsf{M} = m|\mathsf{K}_d = K)}{\mathbb{P}(\mathsf{M} = m)} = \frac{g_d(m, \bar{p}; K, N-K)}{D_d(\mu_d, \bar{p}; K, N-K)}, \quad (19)$$

for $g_d(m, a; k, l) \equiv (a^m)^k (1-a^m)^l$ and $\mathsf{M} \sim \mathrm{Po}(\mu_d)$. Substitution of (19) gives $\mathbb{E}[\mathsf{M}|\mathsf{K}_d = K]$

$$\begin{aligned} &\equiv \sum_{m=0}^{\infty} m \mathbb{P}(\mathsf{M} = m|\mathsf{K}_d = K) \\ &= \frac{\sum_{m=0}^{\infty} m \mathbb{P}(\mathsf{M} = m) g_d(m, \bar{p}; K, N-K)}{D_d(\mu_d, \bar{p}; K, N-K)} \\ &= \frac{\mathbb{E}[\mathsf{M} g_d(\mathsf{M}, \bar{p}; K, N-K)]}{D_d(\mu_d, \bar{p}; K, N-K)} \end{aligned} \quad (20)$$

Application of the binomial theorem to $m g_d$ gives

$$m g_d(m, \bar{p}; K, N-K) = \sum_{j=0}^{N-K} \binom{N-K}{j} (-1)^j m (\bar{p}^{K+j})^m. \quad (21)$$

Application of Lemma 6 in [3] to the case $g(m) = m$ and $\mathsf{M} \sim \mathrm{Po}(\mu_d)$ gives, for $a > 0$:

$$\mathbb{E}[\mathsf{M}(a^b)^{\mathsf{M}}] = \mathrm{e}^{-\mu_d(1-a^b)} a^b \mu_d. \quad (22)$$

Replacing the numerator in (20) with the expectation (with respect to $\mathsf{M} \sim \mathrm{Po}(\mu_d)$) of (21), and substituting (22) (with $a^b = \bar{p}^{K+j}$) yields (18). ■

Following the same argument used in support of (16), we can take an expectation with respect to the conditional distribution of $\mathsf{K}_d(\phi) \sim \mathrm{bin}(N, p_d(\phi))$, to yield the expectation (with respect to $\mathsf{K}_d(\phi)$) of $\mathbb{E}[\mathsf{M}|\mathsf{K}_d(\phi)]$:

$$\mathbb{E}[\mathbb{E}[\mathsf{M}|\mathsf{K}_d]|\Phi = \phi] = \sum_{K=0}^{N} \binom{N}{K} p_d(\phi)^K \bar{p}_d(\phi)^{N-K} \mathbb{E}[\mathsf{M}|\mathsf{K}_d = K]. \quad (23)$$

Use (23) to define the expected error in the estimate of M:

$$\tilde{\eta}_M \equiv \mathbb{E}[|\mathsf{M} - \mathbb{E}[\mathbb{E}[\mathsf{M}|\mathsf{K}_d]|\Phi]|]. \quad (24)$$

The outer expectation may be computed numerically by a Monte-Carlo average over realizations of Φ.[3]

[3]In fact, as $\mathsf{K}_d(\phi)$ depends upon ϕ only through $M(\phi)$, it suffices to simply take the expectation with respect to $\mathsf{M} \sim \mathrm{Po}(\mu_d)$.

VII. Numerical results

The results give the probability of success of the reference transmission under the physical model conditioned on five distinct forms of knowledge of the realized point process of potential interferers, ϕ: $i)$ §II-C zero knowledge of ϕ, aside from the model parameters (λ, p, δ), with $p_{\mathsf{H}}(1)$ in (2); $ii)$ §III full knowledge of ϕ, with $p_{\mathsf{H}|\Phi}(1|\phi)$ (5) in Prop. 1; $iii)$ §IV knowledge of the number of nearby potential interferers $M(\phi) = \phi(b(o, r_{\mathrm{O}}))$, with $p_{\mathsf{H}|\mathsf{M}}(1|M)$ (10) in Prop. 2; $iv)$ §V K successes out of N observations of the reference transmission attempt under the *physical* model, with estimate $p_{\mathsf{H}|\mathsf{K}_h}(1|K)$ (11) in Cor. 1 for a given K, and $\mathbb{E}[p_{\mathsf{H}|\mathsf{K}_h}(1|\mathsf{K}_h)|\Phi = \phi]$ (13) the expectation with respect to the distribution of K_h conditioned on $\Phi = \phi$; and $v)$ §VI K successes out of N observations of the reference transmission attempt under the *protocol* model, with estimate $p_{\mathsf{H}|\mathsf{K}_d}(1|K)$ (14) in Prop. 3 for a given K, and $\mathbb{E}[p_{\mathsf{H}|\mathsf{K}_d}(1|\mathsf{K}_d)|\Phi = \phi]$ (16) the expectation with respect to the distribution of K_d conditioned on $\Phi = \phi$.

A. Figures of merit

We define four figures of merit to quantify the relative value of these different forms of knowledge, with the outer expectation in all cases taken with respect to the random Φ.

$i)$ The expected error in the reference TX success probability under the physical model for no knowledge of ϕ:

$$\eta_\phi \equiv \mathbb{E}[|p_{\mathsf{H}}(1) - p_{\mathsf{H}|\Phi}(1|\Phi)|]. \quad (25)$$

$ii)$ The expectation of the cost in the accuracy of estimating the probability of success of the reference TX under the physical model from knowing only $M(\phi)$, relative to full knowledge of ϕ (observe η_M is distinct from $\tilde{\eta}_M$ in (24)):

$$\eta_M \equiv \mathbb{E}\left[|p_{\mathsf{H}|\mathsf{M}}(1|\mathsf{M}) - p_{\mathsf{H}|\Phi}(1|\Phi)|\right]. \quad (26)$$

$iii)$ The expectation of the cost in the accuracy of estimating the probability of success of the reference TX under the physical model from N *physical* model observations, relative to full knowledge of ϕ:

$$\eta_h \equiv \mathbb{E}\left[|\mathbb{E}[p_{\mathsf{H}|\mathsf{K}_h}(1|\mathsf{K}_h)|\Phi] - p_{\mathsf{H}|\Phi}(1|\Phi)|\right]. \quad (27)$$

$iv)$ The expectation of the cost in the accuracy of estimating the probability of success of the reference TX under the physical model from N *protocol* model observations, relative to full knowledge of ϕ:

$$\eta_d \equiv \mathbb{E}\left[|\mathbb{E}[p_{\mathsf{H}|\mathsf{K}_d}(1|\mathsf{K}_d)|\Phi] - p_{\mathsf{H}|\Phi}(1|\Phi)|\right]. \quad (28)$$

The outer expectation in the four metrics will be computed via Monte-Carlo simulation, as explained below.

B. Simulation methodology

Fix the network dimension $n \in \{1, 2, 3\}$, and restrict the network domain from $\mathbb{R}^n$ to the bounded arena $\mathcal{A}_l^n \equiv [-l/2, l/2]^n$ with side length l, area l^n, and the origin o at the center of the arena. Fix the TX-RX separation distance r_{T}, the pathloss exponent $\alpha > n$, the transmission probability p, the observation radius r_{O}, and the spatial density of potential

TABLE II
SIMULATION PARAMETERS

Number of simulations	S	$10,000$
Network dimension	n	2 (planar)
Arena side length	l	$1,000$ (meters)
Spatial intensity of Φ	λ	$1/1000$
Constant TX-RX separation distance	r_{T}	10 (meters)
Slotted Aloha TX probability	p	$\{1/10, 1/4, 1/2, 1\}$
Large-scale pathloss constant	α	4
SIR threshold	β	5

interferers λ. Let $S \in \mathbb{N}$ be the number of independent realizations of the interference process to be used in the simulation, and let $(\Phi^{(1)}, \ldots, \Phi^{(S)})$ be the IID random realizations of the potential interferer locations, so that $\Phi_k^{(s)}$ gives the random interferers seen at time k under potential interferers $\Phi^{(s)}$. Let the Poisson RV $\mathsf{R}_s \equiv \Phi^{(s)}(\mathcal{A}_l^n) \sim \mathrm{Po}(\lambda l^n)$ count the number of potential interferers from $\Phi^{(s)}$ in $\mathcal{A}_l^n$, and recall that, conditioned on $\mathsf{R}^{(s)}$, the points $\Phi^{(s)} \cap \mathcal{A}_l^n$ are independently and uniformly distributed within $\mathcal{A}_l^n$. Thus each (truncated) random realization is in fact obtained by first generating $\mathsf{R}^{(s)} \sim \mathrm{Po}(\lambda l^n)$, and then placing $\mathsf{R}^{(s)}$ points independently and uniformly at random in $\mathcal{A}_l^n$. Let $(\tilde{\phi}^{(1)}, \ldots, \tilde{\phi}^{(S)})$ denote the realizations of these truncated random PPPs.

For each realization $\tilde{\phi}^{(s)}$ compute: $i)$ $p_{\mathsf{H}|\Phi}(1|\tilde{\phi}^{(s)})$ (5) in Prop. 1 where the number of terms in the product is the (almost surely) finite number $R^{(s)} = |\tilde{\phi}^{(s)}|$; $ii)$ $p_{\mathsf{H}|\mathsf{M}}(1|M^{(s)})$ (10) in Prop. 2 where $M^{(s)} = \tilde{\phi}^{(s)}(b(o, r_{\mathsf{O}}))$; $iii)$ $\mathbb{E}[p_{\mathsf{H}|\mathsf{K}_h}(1|\mathsf{K}_h)|\tilde{\phi}^{(s)}]$ (13); and $iv)$ $\mathbb{E}[p_{\mathsf{H}|\mathsf{K}_d}(1|\mathsf{K}_d)|\tilde{\phi}^{(s)}]$ (16). Next, use these four numbers to compute the quantities inside the outer expectations in (25), (26), (27), and (28). Finally, average each metric over the S realizations to approximate the expectation over Φ.

The simulation results below were obtained with parameters given in Tab. II. We employed a planar network ($n = 2$) with the network arena a square with side length of 1 km, and a spatial intensity of $\lambda = 1/1000$. Thus $\mathsf{R}^{(s)} = \Phi(\mathcal{A}_l^n)$ is Poisson with on average $\mathbb{E}[\mathsf{R}^{(s)}] = 1,000$ potential TX in the arena. $S = 10,000$ independent realizations were generated.

The "zero-knowledge" probability of success of the reference TX under the physical model for the four p values are

p	$1/10$	$1/4$	$1/2$	1
$p_{\mathsf{H}}(1)$	0.8955	0.7589	0.5759	0.3317

. (29)

The decrease of $p_{\mathsf{H}}(1)$ (c.f. (2)) in p is due to the average intensity of the interfering point process, λp increasing in p.

C. Simulation results (1): impact of full knowledge of ϕ (η_ϕ)

Fig. 1 shows histograms of the sucess probability of the reference transmission under the physical model conditioned on knowledge of ϕ (left) and of the absolute value of the change in the estimate relative to the "zero-knowledge" estimate $p_{\mathsf{H}}(1)$. The vertical lines on the left show $p_{\mathsf{H}}(1)$ from (29), while the vertical lines on the right are the average change, i.e., η_ϕ (25). As is evident, the average improvement in the estimate from full knowledge of ϕ, i.e., η_ϕ, is increasing in p, with full knowledge of ϕ affording an average 25% improvement in the estimation of the success probability over $p_{\mathsf{H}}(1)$ for $p = 1$.

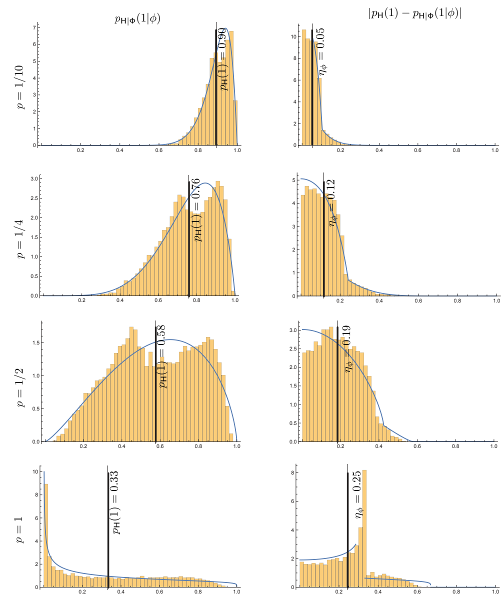

Fig. 1. Histograms of $p_{\mathsf{H}|\Phi}(1|\phi)$ (left) and $|p_{\mathsf{H}}(1) - p_{\mathsf{H}|\Phi}(1|\phi)|$ (right) for TX probabilities $p \in \{1/10, 1/4, 1/2, 1\}$ (four rows), for $S = 10,000$ independent realizations of Φ. The solid curves are the beta distribution approximation of the SIR meta-distribution [5] (left) and the derived distribution of the distance from the mean (right).

The RV $p_{\mathsf{H}|\Phi}(1\Phi)$ has the SIR "meta distribution" [5] (see Acknowledgment), which may be approximated by the beta distribution [5] (§II-F). Fig. 1 shows the approximation provides an excellent match to the histogram. The left probability density function (PDF) in Fig. 1 is the beta distribution $f_{\mathsf{u}}(u; \mu, \beta)$ on $u \in [0, 1]$ with parameters (μ, β) (such that $\mathbb{E}[\mathsf{u}] = \mu$ and $\mathrm{var}(\mathsf{u}) = \frac{\mu\bar{\mu}^2}{\beta + \bar{\mu}})$. The parameters (μ, β) are expressible in terms of the model parameters via $\mu = \mathrm{e}^{-\lambda\kappa_\delta\beta^\delta p}$ and $\beta = \frac{(\mu - M_2)\bar{\mu}}{M_2 - \mu^2}$, for $M_2 = \mathrm{e}^{-\lambda\kappa_\delta\beta^\delta(2p + (\delta - 1)p^2)}$ [5]. The right PDF is $f_{\mathsf{v}}(v; \mu, \beta)$ for $\mathsf{v} \equiv |\mathsf{u} - \mu|$, and is expressible in terms of $f_{\mathsf{u}}(u)$:

$$f_{\mathsf{v}}(v) = \begin{cases} f_{\mathsf{u}}(\mu + v) + f_{\mathsf{u}}(\mu - v), & 0 \le v \le \min\{\mu, \bar{\mu}\} \\ f_{\mathsf{u}}(\mu + v), & \mu \le v \le \bar{\mu} \\ f_{\mathsf{u}}(\mu - v), & \bar{\mu} \le v \le \mu \\ 0, & \max\{\mu, \bar{\mu}\} \le v \le 1 \end{cases} \tag{30}$$

D. Simulation results (2): impact of knowledge of M (η_M)

Recall the discussion of the tradeoff inherent in the selection of the observation radius r_{O} in §IV. Fig. 2 shows the correlation $\rho_{\mathsf{H},\mathsf{D}}(\chi)$ as a function of $\chi = \chi(r_{\mathsf{O}}) = r_{\mathsf{O}}^\alpha/\sigma$ for various spatial intensities λp, with $p \in \{1/10, 1/4, 1/2, 1\}$. The decrease in correlation in p indicates the presence or absence of a "nearby" interferer more strongly correlates with physical model success or failure for small effective spatial intensities. As evident from the figure, while the optimized correlation level is sensitive to χ, there is little sensitivity in the maximizing choice of χ, and $\chi = 1.67$, corresponding to $r_{\mathsf{O}} = 17$ meters, is near-optimal for all p, and as such we henceforth fix this value for all remaining results.

Fig. 3 quantifies the value in knowing $M(\phi)$ in estimating the probability of success of the reference transmission under

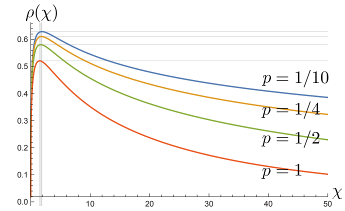

Fig. 2. Correlation $\rho_{\mathsf{H},\mathsf{D}}(\chi)$ of the success indicators under the physical model H and the protocol model D, as a function of $\chi = \chi(r_{\mathrm{O}}) = r_{\mathrm{O}}^{\alpha}/\sigma$ for spatial intensity λp, with $p \in \{1/10, 1/4, 1/2, 1\}$.

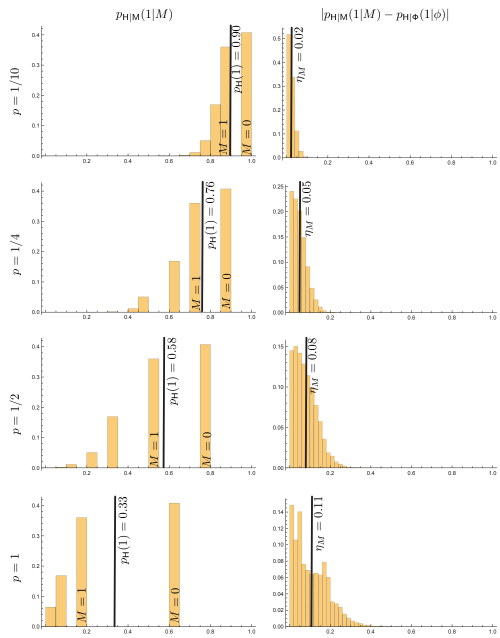

Fig. 3. Histograms of $p_{\mathsf{H}|\mathsf{M}}(1|M)$ (left) and $|p_{\mathsf{H}|\mathsf{M}}(1|M) - p_{\mathsf{H}|\Phi}(1|\phi)|$ (right) for TX probabilities $p \in \{1/10, 1/4, 1/2, 1\}$ (four rows), for $S = 10,000$ independent realizations of Φ.

the physical model. Specifically, the left (right) column shows histograms for $p_{\mathsf{H}|\mathsf{M}}(1|M)$ ($|p_{\mathsf{H}|\mathsf{M}}(1|M) - p_{\mathsf{H}|\Phi}(1|\phi)|$) for each $p \in \{1/10, 1/4, 1/2, 1\}$. The vertical lines on the left are the "blind" estimates $p_{\mathsf{H}}(1)$, while those on the right are η_M in (26). The histogram bars on the left are for each possible value of M (with $M = 0$ and $M = 1$ labeled). The average changes (relative to zero knowledge) in the estimated probability of success due to knowledge of M for the four values of p are:

p	$1/10$	$1/4$	$1/2$	1
$\mathbb{E}[\lvert p_{\mathsf{H}\vert\mathsf{M}}(1\vert M) - p_{\mathsf{H}}(1)\rvert]$	0.048	0.107	0.176	0.241

(31)

The right side shows the histogram of "costs" of only knowing $M(\phi)$ relative to knowing all of ϕ, and their average, η_M. In summary, for $p = 1$ we see that knowledge of $M(\phi)$ results in average improvement in the success probability of 24%, and an average cost due to partial knowledge of 11%.

E. Simulation results (3): physical model observations (η_h)

Fig. 4 shows the impact of physical model observations, as described in §V, on the estimate of the success probability of

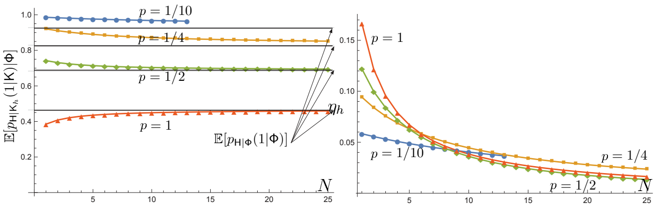

Fig. 4. Impact of physical model observations. **Left:** the average success probability of the reference transmission under the physical model having observed the outcomes of N attempted reference transmissions vs. N, for $p \in \{1/10, 1/4, 1/2, 1\}$; the horizontal gridlines show the average success probability with full knowledge of ϕ. **Right:** the average cost of partial knowledge due to physical model observations η_h relative to full knowledge of ϕ (27) vs. the number of observations N.

the reference transmission under the physical model. In particular, the left plot shows $\mathbb{E}[p_{\mathsf{H}|\mathsf{K}_h}(1|\mathsf{K})|\Phi]$ vs. the number of observations N, for the four values of $p \in \{1/10, 1/4, 1/2, 1\}$, where the horizontal gridlines are $\mathbb{E}[p_{\mathsf{H}|\Phi}(1|\Phi)]$, the average success probability given full knowledge of ϕ. The right plot computes the third peformance measure, η_h (27), the average cost of partial knowledge of ϕ, in the form of the N physical model observations, relative to full knowledge of ϕ. As is evident from the plot, the cost of partial knowledge decays rapidly with the number of observations. For example, with $p = 1$ the gap in the estimate falls from 16.5% for $N = 1$ observation down to 1.5% for $N = 25$ observations. Moreover, the decay in the cost of partial knowledge in N appears to be somewhat insensitive to the value of p, at least for these parameters. Moreover, it is worth noting that a moderate number of physical model observations suffice to achieve near perfect estimation of the success probability of the reference transmission, i.e., $N = 25$ physical model observations are almost as good as full knowledge of ϕ. Finally, the results in Fig. 4 were computed using only $S = 10$ simulations (vs. $S = 10,000$ used in the previous sections), on account of the high computational cost in evaluating (13). The restriction to at most $N = 25$ observations is likewise due to the high computational cost in evaluating (11) for large N.

F. Simulation results (4): protocol model observations (η_d)

Recall from §VI-A that protocol model observations serve to estimate the success probability of the reference transmission under the physical model by giving estimates of $\mathsf{M}(\Phi)$, the number of potential interferers located in the observation ball $b(o, r_{\mathrm{O}})$, and the expected error in this estimate of M was defined in $\tilde{\eta}_M$ in (24). Fig. 5 (left) shows this quantity vs. the number of observations N for the four values of p. As evident from the figure the expected error is in general decreasing in N, with a decreasing marginal value of each observations (measured by the decrease in the expected error) in both N and p. As is intuitive, there is little to no value in additional protocol model observations beyond the first one for $p = 1$. Finally, recall a protocol model observation simply indicates whether or not one or more interferers was active in a time slot, not the number of such interferers, and as such it is natural that the error $\tilde{\eta}_M$ may not go to zero as $N \uparrow \infty$, since knowledge of $\mathbf{1}_{\mathsf{M}>0}$ provides an imperfect estimate of M.

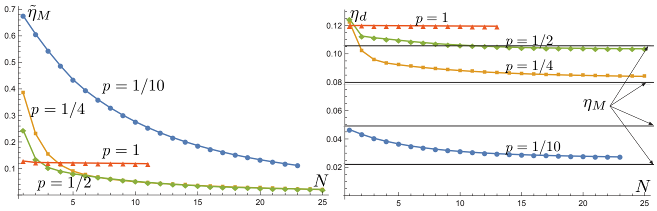

Fig. 5. **Left:** impact of protocol model observations K_d on the estimate of the number of potential nearby interferers, M, measured via the expected error $\tilde{\eta}_M$ in (24) vs. the number of observations N, for $p \in \{1/10, 1/4, 1/2, 1\}$. **Right:** impact of protocol model observations K_d on the estimated probability of success of the reference transmission under the physical model, measured via the expected error $\tilde{\eta}_d$ in (28) vs. the number of observations N, for $p \in \{1/10, 1/4, 1/2, 1\}$. The horizontal gridlines show η_M in (26), the error given full knowledge of M.

Finally, Fig. 5 (right) shows the fourth and final performance metric, η_d, the expected error in the estimate of the probability of success relative to the estimate with full knowledge of ϕ, vs. the number of observations N, for $p \in \{1/10, 1/4, 1/2, 1\}$. The gridlines show η_M in (26), the error given full knowledge of M. Observe η_M is a lower bound on η_d for all N. It appears from the figure that η_d approaches η_M as $N \uparrow \infty$, at least for p small, but this may not hold for larger p, consistent with the discussion of the error in estimating M from $\mathbf{1}_{\mathsf{M}>0}$ in Fig. 5.

VIII. Conclusion

We have $i)$ defined a framework for valuation of network observations, $ii)$ provided estimates of the reference transmission success probability under the various observations, and $iii)$ shown numerical results that quantify observation value.

Acknowledgment

The author wishes to thank Martin Haenggi for comments on a preliminary version of this manuscript, and for observing the RV $p_{\mathsf{H}|\Phi}(1|\Phi)$ has the SIR "meta distribution" [5].

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Appendix

Define the vector of RVs $\mathsf{I}_o^{(N)} \equiv (\mathsf{I}_{o,j}, j \in [N])$ as the interference seen at the reference RX at o at each time $j \in [N]$. Define $\mathsf{I}_o(A) = \sum_{j \in A} \mathsf{I}_{o,j}$ as the sum interference at o over time indices $A \subseteq [N]$ (with $\mathsf{I}_o(\emptyset) \equiv 0$), and $\mathsf{I}_o([k])$ as the sum interference over $k \in [N]$ times. Let $(\Sigma_{o,j}, j \in [N])$ be the SIR at the reference RX at each time j. Let $\mathcal{A}_K^{(N)}$ be the set of all subsets of $[N]$ of cardinality K. Define RVs $p_{h,j}(\mathsf{I}_o^{(N)}) \equiv \mathbb{P}(\Sigma_{o,j} \geq \beta | \mathsf{I}_o^{(N)})$ for $j \in [N]$, and $\bar{p}_{h,j}(\mathsf{I}_o^{(N)}) \equiv 1 - p_{h,j}(\mathsf{I}_o^{(N)})$.

Lemma 1. *The probability of K physical successes in N trials is:*

$$\mathbb{P}(\mathsf{K}_h^{(N)} = K) = \binom{N}{K} \sum_{j=0}^{N-k} (-1)^j \mathcal{L}[\mathsf{I}_o([K+j])](\sigma). \quad (32)$$

The probability of success in trial $N+1$ and K successes in the first N trials is: $\mathbb{P}(\mathsf{H}_{N+1} = 1, \mathsf{K}_h^{(N)} = K)$

$$= \binom{N}{K} \sum_{j=0}^{N-k} (-1)^j \mathcal{L}[\mathsf{I}_o([K+j+1])](\sigma). \quad (33)$$

Proof of Lem. 1: We show (32); the proof of (33) is similar. Condition on the collection of RVs $\mathsf{I}_o^{(N)}$:

$$\mathbb{P}(\mathsf{K}_h^{(N)} = K) = \mathbb{E}[\mathbb{P}(\mathsf{K}_h^{(N)} = K | \mathsf{I}_o^{(N)})]. \quad (34)$$

Conditioned on $\mathsf{I}_o^{(N)}$, the trials are independent but not identically distributed. The probability of K of N successes with success probabilities $(p_{h,j}(\mathsf{I}_o^{(N)}), j \in [N])$ is: $\mathbb{P}(\mathsf{K}_h^{(N)} = K)$

$$= \mathbb{E}\left[\sum_{A \in \mathcal{A}_K^{(N)}} \prod_{j \in A} p_{h,j}(\mathsf{I}_o^{(N)}) \prod_{j \in [N] \setminus A} \bar{p}_{h,j}(\mathsf{I}_o^{(N)}) \right]. \quad (35)$$

Use $p_{h,j}(\mathsf{I}_o^{(N)}) = \mathbb{P}(\mathsf{F}_{o,j} \geq \sigma \mathsf{I}_{o,j} | \mathsf{I}_o^{(N)}) = \mathrm{e}^{-\sigma \mathsf{I}_{o,j}}$, apply the definition of $\mathsf{I}_o(A)$, use linearity of expectation, and use the definition of the Laplace transform: $\mathbb{P}(\mathsf{K}_h^{(N)} = K)$

$$\begin{aligned} &= \mathbb{E}\left[\sum_{A \in \mathcal{A}_K^{(N)}} \mathrm{e}^{-\sigma \mathsf{I}_o(A)} \sum_{B \subseteq [N] \setminus A} (-1)^{|B|} \mathrm{e}^{-\sigma \mathsf{I}_o(B)} \right] \\ &= \sum_{A \in \mathcal{A}_K^{(N)}} \sum_{B \subseteq [N] \setminus A} (-1)^{|B|} \mathcal{L}[\mathsf{I}_o(A \cup B)](\sigma) \end{aligned} \quad (36)$$

The (unconditioned) outcomes at each time *are* identically distributed, and as such $\mathcal{L}[\mathsf{I}_o(A \cup B)](\sigma)$ depends upon the (disjoint) index sets A, B only through their cardinalities of $|A| = K$ and $|B| = j \in \{0, \ldots, N-K\}$. Observe $|\mathcal{A}_K^{(N)}| = \binom{N}{K}$ and there are $\binom{N-K}{j}$ subsets B of $\{K+1, \ldots, N\}$ of each possible size $|B| = j$, proving (32). ■

Proof of Cor. 1: The conditional distribution is the ratio:

$$p_{\mathsf{H}|\mathsf{K}_h}(1|K) = \frac{\mathbb{P}(\mathsf{H}_{N+1} = 1, \mathsf{K}_h^{(N)} = K)}{\mathbb{P}(\mathsf{K}_h^{(N)} = K)}. \quad (37)$$

Substitute expressions from Lem. 1, then use the expression for the log Laplace transform $\log \mathcal{L}[\mathsf{I}_o([k])](s) = -\mu_h(s) p D_h(k, p, \delta)$ [4, Thm. 1], for $\mu_h(s) \equiv \lambda c_n \kappa_\delta s^\delta$, evaluated at $s = \sigma$, to obtain (11). ■