Downlink Multi-User MIMO Scheduling with Performance Guarantees

Narayan Prasad*, Xiao Feng Qi*
* Futurewei, NJ e-mail: {narayan.prasad1, Xiao.Feng.Qi}@huawei.com

***Abstract*—We consider a long-standing open problem pertaining to scheduling over a wideband multi-user downlink. In this problem a base-station (BS) must assign multiple subbands to its served users such that a weighted sum rate metric is maximized subject to sum power and cardinality constraints. On each subband multiple users can simultaneously be scheduled. Such scheduling is complicated by the fact that the rate achieved by a user on any subband assigned to it depends not only on its own channel condition, but on the set of other users co-scheduled on that subband as well. The latter dependence is via the transmission scheme adopted by the BS in order to simultaneously serve multiple users on the same subband. This problem has received wide attention for over a decade and while numerous heuristics have been designed, there is no known algorithm that offers provable constant-factor worst-case guarantee. In this paper we obtain an important result which demonstrates that when the transmitter employs capacity-optimal dirty paper coding, constant-factor approximation guarantee can be achieved via simple algorithms. Indeed, we show that for a wideband scheduling problem in which a permissible set of user groups is specified as input, a simple deterministic algorithm yields a constant-factor approximation guarantee. Further, for the generalized case where any user group subject to a cardinality constraint is permissible, a greedy algorithm yields a constant-factor guarantee over certain practically relevant regimes.**

I. INTRODUCTION

The significant promise of MU-MIMO was demonstrated by the seminal works of [1]–[3] which derived theoretical limits for a broadcast channel, achieved using dirty paper coding (DPC). The ensuing investigations that have been carried out for the past decade have mainly considered more practical linear transmit precoding [5]–[7]. In industry, standardization of MU-MIMO is an ongoing effort being carried out by IEEE and 3GPP. This effort has so far led to precoded pilots being standardized. These pilots are precoded using the same transmit precoder as data symbols and are embedded in the block of data symbols sent to each user. Their key advantage is that transmit precoders no longer have to be selected from any fixed pre-determined codebook, which allows for considerable optimization of linear transmit precoding schemes. In initial field trials as well as detailed system simulations, the performance results of MU-MIMO over networks where each BS is equipped with a small number of cross-polarized transmit antennas (typically 2 or 4) have not met expectations. Fortunately, the advent of massive MIMO, which advocates the use of a large array at each BS, together with more sophisticated transceivers has once again galvanized MU-MIMO, particularly for TDD networks where channel state information can be more readily acquired. Indeed, simultaneous transmission to a multitude of users on the same spectral band is the main benefit promised by massive MIMO which is a key 5G technology [4]. Furthermore, enhanced base-stations that can perform more complex encoding as well as user terminals capable of non-linear and advanced decoding make schemes hitherto of theoretical interest, to be viable candidates for 5G networks. The emphasis now is on realizing efficient scheduling algorithms that can extract most of the gains made possible by such increased sophistication.

In this paper, we consider wideband dynamic downlink MU-MIMO scheduling with DPC, where the BS as well as users have multiple antennas. The problem at hand involves selecting a set of scheduled users and their respective covariance matrices, on each subband (once every subframe), subject to total power and several other constraints, and can be regarded as a generalization of the the popular narrowband user grouping problem in MU-MIMO. The latter problem which has been widely investigated, deals with selecting a group of users on a single subband, where each user sees a frequency-flat channel and has one receive antenna, while the BS employs linear transmit precoding [5], [7]. Due to the intractability of even this seemingly simple problem, several heuristics have been proposed (cf. [8]). A combinatorial optimization view has been adopted in [9] where it has been shown that this problem can be even hard to approximate. To the best of our knowledge, only a recent work in [17] has revealed an exploitable structure in the problem by reformulating it as the maximization of the difference of submodular set functions. However, even the latter approach fails to provide any performance guarantees. In lieu of these somewhat pessimistic results, a germane question is whether any strong analytical guarantee[1] is at all possible for MU-MIMO scheduling over relevant regimes. Our results in this paper answer the latter question in the affirmative and are summarized below.

- We first establish that the narrowband user grouping problem with DPC is a monotone submodular maximization problem in certain relevant regimes. We derive simple conditions to assess whether an input instance falls in such a regime.
- We then consider a wideband scheduling problem in which DPC can be used on each subband on any user group selected from an input set of permissible groups, subject to a wideband sum power constraint as well as multiple well justified constraints. The latter constraints include constraints on the number of assigned subbands per-user, as well as the total number of assigned subbands. We demonstrate that this problem is equivalent to submodular set function maximization subject to matroid and binary knapsack constraints. Thereby, we can design simple constant-factor approximation algorithms.

To the best of our knowledge, *these are the first results establishing submodularity for any problem involving MU-MIMO in the downlink*. On the other hand, results establishing submodularity for downlink single-user or SU-MIMO scheduling (which avoids co-scheduled user interference) are

[1] We exclude here guarantees which linearly depend on system parameters such as number of users, antennas, subbands etc.

available. In particular, [10], [11] reveal submodularity in the SU-MIMO scheduling problem with fixed powers (see also [12]), and a more recent contribution in [18] proved submodularity of waterfilling for sum rate maximization over parallel scalar channels. The latter result was then extended to weighted sum rate maximization in [19]. We note that submodularity has also been shown to hold in formulations considering the user-BS association problem [13], [14]. The formulation in [13] accounts for MU-MIMO scheduling at each BS by invoking limiting SINR expressions, which in turn assume pre-determined multiplexing gains along with channel hardening to remove the impact of short-term fading. Thereby, each user's achievable rate depends only the cardinality of the set of co-scheduled users and not its composition. In contrast, our results consider dynamic MU-MIMO scheduling and do not make any assumption on the channel statistics or system dimension. Consequently, they can be used as building blocks in a variety of resource management settings where submodularity and/or game theoretic tools are used, [15], [16], [20], [21].

In the following sections we will use boldface uppercase (lowercase) alphabets to denote matrices (vectors). Further, $|.|$ is used to denote the determinant of its matrix argument as well as cardinality of its input set. $(.)^\dagger$ is used to denote the conjugate transpose of its matrix argument while $\|.\|$ denotes the Frobenius (ℓ_2) norm of its matrix (vector) argument.

II. System Model

We consider a single-cell downlink in which one base-station (BS) equipped with N_t transmit antennas serves K users, each equipped with N_r receive antennas. Let $\mathcal{U}$ denote the set of users with cardinality $|\mathcal{U}| = K$. We consider a wide-band frequency-selective channel to model the propagation link between each user and the BS. Then, supposing that the BS employs OFDM and that all propagation delays lie within the cyclic prefix, we can model the received observations at the k^{th} user and n^{th} subband, where $n \in \mathcal{N} \triangleq \{1, \cdots, N\}$, as

$$\mathbf{y}_{k,n} = \mathbf{H}_{k,n}^\dagger \mathbf{s}_n + \boldsymbol{\eta}_{k,n}, \quad (1)$$

where $\mathbf{y}_{k,n} \in \mathbb{C}^{N_r \times 1}$ denotes the received vector of observations. $\boldsymbol{\eta}_{k,n}$ denotes the additive noise which is assumed to have $\mathcal{CN}(\mathbf{0}, \mathbf{I})$ distribution. $\mathbf{s}_n \in \mathbb{C}^{N_t \times 1}$ denotes the transmit vector on the n^{th} subband and $\mathbf{H}_{k,n} \in \mathbb{C}^{N_t \times N_r}$ models the channel matrix of user k on the n^{th} subband. A sum power constraint, $\sum_{n \in \mathcal{N}} E[||\mathbf{s}_n||^2] \leq P$, is imposed on the BS along with other practical ones that will be revealed later. We will also refer to P as the transmit SNR. Our focus here is on the design of scheduling algorithms so for simplicity we assume that the BS has perfect knowledge of all $\mathbf{H}_{k,n} \; \forall \, k \in \mathcal{U}, n \in \mathcal{N}$.

III. User Grouping

In this section we consider the by now classical problem of user grouping. This problem considers a narrowband model wherein any subset of $\mathcal{U}$ subject to a cardinality constraint can be scheduled. In order to distill the problem essence, we consider the sum rate objective and any one sub-band in (1). We also suppose that $N_r = 1$ and that $N_t \geq K$. Accordingly, we drop the subband index and denote the channel vector correspoting to user k by $\mathbf{h}_k \;\; \forall \;\; k \in \mathcal{U}$. We note that these assumptions are mainly for ease of exposition and the techniques and results presented in this section indeed extend to the original more general choice of parameters.

Let $\mathbf{H}_\mathcal{G} = [\mathbf{h}_k]_{k \in \mathcal{G}} \; \forall \; \mathcal{G} \subseteq \mathcal{U}$. By invoking uplink-downlink duality, which promises that any set of rates achieved via DPC in the downlink is achieved in the dual uplink and vice-versa [24], we can pose the DPC user grouping problem of interest to us succintly as

$$\max_{\mathcal{G} \subseteq \mathcal{U} : |\mathcal{G}| \leq J} \left\{ \underbrace{\max_{\substack{p_k \in \mathbb{R}_+ \forall \, k \in \mathcal{G} \\ \sum_{k \in \mathcal{G}} p_k \leq P}} \left\{ \log \left| \mathbf{I} + \sum_{k \in \mathcal{G}} \mathbf{h}_k p_k \mathbf{h}_k^\dagger \right| \right\}}_{\triangleq f(\mathcal{G}, P)} \right\}, \quad (2)$$

where J is the input cardinality bound. Notice that (2) is a mixed optimization problem that involves (discrete) subset or user group selection and a continuous optimization over powers for each group. Optimally solving this problem seems intractable so we seek to characterize the set function $f(., P)$ defined in (2). It is easy to see that the set function $f(., P)$ is monotone, i.e., $f(\mathcal{A}, P) \leq f(\mathcal{B}, P) \; \forall \; \mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{U}$ and normalized, i.e., $f(\phi, P) = 0$, where ϕ denotes the empty set. We now proceed to check whether it is submodular (cf. [25]), i.e., whether for any $\mathcal{A} \subset \mathcal{B} \subset \mathcal{U} : |\mathcal{B}| = |\mathcal{A}| + 1$ and any $u \in \mathcal{U} \setminus \mathcal{B}$,

$$f(\mathcal{A} \cup u, P) - f(\mathcal{A}, P) \geq f(\mathcal{B} \cup u, P) - f(\mathcal{B}, P). \quad (3)$$

Note that (3) is readily satisfied by $f(., P)$ when $\mathcal{A} = \phi$ (i.e., $\mathcal{A}$ is the empty set). Thus, we can assert that the set function is submodular when $K = 2$. As a result, we suppose $K \geq 3$ and will determine whether (3) holds when $\mathcal{A} \neq \phi$. A general result that can verify whether or not (3) holds for any given input instance is as yet elusive. Note here that for a given choice of parameters N_t, K, an instance refers to an input channel matrix $\mathbf{H}_\mathcal{U}$, a power budget P and a cardinality bound J. Clearly, a brute-force approach that considers all subsets in order to verify (3) is futile. Consequently, we analytically identify a set of instances over which (3) holds, with the understanding that readily verifiable relations should be derived in order to be able to assert whether any given input instance falls in the identified set or not. We offer our first result.

Theorem 1. *The set function $f(., P)$ defined in (2) is a submodular set function over $\mathcal{U}$ for any instance in which $\mathbf{H}_\mathcal{U}^\dagger \mathbf{H}_\mathcal{U}$ is invertible and which satisfies*

$$K \log\left(1 + \frac{K}{P\alpha}\right) + (K-2) \log\left(1 + \frac{K-2}{P\alpha}\right) - 2 \sum_{i=1}^{K-1} \log\left(1 + \frac{K-1}{P\beta_{(i)}}\right) \leq \Delta(K), \quad (4)$$

such that

$$\Delta(K) = \log\left(\frac{K^K}{(K-1)^{K-1}}\right) - \log\left(\frac{(K-1)^{K-1}}{(K-2)^{K-2}}\right) \quad (5)$$

and $\{\beta_{(i)}\}_{i=1}^{K}$ *is the set* $\{||\mathbf{h}_k||^2\}_{k\in\mathcal{U}}$ *sorted in descending order.* $\alpha = \frac{1}{\max_{k\in\mathcal{U}}\{Q_{k,k}\}}$ *where* $\mathbf{Q} = (\mathbf{H}_{\mathcal{U}}^{\dagger}\mathbf{H}_{\mathcal{U}})^{-1}$.

Proof. We start by noticing that the condition in (4) is readily verifiable. The proof of this theorem is fairly involved and has multiple parts. We suppose $\mathbf{H}_{\mathcal{U}}^{\dagger}\mathbf{H}_{\mathcal{U}}$ to be invertible and will instead derive a sufficient condition under which a stronger relation than (3) holds, where the said stronger relation entails replacing $f(\mathcal{A}\cup u, P), f(\mathcal{B})$ in (3) by their respective lower bounds and $f(\mathcal{B}\cup u, P), f(\mathcal{A})$ by their respective upper bounds. In particular we will show that

$$f^{\rm lb}(\mathcal{A}\cup u, P) - f^{\rm ub}(\mathcal{A}, P) \geq f^{\rm ub}(\mathcal{B}\cup u, P) - f^{\rm lb}(\mathcal{B}, P), \quad (6)$$

whenever (4) holds. In order to derive appropriate bounds, we consider any subset $\mathcal{G}\subseteq\mathcal{U}$ and re-write $f(\mathcal{G}, P)$ as

$$f(\mathcal{G}, P) = \max_{\substack{d_k\in \mathbb{R}_+ \forall\ k\in\mathcal{G}\\ \sum_{k\in\mathcal{G}} d_k\leq 1}} \left\{\log\left|\mathbf{I} + P\sum_{k\in\mathcal{G}}\mathbf{h}_k d_k \mathbf{h}_k^{\dagger}\right|\right\} \quad (7)$$

Then, for any number $\ell\in\{1,\cdots,|\mathcal{G}|\}$, we let $\pi(\mathcal{G},\ell)$ denote any selection of ℓ distinct elements (order being immaterial) from $\mathcal{G}$ and let $\mathbf{H}_{\pi(\mathcal{G},\ell)}$ be the matrix formed by columns of $\mathbf{H}_{\mathcal{U}}$ with indices in $\pi(\mathcal{G},\ell)$. For any choice of power allocation fractions $\{d_k\}_{k\in\mathcal{G}}$ we have the determinant expansion in (8) (cf. [28]). We can then obtain the upper bounds in (9), where we use the AM-GM inequality to ascertain that $\max_{\substack{d_k\in\mathbb{R}_+\forall\ k\in\mathcal{G}:\\ \sum_{k\in\mathcal{G}} d_k\leq 1}}\left\{\left(\prod_{k\in\mathcal{G}} d_k\right)\right\} = (1/|\mathcal{G}|)^{|\mathcal{G}|}$ for all $\mathcal{G}\neq\phi$. Proceeding further we will upper bound the term $s(\mathcal{G}, P)$ in (9). Note that by invoking Schur's complement formula for determinant of block partitioned matrices along with Cholesky decomposition of positive definite matrices [28] we get that

$$\frac{|\mathbf{H}_{\pi(\mathcal{G},\ell)}^{\dagger}\mathbf{H}_{\pi(\mathcal{G},\ell)}|}{|\mathbf{H}_{\mathcal{G}}^{\dagger}\mathbf{H}_{\mathcal{G}}|} = \frac{1}{|\mathbf{H}_{\mathcal{G}\setminus\pi(\mathcal{G},\ell)}^{\dagger}(\mathbf{I} - \mathbf{H}_{\pi(\mathcal{G},\ell)}(\mathbf{H}_{\pi(\mathcal{G},\ell)}^{\dagger}\mathbf{H}_{\pi(\mathcal{G},\ell)})^{-1}\mathbf{H}_{\pi(\mathcal{G},\ell)}^{\dagger})\mathbf{H}_{\mathcal{G}\setminus\pi(\mathcal{G},\ell)}|}$$
$$\leq \frac{1}{\left(\min_{k\in\mathcal{G}}\left\{\mathbf{h}_k^{\dagger}(\mathbf{I} - \mathbf{H}_{\mathcal{G}\setminus k}(\mathbf{H}_{\mathcal{G}\setminus k}^{\dagger}\mathbf{H}_{\mathcal{G}\setminus k})^{-1}\mathbf{H}_{\mathcal{G}\setminus k}^{\dagger})\mathbf{h}_k\right\}\right)^{|\mathcal{G}|-\ell}}$$
$$\leq \frac{\alpha^{\ell}}{\alpha^{|\mathcal{G}|}}$$

Next considering

$$1 + \max_{\substack{d_k\in\mathbb{R}_+\forall\ k\in\mathcal{G}:\\ \sum_{k\in\mathcal{G}} d_k\leq 1}}\left\{\sum_{\ell=1}^{|\mathcal{G}|-1}(P\alpha)^{\ell}\sum_{\pi(\mathcal{G},\ell)}\left(\prod_{k\in\pi(\mathcal{G},\ell)} d_k\right)\right\}$$

we express it as

$$\max_{\substack{d_k\in\mathbb{R}_+\forall\ k\in\mathcal{G}:\\ \sum_{k\in\mathcal{G}} d_k\leq 1}}\left\{\prod_{k\in\mathcal{G}}(1 + P\alpha d_k) - (P\alpha)^{|\mathcal{G}|}\prod_{k\in\mathcal{G}} d_k\right\}$$

Then, we invoke Lemma 1 stated in the appendix to obtain an upper bound

$$s(\mathcal{G}, P) \leq \left(1 + \frac{|\mathcal{G}|}{P\alpha}\right)^{|\mathcal{G}|} - 1. \quad (10)$$

Next, in order to obtain a lower bound we consider the expansion in (8) and fix a choice $d_k = 1/|\mathcal{G}|\ \forall\ k\in\mathcal{G}$. This yields the lower bounds in (11). Again invoking Schur's complement formula for determinant of block partitioned matrices along with Hadamard's determinant inequality for positive definite matrices we get that

$$\frac{|\mathbf{H}_{\pi(\mathcal{G},\ell)}^{\dagger}\mathbf{H}_{\pi(\mathcal{G},\ell)}|}{|\mathbf{H}_{\mathcal{G}}^{\dagger}\mathbf{H}_{\mathcal{G}}|} \geq \frac{1}{\prod_{k\in\mathcal{G}\setminus\pi(\mathcal{G},\ell)}||\mathbf{h}_k||^2} = \frac{\prod_{k\in\pi(\mathcal{G},\ell)}||\mathbf{h}_k||^2}{\prod_{k\in\mathcal{G}}||\mathbf{h}_k||^2}$$

which when used in (11) yields a lower bound

$$\tilde{s}(\mathcal{G}, P) \geq \frac{\prod_{k\in\mathcal{G}}(1 + P||\mathbf{h}_k||^2/|\mathcal{G}|)}{\prod_{k\in\mathcal{G}} P||\mathbf{h}_k||^2/|\mathcal{G}|} - 1$$
$$= \prod_{k\in\mathcal{G}}\left(1 + \frac{|\mathcal{G}|}{P||\mathbf{h}_k||^2}\right) - 1 \geq \prod_{k=1}^{|\mathcal{G}|}\left(1 + \frac{|\mathcal{G}|}{P\beta_{(k)}}\right) - 1. \quad (12)$$

Putting all these together we can further bound the upper bounds in (9) and lower bounds in (11) to obtain (13). At this point we make a key observation that since the set-function $\tilde{r}(\mathcal{G}) = \log|\mathbf{H}_{\mathcal{G}}^{\dagger}\mathbf{H}_{\mathcal{G}}|\ \forall\ \mathcal{G}$ is a submodular set function (cf [26]) we must have that

$$\log|\mathbf{H}_{\mathcal{A}\cup u}^{\dagger}\mathbf{H}_{\mathcal{A}\cup u}| - \log|\mathbf{H}_{\mathcal{A}}^{\dagger}\mathbf{H}_{\mathcal{A}}| \geq \log|\mathbf{H}_{\mathcal{B}\cup u}^{\dagger}\mathbf{H}_{\mathcal{B}\cup u}| - \log|\mathbf{H}_{\mathcal{B}}^{\dagger}\mathbf{H}_{\mathcal{B}}|. \quad (14)$$

Then, define a function $r : \mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ such that $r(x,z) = x\log(1 + x/(Pz)) - x\log(x)\ \forall\ x, z > 0$. Using the bounds in (13), where we recall that $|\mathcal{B}| = |\mathcal{A}| + 1$, along with the key observation stated above, we can deduce that the stronger condition in (6) (and hence (3)) holds true whenever the following condition holds for all $n : 3\leq n\leq K$.

$$r(n,\alpha) + r(n-2,\alpha) + 2(n-1)\log(n-1) - 2\sum_{i=1}^{n-1}\log\left(1 + \frac{n-1}{P\beta_{(i)}}\right) \leq 0. \quad (15)$$

Finally to prove (15) we first re-write it as

$$r(n,\alpha) + r(n-2,\alpha) - 2r(n-1,\alpha) + 2\sum_{i=1}^{n-1}\left(\log\left(1 + \frac{n-1}{P\alpha}\right) - \log\left(1 + \frac{n-1}{P\beta_{(i)}}\right)\right) \leq 0. \quad (16)$$

We can now invoke Lemma 2, also stated in the appendix, along with the observation that the summation in the LHS of (16) is increasing in n since $\beta_{(i)}\geq\alpha\ \forall\ i$ to deduce that the LHS of (16) is maximized at $n = K$. Thus, (15) holds whenever

$$r(K,\alpha) + r(K-2,\alpha) + 2(K-1)\log(K-1) - 2\sum_{i=1}^{K-1}\log\left(1 + \frac{K-1}{P\beta_{(i)}}\right) \leq 0. \quad (17)$$

(17) is indeed the condition stated in the theorem, which thereby proves it. □

We also have the following corollary which provides a simpler condition than (4) albeit which is harder to satisfy. It can be proved by using the convexity of $\log(1+\gamma/x)$ in $x > 0$ for any $\gamma > 0$, along with Jensen's inequality in (4).

$$\left|\mathbf{I}+P\sum_{k\in\mathcal{G}}\mathbf{h}_k d_k \mathbf{h}_k^\dagger\right| = 1+\sum_{\ell=1}^{|\mathcal{G}|-1}\sum_{\pi(\mathcal{G},\ell)}\left(P^\ell \prod_{k\in\pi(\mathcal{G},\ell)} d_k\right)|\mathbf{H}^\dagger_{\pi(\mathcal{G},\ell)}\mathbf{H}_{\pi(\mathcal{G},\ell)}| + P^{|\mathcal{G}|}\left(\prod_{k\in\mathcal{G}} d_k\right)|\mathbf{H}^\dagger_{\mathcal{G}}\mathbf{H}_{\mathcal{G}}| \tag{8}$$

$$\begin{aligned}
f(\mathcal{B}\cup u,P) &\leq \log\left(1+\max_{\substack{d_k\in I\!R_+\forall\, k\in\mathcal{B}\cup u:\\ \sum_{k\in\mathcal{B}\cup u} d_k\leq 1}}\left\{\sum_{\ell=1}^{|\mathcal{B}|}\sum_{\pi(\mathcal{B}\cup u,\ell)} P^\ell\left(\prod_{k\in\pi(\mathcal{B}\cup u,\ell)} d_k\right)|\mathbf{H}^\dagger_{\pi(\mathcal{B}\cup u,\ell)}\mathbf{H}_{\pi(\mathcal{B}\cup u,\ell)}|\right\}\right.\\
&\qquad \left. + P^{|\mathcal{B}|+1}\max_{\substack{d_k\in I\!R_+\forall\, k\in\mathcal{B}\cup u:\\ \sum_{k\in\mathcal{B}\cup u} d_k\leq 1}}\left\{\left(\prod_{k\in\mathcal{B}\cup u} d_k\right)\right\}|\mathbf{H}^\dagger_{\mathcal{B}\cup u}\mathbf{H}_{\mathcal{B}\cup u}|\right)\\
&= \log(1+s(\mathcal{B}\cup u,P))+\log\left(P^{|\mathcal{B}|+1}|\mathbf{H}^\dagger_{\mathcal{B}\cup u}\mathbf{H}_{\mathcal{B}\cup u}|\right)-(|\mathcal{B}|+1)\log(|\mathcal{B}|+1)\\
f(\mathcal{A},P) &\leq \log(1+s(\mathcal{A},P))+\log\left(P^{|\mathcal{A}|}|\mathbf{H}^\dagger_{\mathcal{A}}\mathbf{H}_{\mathcal{A}}|\right)-(|\mathcal{A}|)\log(|\mathcal{A}|)\\
s(\mathcal{G},P) &= \frac{1+\max_{\substack{d_k\in I\!R_+\forall\, k\in\mathcal{G}:\\ \sum_{k\in\mathcal{G}} d_k\leq 1}}\left\{\sum_{\ell=1}^{|\mathcal{G}|-1}\sum_{\pi(\mathcal{G},\ell)}P^\ell\left(\prod_{k\in\pi(\mathcal{G},\ell)} d_k\right)|\mathbf{H}^\dagger_{\pi(\mathcal{G},\ell)}\mathbf{H}_{\pi(\mathcal{G},\ell)}|\right\}}{(P/|\mathcal{G}|)^{|\mathcal{G}|}|\mathbf{H}^\dagger_{\mathcal{G}}\mathbf{H}_{\mathcal{G}}|}\quad\forall\,\mathcal{G}\subseteq\mathcal{U}\;\&\;\mathcal{G}\neq\phi.
\end{aligned}\tag{9}$$

$$\begin{aligned}
f(\mathcal{A}\cup u,P) &\geq \log\left(1+\sum_{\ell=1}^{|\mathcal{A}\cup u|-1}\sum_{\pi(\mathcal{A}\cup u,\ell)}(P/|\mathcal{A}\cup u|)^\ell\,|\mathbf{H}^\dagger_{\pi(\mathcal{A}\cup u,\ell)}\mathbf{H}_{\pi(\mathcal{A}\cup u,\ell)}|+(P/|\mathcal{A}\cup u|)^{|\mathcal{A}\cup u|}|\mathbf{H}^\dagger_{\mathcal{A}\cup u}\mathbf{H}_{\mathcal{A}\cup u}|\right)\\
&= \log(1+\tilde{s}(\mathcal{A}\cup u,P))+\log\left(P^{|\mathcal{A}|+1}|\mathbf{H}^\dagger_{\mathcal{A}\cup u}\mathbf{H}_{\mathcal{A}\cup u}|\right)-(|\mathcal{A}|+1)\log(|\mathcal{A}|+1)\\
f(\mathcal{B},P) &\geq \log(1+\tilde{s}(\mathcal{B},P))+\log\left(P^{|\mathcal{B}|}|\mathbf{H}^\dagger_{\mathcal{B}}\mathbf{H}_{\mathcal{B}}|\right)-|\mathcal{B}|\log(|\mathcal{B}|)\\
\tilde{s}(\mathcal{G},P) &= \frac{1+\sum_{\ell=1}^{|\mathcal{G}|-1}\sum_{\pi(\mathcal{G},\ell)}(P/|\mathcal{G}|)^\ell|\mathbf{H}^\dagger_{\pi(\mathcal{G},\ell)}\mathbf{H}_{\pi(\mathcal{G},\ell)}|}{(P/|\mathcal{G}|)^{|\mathcal{G}|}|\mathbf{H}^\dagger_{\mathcal{G}}\mathbf{H}_{\mathcal{G}}|}\quad\forall\,\mathcal{G}\subseteq\mathcal{U}\;\&\;\mathcal{G}\neq\phi.
\end{aligned}\tag{11}$$

$$\begin{aligned}
f(\mathcal{B}\cup u,P) &\leq (|\mathcal{B}|+1)\log\left(1+\frac{|\mathcal{B}|+1}{P\alpha}\right)+\log\left(P^{|\mathcal{B}|+1}|\mathbf{H}^\dagger_{\mathcal{B}\cup u}\mathbf{H}_{\mathcal{B}\cup u}|\right)-(|\mathcal{B}|+1)\log(|\mathcal{B}|+1)\\
f(\mathcal{A},P) &\leq |\mathcal{A}|\log\left(1+\frac{|\mathcal{A}|}{P\alpha}\right)+\log\left(P^{|\mathcal{A}|}|\mathbf{H}^\dagger_{\mathcal{A}}\mathbf{H}_{\mathcal{A}}|\right)-(|\mathcal{A}|)\log(|\mathcal{A}|)\\
f(\mathcal{A}\cup u,P) &\geq \sum_{i=1}^{|\mathcal{A}|+1}\log\left(1+\frac{|\mathcal{A}|+1}{P\beta_{(i)}}\right)+\log\left(P^{|\mathcal{A}|+1}|\mathbf{H}^\dagger_{\mathcal{A}\cup u}\mathbf{H}_{\mathcal{A}\cup u}|\right)-(|\mathcal{A}|+1)\log(|\mathcal{A}|+1)\\
f(\mathcal{B},P) &\geq \sum_{i=1}^{|\mathcal{B}|}\log\left(1+\frac{|\mathcal{B}|}{P\beta_{(i)}}\right)+\log\left(P^{|\mathcal{B}|}|\mathbf{H}^\dagger_{\mathcal{B}}\mathbf{H}_{\mathcal{B}}|\right)-(|\mathcal{B}|)\log(|\mathcal{B}|)
\end{aligned}\tag{13}$$

Corollary 1. *The set function $f(.,P)$ defined in (2) is a submodular set function over $\mathcal{U}$ for any instance in which $\mathbf{H}^\dagger_{\mathcal{U}}\mathbf{H}_{\mathcal{U}}$ is invertible and which satisfies*

$$K\log\left(1+\frac{K}{P\alpha}\right)+(K-2)\log\left(1+\frac{K-2}{P\alpha}\right) -2(K-1)\log\left(1+\frac{K-1}{P\beta}\right)\leq\Delta(K), \tag{18}$$

where β is the arithmetic mean of $\{\beta_{(i)}\}_{i=1}^{K-1}$.

Remark 1. *Note that for any $K\geq 3$, $\Delta(K)$ defined in (5) strictly positive and thus the condition in (4) is always satisfied at high transmit power (hight transmit SNR) P. Moreover, considering (18) we notice that it always holds when $\beta=\alpha$. In general, the ratio $\frac{\beta}{\alpha}$ lies between one and the condition number of $\mathbf{H}^\dagger_{\mathcal{U}}\mathbf{H}_{\mathcal{U}}$. Therefore (18) is more readily satisfied by well-conditoned channels with large norms. The latter condition certainly holds in the massive MIMO regime but is also seen to be often met with not-so-large number of antennas.*

The following result captures the utility of the results derived in Theorem 1 and directy follows from using the classical result of [25] proving the approximate optimality of the greedy algorithm for maximizing a monotone submodular set function subject to a cardinality constraint. To clarify, we will say that an algorithm offers a γ guarantee for (2), for some scalar $\gamma\in I\!R_+$, if for every input instance it provides an output whose corresponding sum-rate is at-least as large as γ times the optimal sum rate for that instance.

Theorem 2. *For any instance satisfying (4), a simple greedy algorithm guarantees $1-1/e$ approximation for (2).*

IV. Simulation Results

We first evaluate the performance of different algorithms over the user selection problem in (2). We consider a narrow-band model with $N_t = 32$ (equivalently 32 receive antennas in the dual uplink) and varying numbers of users, at varying power budgets of $10,\ 16,\ 18.45\ \&\ 20\ dB$, respectively. In each trial each user's channel vector is generated indepently using the i.i.d. $\mathcal{CN}(0, \mathbf{I})$ distribution. To evaluate any tentative choice of user selection we implemented the iterative water-filling algorithm to optimize the sum rate [23]. We tried:

• *Exhaustive search:* We evaluate all $\binom{K}{J}$ user group choices and select the best one. Due to its exponentially scaling complexity this method is tried only for small K.

• *Greedy:* We implement the natural greedy method in which the locally best user (offering the largest sum rate improvement) is selected from the pool of un-selected users in each step, until the cardinality limit is reached.

• *Lazy Greedy:* We implement the lazy implementation [27] assuming submodularity. In particular, we assume that $f(\mathcal{O} \cup u, P) - f(\mathcal{O}, P) \leq f(\tilde{\mathcal{O}} \cup u, P) - f(\tilde{\mathcal{O}}, P)$ for any unselected user $u \in \mathcal{U} \setminus \mathcal{O}$, where $\mathcal{O}$ is the set of users selected so far while $\tilde{\mathcal{O}} \subset \mathcal{O}$ is a user set that was selected upto any previous iteration. With the partial ordering provided by this assumption we can avoid evaluating the rate improvements of user choices whose past improvement is below the current improvement evaluated for any user. Further, in any instance where submodularity indeed holds we can assert that the lazy greedy will perform identical to greedy.

In Fig. 1 we plot the achieved sum rates (averaged over 1000 realizations) versus the user group cardinality J. In each case we set the number of users as $K = \min\{32, 2J\}$. Comparing the exhaustive search (which is plotted only for $J = 4\ \&\ 8$) and greedy we see that the curves are virtually indistinguishable and greedy achieves almost identical performance. Interestingly greedy and lazy greedy were seen to achieve exactly *identical* performance over all instances which suggests submodularity holds in all instances for (2). More importantly, Fig. 2 plots the reduction in number of tentative choice evaluations (directly proportional to complexity reduction) obtained using lazy greedy over greedy. We see that as problem size grows the complexity reduction becomes significant (about 30% reduction).

In Fig. 3 we test the regimes over which the sufficient condition derived in Theorem 1 holds. We employ the setup used in above examples and consider varying transmit powers with $K = 4,\ 6\ \&\ 8$ users. In each case for ten thousand instances we compute the frequency with which the sufficient condition for submodularity derived in Theorem 1 is satisfied. From the plot we see that this condition is highly likely to be met in asymmetric configurations (where the number of users is significantly dominated by the number of BS antennas) or at high transmit SNRs. Both these scenarios are practical.

V. Wideband Scheduling

We consider the original model in (1) and assume that the BS can employ the capacity achieving DPC on each subband. Accordingly, we compute the weighted sum rate that can be achieved on any subband for any given user-grouping and power budget. Towards this end, let w_k denote the given positive weight assigned to user $k \in \mathcal{U}$. [2] Next, consider any subband n, subband power budget P and any user group $\mathcal{G} = \{k_1, \cdots k_\ell\} \subseteq \mathcal{U}$ and without loss of generality suppose that $w_{k_1} \geq w_{k_2} \geq \cdots \geq w_{k_\ell}$. In order to deduce the optimal weighted sum rate that can be achieved in the downlink via DPC for the given power budget and user group, we can equivalently consider the dual multiple access channel for the same sum power budget [24]. Consequently, considering the dual MAC let us define a function $g : 2^{\mathcal{U}} \times \mathcal{N} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(\mathcal{G}, n, P')$, for any user subset $\mathcal{G} \subseteq \mathcal{U}$, subband $n \in \mathcal{N}$ and subband power budget P', yields the optimal weighted sum rate. Letting $\boldsymbol{\mathcal{S}}_+^{N_r}$ denote the set of all $N_r \times N_r$ positive semidefinite matrices, we can determine $g(\mathcal{G}, n, P')$ explicitly as in (19) (cf. [23]).

In order to pose our scheduling problem, we define a family of permissible subsets of $\mathcal{U}$ denoted by $\underline{\mathcal{I}}$. In particular, it is a family of subsets of $\mathcal{U}$ such that each of its member can be selected as a user group on any subband. We suppose that $\underline{\mathcal{I}}$ is given as an input and that its cardinality is much smaller than 2^K. Indeed, any scheduling algorithm whose complexity scales polynomially in $|\underline{\mathcal{I}}|$ is deemed feasible. [3] We can now pose our main scheduling problem in (20). In (20) the binary valued variable $x_{\mathcal{G},n}$ is one if group $\mathcal{G}$ is chosen on subband n and zero otherwise, so that the first set of constraints in (20) ensures that at-most one member of $\underline{\mathcal{I}}$ is chosen on each subband. Further, the second constraint enforces that the total number of occupied subbands cannot exceed $\bar{N}$, for some given $\bar{N} : 1 \leq \bar{N} \leq N$. Indeed, this constraint is meaningful whenever $N - \bar{N}$ subbands have to reserved for some (lower priority) application. The third constraint enforces the total sum power budget and the final set of constraints (in conjunction with the first set of constraints) enforces that for each user u in some given user subset $\mathcal{U}'$, the total number of subbands on which that user is scheduled cannot exceed some given number J_u. Such a constraint is very useful to impose assigned bandwidth constraint on any specified subset of users. Note here that any choice of indicator variables in (20) defines a collection of tuples, where each tuple comprises of a member (user subset) in $\underline{\mathcal{I}}$ and a subband from $\mathcal{N}$. Accordingly, let us define a ground set of all possible such tuples as

$$\underline{\boldsymbol{\mathcal{F}}} \triangleq \{(\mathcal{G}, n) : \mathcal{G} \in \underline{\mathcal{I}}, n \in \mathcal{N}\}. \tag{21}$$

We now proceed to characterize the problem in (20). Notice first that this problem is a mixed optimization problem involving discrete indicator variables, continuous non-negative power budget variables, as well as (implicit) positive semi-definite matrix variables. Before we offer our key result let us define an input instance. For a given choice of system parameters K, N_t, N_r and $|\mathcal{I}|$, an input instance for (20) comprises of channels $\{\mathbf{H}_{k,n}\}_{k \in \mathcal{U}, n \in \mathcal{N}}$, family $\underline{\mathcal{I}}$, sum power budget P, user subset $\mathcal{U}'$ and bounds $\{J_u\}_{u \in \mathcal{U}'}$ and $\bar{N}$. Our key result follows.

[2] These weights can be updated based on the obtained scheduling decision, to optimize a desired utility over a coarser time-scale (cf. [10]).

[3] The burden of choosing an effective family $\underline{\mathcal{I}}$ is not addressed here. Nevertheless, results of Section III indicate that simple greedy methods used on "average" channels that capture spatial correlations might be useful.

$$g(\mathcal{G},n,P') = \max_{\substack{\mathbf{Q}_k \in \mathcal{S}_+^{N_r} \forall\, k\in\mathcal{G} \\ \sum_{k\in\mathcal{G}} \mathrm{tr}(\mathbf{Q}_k)\le P'}} \underbrace{\left\{ \sum_{j=1}^{\ell-1} (w_{k_j} - w_{k_{j+1}}) \log \left| \mathbf{I} + \sum_{q=1}^{j} \mathbf{H}_{k_q,n}\mathbf{Q}_{k_q}\mathbf{H}_{k_q,n}^\dagger \right| + w_{k_\ell} \log \left| \mathbf{I} + \sum_{q=1}^{\ell} \mathbf{H}_{k_q,n}\mathbf{Q}_{k_q}\mathbf{H}_{k_q,n}^\dagger \right| \right\}}_{\triangleq h(\mathcal{G},n,\{\mathbf{Q}_k\}_{k\in\mathcal{G}})} \quad (19)$$

$$\max_{x_{\mathcal{G},n}\in\{0,1\}, P_n\in\mathbb{R}_+ \;\forall\; \mathcal{G}\in\underline{\mathcal{I}}, n\in\mathcal{N}} \left\{ \sum_{n\in\mathcal{N}} \sum_{\mathcal{G}\in\underline{\mathcal{I}}} x_{\mathcal{G},n} g(\mathcal{G},n,P_n) \right\}$$
$$\text{s.t.} \sum_{\mathcal{G}\in\underline{\mathcal{I}}} x_{\mathcal{G},n} \le 1 \;\forall\; n\in\mathcal{N};\; \sum_{n\in\mathcal{N}} \sum_{\mathcal{G}\in\underline{\mathcal{I}}} x_{\mathcal{G},n} \le \bar{N};\; \sum_{n\in\mathcal{N}} P_n \le P;\; \sum_{n\in\mathcal{N}} \sum_{\mathcal{G}\in\underline{\mathcal{I}}:u\in\mathcal{G}} x_{\mathcal{G},n} \le J_u \;\forall u\in\mathcal{U}'. \quad (20)$$

Theorem 3. *The problem in (20) is equivalent to the maximization of a normalized monotone submodular set function subject to one matroid and $|\mathcal{U}'|$ knapsack constraints.*

Proof. We begin by definining any collection of tuples,

$$\boldsymbol{\mathcal{A}} = \cup_{j=1}^{m} (\mathcal{G}_{(j)}, n_{(j)}),\; \mathcal{G}_{(j)} \in \underline{\mathcal{I}}, n_{(j)} \in \mathcal{N} \;\forall\; 1\le j\le m \quad (22)$$

where we do allow for repeated subbands or arbitrarily overlapping subsets across tuples in $\boldsymbol{\mathcal{A}}$, i.e., we do not restrict our attention to only collections feasible for (20). For any such non-empty collection $\boldsymbol{\mathcal{A}}$ and any sum power budget P let us compute the weighted sum rate as

$$f(\boldsymbol{\mathcal{A}},P) = \max_{\substack{P_{(j)}\in\mathbb{R}_+ \;\forall\; j \\ \sum_{j=1}^{m} P_{(j)}\le P}} \left\{ \sum_{j=1}^{m} g(\mathcal{G}_{(j)}, n_{(j)}, P_{(j)}) \right\} \quad (23)$$

and where $f(\phi,P)=0$ for the empty set ϕ. Notice here that in (23) for the j^{th} tuple and any tentative choice of power budget for that tuple, $P_{(j)}$, $g(\mathcal{G}_{(j)}, n_{(j)}, P_{(j)})$ is evaluated using (19) assuming subband $n_{(j)}$, user group $\mathcal{G}_{(j)}$ and sum power $P_{(j)}$. Further, the optimization problem in (23) is a convex optimization problem and can be efficiently solved [23]. Similarly let $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{A}} \cup (\mathcal{G}_{(m+1)}, n_{(m+1)}) \subset \underline{\boldsymbol{\mathcal{F}}}$ so that $\boldsymbol{\mathcal{B}}$ includes all tuples in $\boldsymbol{\mathcal{A}}$ and one extra tuple. Further, let $\boldsymbol{\varepsilon} = (\mathcal{G}_{(m+2)}, n_{(m+2)}) \in \underline{\boldsymbol{\mathcal{F}}}$ denote a tuple that is not present in $\boldsymbol{\mathcal{B}}$. We will prove that submodularity holds, i.e.,

$$f(\boldsymbol{\mathcal{A}}\cup\boldsymbol{\varepsilon},P) - f(\boldsymbol{\mathcal{A}},P) \ge f(\boldsymbol{\mathcal{B}}\cup\boldsymbol{\varepsilon},P) - f(\boldsymbol{\mathcal{B}},P). \quad (24)$$

In order to prove (24) we consider any optimal solution (of powers and covariance matrices) under $\boldsymbol{\mathcal{B}}\cup\boldsymbol{\varepsilon}$ and let $\hat{P}_{(m+1)}$ and $\hat{P}_{(m+2)}$ denote the powers assigned to tuples $(\mathcal{G}_{(m+1)}, n_{(m+1)})$ and $(\mathcal{G}_{(m+2)}, n_{(m+2)})$, respectively. Clearly, the relations in (25) must hold true. In particular, equality there follows from the definition in (23) whereas both inequalities follow from the simple observation that fixing the choice of power for any tuple and optimizing over the remaining ones will lead to a sub-optimal solution. Thus, in lieu of (25), to prove (24) it suffices to prove

$$\begin{aligned} f(\boldsymbol{\mathcal{A}}, P - \hat{P}_{(m+2)} - \hat{P}_{(m+1)}) - f(\boldsymbol{\mathcal{A}}, P - \hat{P}_{(m+1)}) \\ \le f(\boldsymbol{\mathcal{A}}, P - \hat{P}_{(m+2)}) - f(\boldsymbol{\mathcal{A}}, P). \end{aligned} \quad (26)$$

Moreover, (26) clearly follows if we are able to show that for any collection of tuples $\boldsymbol{\mathcal{A}}$, $f(\boldsymbol{\mathcal{A}},P)$ is concave in P. Towards that end, letting $P^{(1)}$ and $P^{(2)}$ denote any two power budgets, we will establish that for each $\lambda\in(0,1)$,

$$\begin{aligned} f(\boldsymbol{\mathcal{A}}, \lambda P^{(1)} + (1-\lambda)P^{(2)}) \\ \ge \lambda f(\boldsymbol{\mathcal{A}}, P^{(1)}) + (1-\lambda) f(\boldsymbol{\mathcal{A}}, P^{(2)}). \end{aligned} \quad (27)$$

Finally, to show (27) let $\{\mathbf{Q}_{u,(j)}^{(1)}\}_{\substack{u\in\mathcal{G}_{(j)}\\1\le j\le m}}$ and $\{\mathbf{Q}_{u,(j)}^{(2)}\}_{\substack{u\in\mathcal{G}_{(j)}\\1\le j\le m}}$ denote any optimal sets of covariance matrices under sum power budgets $P^{(1)}$ and $P^{(2)}$, respectively. Notice here that the optimization problems in (23) and (19) have continuous objectives and for any finite sum power budget their respective constraint sets are compact, so an optimal solution (set of per-subband sum powers and covariance matrices) must exist. Clearly then $\{\lambda\mathbf{Q}_{u,(j)}^{(1)} + (1-\lambda)\mathbf{Q}_{u,(j)}^{(2)}\}_{\substack{u\in\mathcal{G}_{(j)}\\1\le j\le m}}$ is a feasible choice under budget $\lambda P^{(1)} + (1-\lambda)P^{(2)}$ in that the sum power constraint is satisfied and each matrix in that set is positive semi-definite. Then invoking the joint concavity of each $\log|.|$ term of (19) in its input covariance matrices, we can deduce that

$$\begin{aligned} h(\mathcal{G}_{(j)}, n_{(j)}, \{\lambda\mathbf{Q}_{u,(j)}^{(1)} + (1-\lambda)\mathbf{Q}_{u,(j)}^{(2)}\}_{u\in\mathcal{G}_{(j)}}) \ge \\ \lambda h(\mathcal{G}_{(j)}, n_{(j)}, \{\mathbf{Q}_{u,(j)}^{(1)}\}_{u\in\mathcal{G}_{(j)}}) \\ +(1-\lambda) h(\mathcal{G}_{(j)}, n_{(j)}, \{\mathbf{Q}_{u,(j)}^{(2)}\}_{u\in\mathcal{G}_{(j)}}) \end{aligned}$$

so that the relations in (28) must all hold. From (28), which demonstrates concavity, we can conclude that the property in (26) and hence (24) indeed holds true. Thus, we have proved that the set function $f(.,P)$ is a normalized submodular set function over the ground set $\underline{\boldsymbol{\mathcal{F}}}$ defined in (21). Clearly, this set function is also monotone non-decreasing. Finally, a feasible collection $\boldsymbol{\mathcal{A}}$ must have no more that $\bar{N}$ tuples and no two tuples in it can have identical subband indices. It can be verified that the latter two constraints together specify one matroid. Finally, the set of bandwidth limiting constraints (one for each $u\in\mathcal{U}'$) are simply knapsack constraints defined on the ground set. □

VI. Approximation Algorithm

We begin our quest to obtain an approximation algorithm for (20) by defining a vector of indicator variables $\mathbf{x} = [x_{\mathcal{G},n}],\; \forall\; \mathcal{G}\in\underline{\mathcal{I}}\; n\in\mathcal{N}$. Then, note that the constraints in (20) can be represented as $\mathbf{Ax}\le\mathbf{b}$ where the inequality is componentwise and $\mathbf{A}\in\{0,1\}^{(|\mathcal{U}'|+N+1)\times|\underline{\mathcal{I}}|N}$ is a binary-valued matrix corresponding to the $|\mathcal{U}'|+N+1$ constraints (all constraints in (20) involving indicator variables) and the

$$f(\boldsymbol{\mathcal{B}}\cup\varepsilon,P) = f(\boldsymbol{\mathcal{A}},P-\hat{P}_{(m+1)}-\hat{P}_{(m+2)})+g(\mathcal{G}_{(m+1)},n_{(m+1)},\hat{P}_{(m+1)})+g(\mathcal{G}_{(m+2)},n_{(m+2)},\hat{P}_{(m+2)}),$$

$$f(\boldsymbol{\mathcal{B}},P) \geq f(\boldsymbol{\mathcal{A}},P-\hat{P}_{(m+1)})+g(\mathcal{G}_{(m+1)},n_{(m+1)},\hat{P}_{(m+1)})$$

$$f(\boldsymbol{\mathcal{A}}\cup\varepsilon,P) \geq f(\boldsymbol{\mathcal{A}},P-\hat{P}_{(m+2)})+g(\mathcal{G}_{(m+2)},n_{(m+2)},\hat{P}_{(m+2)}) \quad (25)$$

$$\begin{aligned} f(\boldsymbol{\mathcal{A}},\lambda P^{(1)}+(1-\lambda)P^{(2)}) &\geq \sum_{j=1}^{m} h(\mathcal{G}_{(j)},n_{(j)},\{\lambda\mathbf{Q}_{u,(j)}^{(1)}+(1-\lambda)\mathbf{Q}_{u,(j)}^{(2)}\}_{u\in\mathcal{G}_{(j)}}) \quad (28)\\ &\geq \lambda\sum_{j=1}^{m} h(\mathcal{G}_{(j)},n_{(j)},\{\mathbf{Q}_{u,(j)}^{(1)}\}_{u\in\mathcal{G}_{(j)}})+(1-\lambda)\sum_{j=1}^{m} h(\mathcal{G}_{(j)},n_{(j)},\{\mathbf{Q}_{u,(j)}^{(2)}\}_{u\in\mathcal{G}_{(j)}})\\ &= \lambda f(\boldsymbol{\mathcal{A}},P^{(1)})+(1-\lambda)f(\boldsymbol{\mathcal{A}},P^{(2)}). \end{aligned}$$

vector $\mathbf{b}$ is a vector enforcing the respective budgets. Further let $L_{\max}$ defined as

$$L_{\max}=\max\{|\mathcal{G}\cap\mathcal{U}'|:\mathcal{G}\in\underline{\mathcal{I}}\} \quad (29)$$

denote the maximal number of users in $\mathcal{U}'$ present in any member of $\underline{\mathcal{I}}$, which typically would be a small constant.

Theorem 4. *There exists a simple approximation algorithm yielding $C/(L_{\max}+2)$ approximation for (20), where C is a positive constant invariant across all input instances as well as system parameters.*

Proof. The key observations to prove this theorem are to invoke the arguments made in this section above along with Theorem 3. We can then deduce that (20) can be reformulated, with some abuse of notation, as

$$\max_{\mathbf{x}\in\{0,1\}^{|\underline{\mathcal{I}}|N}}\{f(\mathbf{x},P)\} \text{ s.t. } \mathbf{A}\mathbf{x}\leq\mathbf{b}, \quad (30)$$

where the set function $f(.,P)$ is a normalized monotone submodular set function over $\underline{\boldsymbol{\mathcal{F}}}$ and the constraints are binary knapsack constraints. Further, these constraints are column-sparse in that each column of $\mathbf{A}$ has at most $L_{\max}+2$ positive elements (each equal to unity). This allows us to assert that the simple greedy algorithm proposed in [22] to maximize any monotone submodular set function subject to binary column-sparse knapsack constraints, will offer the claimed guarantee for (30) and thus for (20). This proves the theorem. □

We note here that (20) represents a significant generalization of the basic waterfilling problem considered in [18] so that Theorem 3 substantially expands the result therein. Moreover, we demonstrate in [29] that our formulation and analysis can also incorporate minimum rate and power constraints.

In order to test our proposed algorithm for wideband scheduling, we now conduct simulations over a simple setup. We consider a single-cell downlink with $N=4$ subbands and $K=5$ single receive antenna users, and consider varying number of antennas at the BS (in particular $N_t=2,3$ & 4) as well as several values for the transmit power budget ($P=[10,16,18.45,20]$ dB). We further suppose that the family of permissible subsets, $\underline{\mathcal{I}}$, comprises of all subsets of $\mathcal{U}$ whose cardinalities are identical to $J=2$. We impose no other constraints apart from the wideband sum power constraint and the per-subband ones pertaining to restricting the chosen subset to lie in $\underline{\mathcal{I}}$. As a result, the approximation algorithm proposed in Section VI reduces to the natural greedy method. To benchmark the latter algorithm, we compare it against an upper bound obtained by allowing all K users to be scheduled on each subband. We note that this upper bound becomes progressively looser as N_t becomes larger compared to J, since it can exploit additional multiplexing gains that are not available to any feasible solution. Indeed, this observation is reflected in Fig. 4 where we have plotted the performance (sum rate) of the greedy method as a fraction of the upper bound. We see that as long as the family $\underline{\mathcal{I}}$ is sufficiently rich (in terms of candidate user groups) the greedy method captures a significant portion of available gains.

VII. Conclusions

We proved that the user gouping problem under DPC can be cast as a monotone submodular maximization under fairly mild conditions. Further, we showed that a relevant wideband scheduling problem with DPC and an input set of permissible user groups is equivalent to monotone submodular maximization subject to binary sparse knapsack constraints. This enables design of constant-factor approximation algorithms. The simulation results of Section IV strongly suggest that submodularity in (2) indeed holds for all instances. Towards this end, we have been able to identify two other sets of input instances for which submodularity provably holds. A unified approach that can cover all instances is an interesting avenue for further research.

Appendix

The proofs of the following two lemmas are given in [29].

Lemma 1. *For any $\Theta\geq 0$ and any integer $L\geq 1$ we have that*

$$\max_{\substack{\{c_\ell\}_{\ell=1}^{L}\in\mathbb{R}_+^L:\\ \sum_{\ell=1}^{L}c_\ell\leq 1}}\left\{\prod_{\ell=1}^{L}(1+\Theta c_\ell)-(\Theta)^L\prod_{\ell=1}^{L}c_\ell\right\} = (1+\Theta/L)^L-(\Theta/L)^L \quad (31)$$

Lemma 2. *For any $\beta\geq\alpha>0$ we have that*

$$r(x,\alpha)+r(x-2,\alpha)-2r(x-1,\beta) \quad (32)$$

is increasing in x for all $x\geq 3$.

REFERENCES

[1] G. Caire, and S. Shamai, "On the achievable throughput of a multiantenna gaussian broadcast channel," in *IEEE Transactions on Information Theory*, 2003.

[2] S. Vishwanath, et. al., "Duality, achievable rates, and sum-rate capacity of Gaussian MIMO broadcast channels," *IEEE Info. Theory*, vol. 49, no. 10, pp. 2658 2668, Oct. 2003.

[3] H. Weingarten, et. al., "The Capacity Region of the Gaussian Multiple-Input Multiple-Output Broadcast Channel", in *IEEE Trans. Inf. Theory*, Sep. 2006.

[4] J. Hoydis, et. al., "Massive MIMO in the UL/DL of cellular networks: How many antennas do we need?," in *IEEE JSAC 2013*.

[5] G. Dimic and N. Sidiropoulos, "On downlink beamforming with greedy user selection: performance analysis and a simple new algorithm," in *IEEE Trans. Sig. Proc.*, Oct. 2005.

[6] J. Wang, et. al., "User selection with zeroforcing beamforming achieves the asymptotically optimal sum rate," in *IEEE Trans. on Sig. Proc.*, 2008.

[7] A. Bayesteh and A. K. Khandani, "On the user selection for MIMO broadcast channels," *IEEE Trans. Inf. Theory*, vol. 54, no. 3, Mar. 2008.

[8] W. Shen, et. al., "Sieve: Scalable user grouping for large mu-mimo systems," in *IEEE Infocom*, 2015.

[9] G. Zois, et. al., "Non-asymptotic performance bounds for downlink MU-MIMO scheduling," in *13th Annual Conference on Wireless On-demand Network Systems and Services WONS*, 2016.

[10] M. Andrews and L. Zhang, "Scheduling Algorithms for Multi-carrier Wireless Data Systems", in *ACM Mobicom*, Sep. 2007.

[11] H. Zhang, et. al., "MIMO downlink scheduling in LTE systems", in *IEEE Infocom*, 2012.

[12] S. Zuo, et. al., "Joint Rate Control and Scheduling for Real-Time Wireless Networks," in *IEEE Trans. on Wireless Comm (accepted)* 2017.

[13] W. C. Ao and K. Psounis "Approximation Algorithms for Online User Association in Multi-Tier Multi-Cell Mobile Networks", in *IEEE Trans. Net.* 2017

[14] V. Singh, et.al.,"Optimizing user association and activation fractions in heterogeneous wireless networks," in *IEEE WiOpt*, June. 2015.

[15] A. Clark, et.al., "Scalable and Distributed Submodular Maximization with Matroid Constraints", IEEE WiOpt 2015.

[16] N. Golrezaei et. al., "FemtoCaching: Wireless Video Content Delivery through Distributed Caching Helpers", in *IEEE Infocom* 2012.

[17] Y. Ghasempour, et. al., "Novel combinatorial results on downlink mu-mimo scheduling with applications," in *14th Annual Conference on Wireless On-demand Network Systems and Services WONS*, 2017

[18] K. Thekumparampil, et.al., "Combinatorial Resource Allocation Using Submodularity of Waterfilling," in *IEEE Trans. Wireless Comm.*, Jan. 2016.

[19] Y. Ghasempour, et. al., "Managing analog beams in mmWave networks," in *IEEE Asilomar*, 2017.

[20] M. Hong and A. Garcia, "Mechanism design for base station association and resource allocation in downlink OFDMA network," *IEEE J. Sel. Areas Commun.*, vol. 30, no. 11, pp. 22382250, Dec. 2012.

[21] E. Altman, et.al., "Spatial sinr games of base station placement and mobile association," *IEEE Infocom.*, 2009.

[22] Y. Azar and I. Gamzu, "Efficient submodular function maximization under linear packing constraints", in *ICALP* 2012.

[23] M. Kobayashi and G. Caire, "An iterative water-filling algorithm for maximum weighted sum-rate of Gaussian MIMO-BC," *IEEE J. Select. Areas Commun.*, vol. 24, no. 8, pp. 16401646, Aug. 2008.

[24] P. Viswanath and D. N. C. Tse, "Sum capacity of the vector gaussian channel and uplink-downlink duality," in *IEEE Trans. Inf. Theory*, Aug. 2003.

[25] G. L. Nemhauser et. al., "An analysis of approximations for maximizing submodular set functions-I", in *Mathematical Programming*, July 1978.

[26] J. Lee, et.al., "Non-monotone submodular maximization under matroid and knapsack constraints," in *STOC*, 2009.

[27] M. Minoux, "Accelerated greedy algorithms for maximizing submodular set functions," in *Optimization Techniques, LNCS*, 1978.

[28] R. A. Horn and C.R. Johnson, "Matrix Analysis," in *2nd Ed., Cambridge University Press*, NY 2012.

[29] "Downlink Multi-User MIMO Scheduling with Performance Guarantees," Extended version: goo.gl/3Rjgia

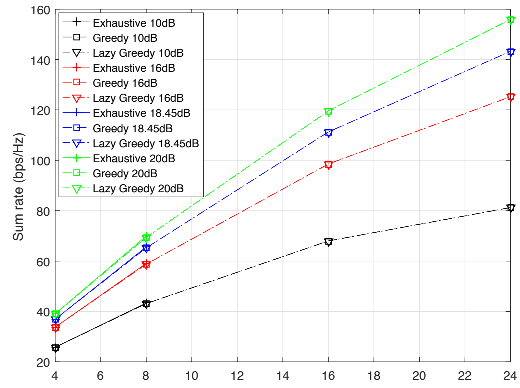

Fig. 1. Comparison of user grouping algorithms

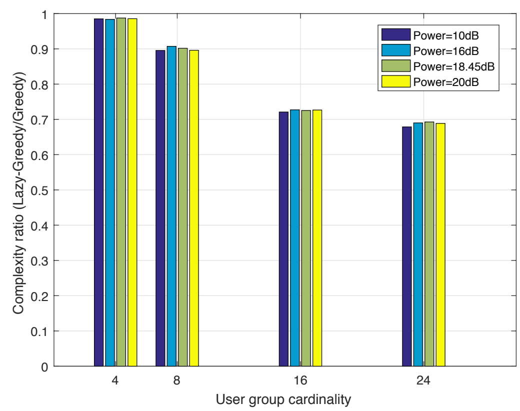

Fig. 2. Complexity reduction of Lazy Greedy over Greedy

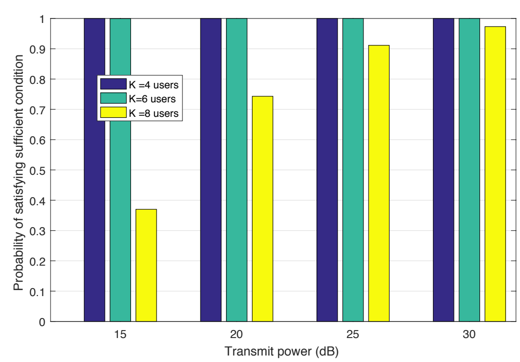

Fig. 3. Probability of ensuring submodularity

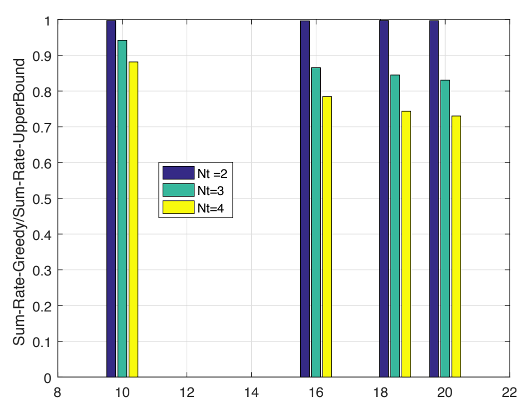

Fig. 4. Performance of Wideband Scheduling