How Much to Share in Resource Pooling

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***Abstract*—Resource pooling between two service providers, each having a fixed number of resources is considered. Each of the providers commit some of their resources to the common pool, and use the other providers' resources (committed by it to the common pool) if available to service their clients if all of their own resources are completely occupied. Each such request requires a fixed payment to the other provider. The standard model of exponentially distributed occupancy time for each resource request is assumed. Assuming transferable utilities, Shapley value based inter-provider payoff keeps the coalition stable, and the problem is to find the optimal number of resources to commit to the common pool by each provider that maximizes the sum of the revenues of the two service providers. An exact solution is derived for the problem that depends on the prices charged by each provider to its clients. In case of equal prices, either full sharing or no sharing is shown to be optimal. Otherwise, the service provider with lower price commits all its resources, while the other provider commits resources depending on the solution of a recursive equation.**

***Index Terms*—Resource pooling, cooperative game, Shapley value**

I. Introduction

Resource pooling in the presence of multiple service providers is a classical problem with a variety of applications, such as hospitals that pool beds, maintenance firms that pool repairmen, spectrum sharing in wireless networks, bandwidth sharing in internet service providers, shared office spaces, call centres, bike and car rental services, etc., and has been studied for various models. The basic problem with resource pooling is: how many resources should be committed to the common pool, where the common pool is accessible for all providers to serve their clients if their own resources are completely occupied. Clearly, many different modelling questions arise, such as: whether the providers are in *cooperative* or *competitive* mode, whether payment is made between providers if any of them uses the others' resources or not, how to share revenue, and several others.

The basic premise of pooling is to improve occupancy rate or decrease idleness of resource usage across different providers, but comes at a cost of potentially degrading QoS for individual providers for its own clients. There is an inherent tradeoff from individual providers' points of view - sharing more of one's own resources will lead to decreased availability for one's own clients, but not sharing enough resources could lead to other providers reciprocating and reducing the benefits intended from pooling.

The main classification in theoretical work on pooling depends on whether the providers are in competitive or cooperative mode. In the competitive setting, individual providers are hoping to get maximum selfish benefit and concepts like Nash bargaining solutions are useful ([1], [2], [3]). In the cooperative setting, however, the question of interest is how to fairly allocate the increased revenue so that the *core* remains non-empty ([4], [5], [6], [7], [8]), where primarily Shapley value [16] based revenue sharing is considered.

One specific example where resource pooling has been studied widely is wireless network providers, because of scarce available spectrum (see [1], [2], [3], [9], [10], [11], [12], [13] and references therein). The majority of the work in this field has concentrated on *dynamic* spectrum sharing - where the excess spectrum of a single provider is auctioned/allocated to other providers dynamically over time [14] [2], [3], [13]. How providers in a coalition will share their revenues has been considered in [15], while strategic behaviour of providers has been considered in [11] - [12], where [12] also considers a pricing game.

In this paper, we consider a generic problem that is suitable for many applications. There are two providers, and provider i has N_i resources out of which it commits a subset of size k_i to the common pool. Resource requests arrive for each provider as a Poisson process with mean $\lambda_i, i = 1, 2$, and have exponentially distributed occupancy/service times with unit mean at both the providers. A resource request arriving to provider i is served by it if provider i has at least one free resource, or by the other provider if there is at least one free resource among the ones that it has committed to the common pool, and denied/blocked otherwise. We consider no queuing to keep the model simple.

Each provider charges its clients a price p_i (fixed ahead of time) for a successfully completed request, and makes inter-provider payment to the other provider equal to the charge levied by each provider to the other for usage of its resources that it has committed to the common pool. Clearly, the revenue that each provider makes is a function of its own resources, the resources committed by the other provider to the common pool, and inter-provider payments. Thus, it is natural to assume that each provider is going to behave in a strategic manner to maximize its own revenue, and consider a *pooling* game.

We consider a cooperative model for providers, and consider stable coalition formation among them. In particular, we con-

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sider transferable utilities between providers, and use Shapley value ([16]) based inter-provider payoff that keeps the coalition stable. With Shapley value based inter-provider payment, the **main problem** we address in this paper is : what is the optimal number of resources committed by the two providers in the common pool that maximizes the sum of the revenue made by the two providers, which is rather challenging to solve.

To obtain our results, we make a correspondence between the considered game and stochastic loss networks, that have been well-studied in multiplexing literature. We show that whenever a coalition is formed, in the optimal solution, at least one provider must fully share its resources (and in certain cases, both providers should) in order to maximize individual revenues via maximizing the total revenue. In particular, we show that when the prices charged by the two providers are identical ($p_1 = p_2$), either both commit all their resources to the common pool or no one commits any resources. When prices are unequal (say $p_1 < p_2$), then the cheaper provider commits all its resources to the common pool, while the costlier provider commits the number of resources that is given by a solution of a recursive equation. The solution definitely has intuitive appeal, however, making a rigorous argument requires non-trivial work using the concept of shadow prices.

Extending the presented results to more than two providers is possible, but currently outside the scope of this paper.

The studied problem is a building block for solving a more general problem, where providers will also select their prices that maximize their respective revenues. This was considered in [12], albeit when the pooled part of the resources was fixed ahead of time. Choosing optimal prices for providers entails anticipating the client traffic that will be at Wardrop equilibrium, depending on the utility function (of the QoS and the price), and given client traffic arrival rates and prices, the results found in this paper will determine the pooling configuration. Iterating over price choices, providers can find the optimal ones. Clearly, this is a more challenging problem, which is part of ongoing work.

II. Model

There are two service providers, who enter into an agreement, or form a coalition, to share portions of their resources. It is assumed that all the resources are of equal size, one of which is used to service a single resource request. Let the number of total resources with provider i be N_i, $i \in \{1, 2\}$, out of which k_i are shared or committed to the common pool. Provider i chooses to *commit* k_i of its resources in the common pool. This implies that a total of $N_i + k_j, j \neq i$ resources are available for provider i's use. Note that there is no guarantee that all k_j resources will be free for provider i's use - provider j still treats its committed k_j resources as it would any exclusive resource for its own requests. Either provider will use its own resources before starting to use the shared resources. If any of provider i's own resources free up while one of its requests is being serviced by a shared resource of provider j's, the request will be instantaneously switched to provider i's own resources.

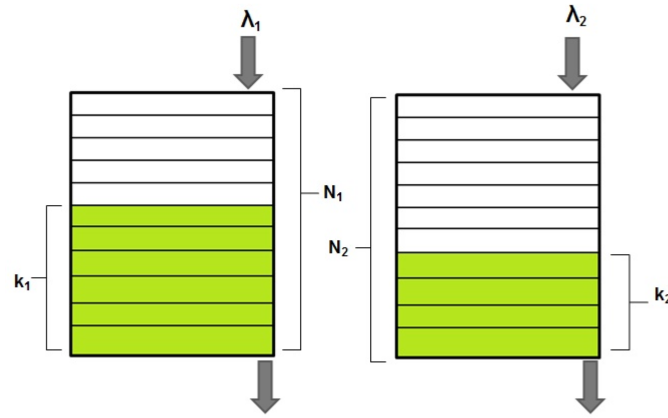

Fig. 1. Sharing model, where the colored set of resources are part of the common pool.

Requests arrive for each provider as a Poisson process with mean λ_i, and have unit mean (without loss) exponentially distributed occupancy time at any of the providers. Each request takes up the entirety of a single resource, i.e., we preclude the possibility of a resource being used for multiple requests. When all the resources available for the use of a provider are occupied, any incoming requests do not wait for a resource to free up, and are denied service and is defined to be *blocked*. Customers are charged a price p_i by provider i only for a successfully fulfilled request, i.e., one that is not blocked.

No provider is expected to be altruistic and hence requires payment from the other provider when it uses any of its shared resources. We assume that provider i is paid an amount r_j by provider j for the use of provider i's resources, and vice versa.

The quality of service that a provider offers to customers while in a coalition is measured using the provider's blocking probability, B_i, defined as the probability that an incoming request from any of its own customers is blocked, and correspondingly the price provider i charges its own customers per successful request is p_i.

While acting by itself (i.e, before forming the coalition of interest to us), we denote the provider's blocking probability as $\tilde{B}_i$, and the price it charges per successful request as p_i - we assume that these two quantities are fixed and predetermined beforehand, and do not concern ourselves with how their values are set.
We now characterize the throughput of each provider, i.e, the average number of successfully served requests.

Lemma II.1. *The time-averaged numbers of requests being served by provider i, T_i, is given by the equation:*

$$T_i = \lambda_i(1 - B_i). \quad (1)$$

Using Lemma II.1, the time-averaged revenue R_i that a provider receives while sharing resources can be expressed as:

$$R_i = p_i\lambda_i(1 - B_i), \tag{2}$$

while the time-averaged revenue a provider receives when serving customers without sharing resources is:

$$\tilde{R}_i = p_i\lambda_i(1 - \tilde{B}_i). \tag{3}$$

We assume that provider i receives revenue R_i (as defined in (2)) from its customers, and is left with an effective revenue f_i after paying r_i to provider j and receiving a payment r_j from provider j. Hence, the effective revenues for the two providers are

$$\begin{aligned} f_1 &= R_1 - r_1 + r_2, \\ f_2 &= R_2 - r_2 + r_1. \end{aligned} \tag{4}$$

Note that we can simplify the inter-provider payments by defining

$$r_{net} = r_1 - r_2, \tag{5}$$

in which case (4) can be written as

$$\begin{aligned} f_1 &= R_1 - r_{net}, \\ f_2 &= R_2 + r_{net}, , \end{aligned} \tag{6}$$

where now (without loss of generality) provider 1 makes a net payment of r_{net} to provider 2, which could be positive or negative (which signifies a payment of $-r_{net}$ from provider 2 to provider 1).
Note also that

$$f_1 + f_2 = R_1 + R_2 = R_{total}, \tag{7}$$

where we define R_{total} to be the total revenue earned by the coalition.

Next, using concepts of cooperative game theory, we state some simple properties for forming coalitions in the two providers case, and describe Shapley value based revenue sharing to keep the coalition stable. Using these preliminaries, we will state the main problem in (14).

III. Inter-provider Payments (revenue Sharing)

We will now provide a formal definition of the considered game between two providers ([17], [18]). The game, which we will called POOLING, can be represented as $(v, \mathcal{P})$, where $\mathcal{P}$ is the set of players, and v is the characteristic function, defined later. Let the two providers be defined by the set $\mathcal{P} = \{P_1, P_2\}$, and let the power set of $\mathcal{P}$ be $2^{\mathcal{P}}$. Any non-empty subset $\mathcal{S}$ of $\mathcal{P}$ is called a coalition, and $\mathcal{P}$ is called the grand coalition. Each coalition is assigned a value or revenue, defined by the characteristic function $v : 2^{\mathcal{P}} \mapsto \mathbb{R}^+$, that denotes the value of the coalition. For POOLING,

$$\begin{aligned} &v(\{P_i\}) = \tilde{R}_i, \\ &v(\{\mathcal{P}\}) = R_1 + R_2 = R_{total}, \\ &v(\emptyset) = 0, \end{aligned} \tag{8}$$

where R and $\tilde{R}$ are defined in (2) and (3).
We assume that the game has a transferrable utility, i.e, revenue from the game can be shared freely among members of a coalition. These assumptions are natural for 2 providers with customer bases and revenue obtained from customers.

Definition III.1. Stable Coalition A stable coalition is one in which no member has an incentive to leave the coalition. Let the payoff that provider i receives be g_i out of $v(\{\mathcal{P}\})$. A coalition is defined to be stable if

$$g_i \geq v(\{P_i\}), i \in \{1, 2\}, \tag{9}$$

i.e., each provider does not receive less revenue in the coalition than if it acted on its own.

We now state a useful property of POOLING that is easy to prove.

Lemma III.1. *For* POOLING, *the characteristic function v is superadditive ([18]), i.e., for every $S, T \in 2^{\mathcal{P}}$ such that $S \cap T = \emptyset$,*

$$v(S \cup T) \geq v(S) + v(T). \tag{10}$$

While superadditivity is a necessary condition for a stable coalition (one whose members have no incentive to split from the coalition and act individually), it is not sufficient. To ensure stability, we must find a suitable payoff vector $[g_1\ g_2]^T$ satisfying (9). Such a payoff vector is said to belong to the core of the game being played.

Definition III.2. Core. The core of a coalitional game with two players is defined as the set of all payoff vectors satisfying (9), i.e, the set of stable payoff vectors - those that will ensure that the players of the game have no incentive to leave the grand coalition, $\mathcal{P} = \{P_1, P_2\}$.

Definition III.3. Shapley Value ([18]) For a game $(v, \mathcal{P})$, the Shapley value payoff vector is defined as:

$$g_i(v) = \sum_{S \subseteq \mathcal{P} \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!}(v(S \cup \{i\}) - v(S)), \tag{11}$$

where $n = |\mathcal{P}|$.

For POOLING, the Shapley value g_1 can be computed as follows:

$$\begin{aligned} g_1(v) &= \sum_{S \in \{\emptyset, \{P_2\}\}} \frac{|S|!(n - |S| - 1)!}{n!}(v(S \cup \{P_1\}) - v(S)), \\ &= \frac{1}{2}(v(\{P_1, P_2\}) - v(\{P_2\})) + \frac{1}{2}((v\{P_1\}) - v(\emptyset)), \\ &= \frac{1}{2}(v\{\mathcal{P}\} - v(\{P_2\}) + v(\{P_1\})), \\ &= \frac{1}{2}(R_1 + R_2 - \tilde{R}_2 + \tilde{R}_1). \end{aligned}$$

Similarly, we get the expression for Shapley value g_2, and the Shapley value payoffs for POOLING are:

$$\begin{aligned} g_1 &= \frac{1}{2}(R_1 + R_2 + \tilde{R}_1 - \tilde{R}_2), \\ g_2 &= \frac{1}{2}(R_1 + R_2 + \tilde{R}_2 - \tilde{R}_1), \end{aligned} \tag{12}$$

where R_i and $\tilde{R}_i$ are the revenues that provider i receives from its customers after and before joining the coalition respectively.

Definition III.4. Convex Game. A game is said to be convex if $v(S) + v(T) - v(S \cap T) \leq v(S \cup T)$, where $S \in 2^{\mathcal{P}}, T \in 2^{\mathcal{P}}, S \neq T$.

We state one result from [17] that will be useful:

Proposition III.2. If a game is convex, then the payoff vector given by the Shapley value belongs to the core of the game.

Using Proposition III.2, we the following property of POOLING.

Proposition III.3. The payoff vector given by the Shapley value belongs to the core of POOLING.

Proof. It is easy to check that POOLING is convex and the result follows from Lemma III.2. ∎

Thus, if we make the effective revenue f_i defined in (4) such that $f_i = g_i$ (12), then the effective revenue vector $[f_1, f_2]$ belongs to the core of POOLING, and thus ensures a stable coalition. Hence, we get the following result on how to find the inter-provider payment r_{net} such that the coalition is stable,

Proposition III.4. The value of inter-provider payment r_{net} (6) that ensures a stable coalition for POOLING is

$$r_{net} = R_1 - f_1 = \frac{1}{2}(R_1 - R_2 - \tilde{R_1} + \tilde{R_2}). \tag{13}$$

Another important implication of using $f_i = g_i$ is that since $\tilde{R_1}, \tilde{R_2}$ are pre-determined, selfishly maximizing f_1 and/or f_2 for provider 1 or 2 is equivalent to maximizing the $R_{total} = R_1 + R_2$. Thus, we next proceed to find the optimal sharing strategy (k_1, k_2) that maximizes R_{total}.

With this background, the **main problem** we want to solve is

$$\max_{k_1,k_2,k_1 \leq N_1, k_2 \leq N_2} R_{total},$$

where (14)

$$R_{total} = R_1 + R_2 = p_1\lambda_1(1 - B_1) + p_2\lambda_2(1 - B_2).$$

Note that solving (14) is not trivial, since it involves the blocking probabilities B_i defined in (17) that are not in closed form.

We begin by deriving equations describing the blocking probabilities B_i of the providers while in the coalition. Let $\mathbf{u} = (u_1, u_2)$ define the state of the system at any time instant, where u_i is the number of requests of provider i's customers that are being serviced in the system. Then, the feasible set of such ordered pairs is given by:

$$\mathcal{F} = \{\mathbf{u} \mid u_1 \leq N_1 + k_2, u_2 \leq N_2 + k_1, u_1 + u_2 \leq N_1 + N_2\}. \tag{15}$$

The number of active requests being serviced by the providers can now be modeled as a continuous time Markov process, with state space $\mathcal{F}$. Analysis now is possible, and results in the following stationary distribution (see [1]).

Lemma III.5. *[1] The stationary distribution for a state $\mathbf{u} \in \mathcal{F}$ can be expressed as*

$$\pi(\mathbf{u}) = \frac{1}{\mathcal{D}} \frac{\lambda_1^{u_1}}{u_1!} \frac{\lambda_2^{u_2}}{u_2!}, \quad \mathcal{D} = \sum_{\mathbf{u} \in \mathcal{F}} \frac{\lambda_1^{u_1}}{u_1!} \frac{\lambda_2^{u_2}}{u_2!}. \tag{16}$$

The blocking probability for provider i can now be expressed as:

$$B_i = \frac{1}{\mathcal{D}} \sum_{\mathbf{u} \in \mathcal{B}_i} \frac{\lambda_1^{u_1}}{u_1!} \frac{\lambda_2^{u_2}}{u_2!}, \tag{17}$$

where

$$\mathcal{B}_i = \{\mathbf{u} \mid u_i = N_i + k_j\} \cup \{\mathbf{u} \mid u_1 + u_2 = N_1 + N_2\}. \tag{18}$$

It is worthwhile to note that the expressions for blocking probabilities seen in (17) are identical to the expressions for blocking probabilities of routes on an appropriately chosen stochastic loss network (see [19]), described next, which will help us analyze the system further.

A. *Stochastic Loss Networks*

Stochastic loss networks are defined by a set of nodes, $\mathcal{N}$, and a set of *links*, $\mathcal{J}$ between nodes. Routes, ($R_i \in \mathcal{R}$, $i \in \{1, 2, 3...\}$ are defined as pre-determined subsets of links in the network that form a path between a source node and a destination node. Each link has a resource capacity C_i allotted to it. A request that arrives for a particular route blocks one resource on each link that belongs to that route, essentially reducing the capacity of all links on that route by one for as long as that request is active. Request arrivals & request service times for each of the routes are modeled as independent random processes. If the number of requests being served by any link on a route is equal to its capacity, all incoming requests to that route will subsequently be blocked. (See [20] for more on loss networks). We will now construct a loss network where the blocking probabilities on routes are defined by the same expressions as in (16).

The loss network whose routes' blocking probabilities will correspond to the blocking probabilities (16) will have 3 links, indexed by $\{1, 2, 3\}$ and two routes: route 1 = {link 1, link 3}, corresponding to provider 1 and route 2 = {link 2, link 3}, corresponding to provider 2 (see Fig. 2). The capacities of the three links are given by:

$$\begin{aligned} C_1 &= N_1 + k_2, \\ C_2 &= N_2 + k_1, \\ C_3 &= N_1 + N_2. \end{aligned} \tag{19}$$

We assume requests arrive at routes 1 and 2 according to independent Poisson processes with parameters λ_1 and λ_2 respectively, and the service times are i.i.d. and exponentially distributed with unit mean.

Let $\mathbf{t} = (t_1, t_2)$ define the state of the loss network at any

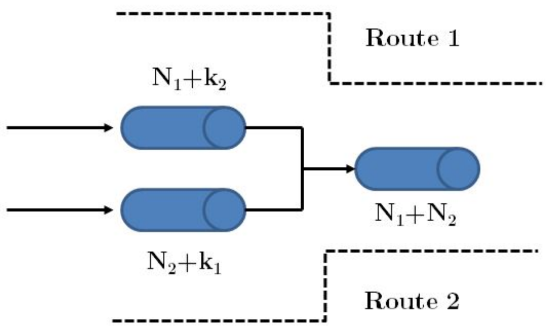

Fig. 2. Equivalent loss network

time instant, where t_i is the number of requests of provider i's customers (equivalently, the number of requests active in the route corresponding to provider i) that are being serviced in the network. It follows from the structure of the loss network that the description of the feasible set of $\mathbf{t}$ is identical to the description of the feasible set of $\mathbf{u}$ in (15), i.e.,

$$\mathcal{F}' = \{\mathbf{t} \mid t_1 \leq N_1 + k_2, t_2 \leq N_2 + k_1, t_1 + t_2 \leq N_1 + N_2\}. \tag{20}$$

Since the request arrival rates and service times are identical for both the considered model and the described loss network, modelling the behaviour of $\mathbf{t}$ as a continuous time Markov process with state space $\mathcal{F}'$ results in a stationary distribution for $\mathbf{t}$ identical to the stationary distribution for $\mathbf{u}$ described in (16). Hence, the expressions for blocking probabilities along the routes (equivalently, the blocking probabilities for the 2 providers) are identical to the expressions in (17).

IV. Quantum of Sharing

We state our main results on solving (14) in the next two theorems depending on p_1, p_2 (the price charged by the two providers).

Theorem IV.1. *If $p_1 = p_2$, the optimal solution to (14) is either $({k_1}^*, {k_2}^*) = (0, 0)$ or $({k_1}^*, {k_2}^*) = (N_1, N_2)$.*

Thus, either no coalition is formed by providers or providers share their resources completely when prices are equal.

Let $p_1 \neq p_2$, and define $T = \frac{p_2}{p_1} > 1$.

Theorem IV.2. *The optimal solution to (14) when $p_1 \neq p_2$ is either $({k_1}^*, {k_2}^*) = (0, 0)$ or ${k_1}^* = N_1$ and ${k_2}^* \leq N_2$, where the optimal k_2^* depends on the value of T and descibed by the solution of* (39).

Thus, either no coalition is formed by providers or the cheaper provider shares its resources fully, while provider 2 shares some of its resources, but not necessarily all of it.

To prove these results, we consider the equivalent loss network where finding an optimal sharing policy reduces to allotting optimal capacities, or *dimensioning* links 1 & 2 in the loss network previously described. The optimal dimensioning of loss networks is a problem that has been considered previously in the context of circuit-switched phone networks (see [21]). As is standard in loss network literature, we use what is called as the Erlang fixed-point approximation, used to approximate blocking probabilities for loss networks, as follows

The Erlang approximation assumes that the events that correspond to requests being blocked in a loss network are independent from link to link in a given route ([22]). Let the probability of a request being blocked by a single standalone link with capacity C when incoming requests are modelled as a Poisson process with intensity λ, and request service times are exponentially distributed with mean 1, be $E(\lambda, C)$, where

$$E(\lambda, C) = \frac{\frac{\lambda^C}{C!}}{\sum_{i=0}^{C} \lambda^i / i}. \tag{21}$$

i.e, $E(\lambda, C)$ can be described using the well-known Erlang-B function from queueing theory. The dependence of the blocking probability of a link on the states of the other links in a loss network is taken into account by the approximation using the concept of 'reduced traffic'. The traffic offered to each link on a route is approximated and assumed to be a Poisson process, but one whose intensity is decreased by the effects of blocking in the other links on that route - resulting in a *thinned* Poisson process.

For Fig. 2, that describes the equivalent loss network for our model, let b_i denote the blocking probability (i.e, the probability that a link's full capacity is being utilized to serve requests on the various routes that it is a part of) for link i and B_i denote the blocking probability for route i. Recall that routes correspond to providers, and that links 1 & 3 make up route 1, and links 2 & 3 make up route 2 (ref. Fig. 2). Then, by the independence assumption mentioned above, and reiterating that if any link on a route is serving requests at full capacity, all incoming requests to that route will subsequently be blocked, we can first write

$$\begin{aligned} B_1 &= 1 - (1 - b_1)(1 - b_3), \\ B_2 &= 1 - (1 - b_2)(1 - b_3). \end{aligned} \tag{22}$$

Then the Erlang fixed-point approximation ([22]) for our loss network can be expressed as:

$$\begin{aligned} b_1 &= E(\lambda_1(1 - b_3), N_1 + k_2), \\ b_2 &= E(\lambda_2(1 - b_3), N_2 + k_1), \\ b_3 &= E(\lambda_1(1 - b_1) + \lambda_2(1 - b_2), N_1 + N_2). \end{aligned} \tag{23}$$

Note that a request is blocked on a route when any link on the route is at full capacity. To understand (23), e.g., consider link 1, and its blocking probability b_1. Consider every route that the link 1 is a part of and those routes' incoming traffic intensities (link 1 is part of only route 1, which has incoming traffic intensity λ_1). We *thin* each of these intensities by removing the requests blocked by other links on each of these routes - temporarily assuming knowledge of other link blocking probabilities, and ultimately leading to the recursive nature of (23). This thinning is carried out by multiplying the traffic intensities by the complements of the blocking probabilities (i.e, probabilities associated with a request being accepted by a link, $1 - b_i$) of all other links on each route (in our example, we multiply λ_1 with $(1 - b_3)$, since link 3 is the only other link on route 1). Finally, we sum all associated traffic intensities to obtain the total reduced intensity that is used in the Erlang-B function to obtain an expression for link blocking probabilities (our example hence yields $b_1 = E(\lambda_1(1 - b_3), N_1 + k_2)$).

Kelly ([23]) proved that the fixed point of link blocking probabilities (i.e, the solution to (23)) both exists and is unique via Brouwer's fixed point theorem. Hence, it is always possible to iteratively compute approximate blocking probabilities from (23). For our purpose, we will now use equations (23) to derive analytical results about optimal k_1 and k_2, via the concept of shadow prices.

Definition IV.1. Shadow price. The shadow price of a link in a loss network is defined as the derivative of the total revenue generated by the loss network with respect to the capacity of that link. The shadow price s_j of a link j for our model is :

$$s_j = \frac{dR_{total}}{dC_j}, \tag{24}$$

where $R_{total} = R_1 + R_2$ is the total revenue (14), and R_i's are defined in (2).

Note that if we consider the capacity of a single link in a loss network, while fixing the capacities of all other links in the network, we can state the following: if the shadow price for a link is positive, the revenue earned by the entire loss network can be increased by increasing the capacity of that link, and if negative, revenue can be increased by decreasing capacity. If the shadow price is zero, then the capacity of that link is at its optimal value. This is how shadow prices are used to dimension loss networks, and we will use this concept to find the optimal sharing strategy for the two providers.

Shadow prices can be calculated via a set of equations that yield a unique fixed-point solution to the values of the shadow prices, given the dimensioning of and the traffic incoming to the loss network. These equations for a general loss network were derived by Kelly in [23], and the set of equations that yield shadow prices for our specific loss network are:

$$\begin{aligned} s_1 &= \eta_1\lambda_1(1-b_3)(p_1 - s_3), \\ s_2 &= \eta_2\lambda_2(1-b_3)(p_2 - s_3), \\ s_3 &= \eta_3(\lambda_1(1-b_1)(p_1 - s_1) + \lambda_2(1-b_2)(p_2 - s_2)), \end{aligned} \tag{25}$$

where $\eta_i = \eta(\rho_i, C_i) = E(\rho_i, C_i - 1) - E(\rho_i, C_i)$, and

$$\begin{aligned} \rho_1 &= \lambda_1(1-b_3), \\ \rho_2 &= \lambda_2(1-b_3), \\ \rho_3 &= \lambda_1(1-b_1) + \lambda_2(1-b_2), \end{aligned} \tag{26}$$

where $\{b_1, b_2, b_3\}$ are as defined in (23).
Finally, we are ready to prove Theorem IV.1 and IV.2 in the next two subsections.

V. Proofs

A. Equal price $p_1 = p_2$

Proof of Theorem IV.1. We prove here that if $({k_1}^*, {k_2}^*) \neq (0,0)$ then $({k_1}^*, {k_2}^*) = (N_1, N_2)$. With $p_1 = p_2 = p$, the shadow price equations (as defined in eq. (25)) simplify to

$$s_1 = \eta_1\lambda_1(1-b_3)(p - s_3), \tag{27a}$$

$$s_2 = \eta_2\lambda_2(1-b_3)(p - s_3), \tag{27b}$$

$$s_3 = \eta_3(\lambda_1(1-b_1)(p - s_1) + \lambda_2(1-b_2)(p - s_2)), \tag{27c}$$

Since $\eta_i > 0$ and $\lambda_i > 0$, (27a, b) implies that s_1 and s_2 will both either be positive, negative or zero. We assume that the loss network starts from an under-dimensioned state, i.e, initially, $k_1 = 0$ and $k_2 = 0$. We have assumed that this is not the optimal solution to (14), and hence it is optimal for the providers to share some of their resources. Hence, we have $s_1 > 0, s_2 > 0$ at $k_1 = 0$ and $k_2 = 0$.

To prove the theorem, we will prove that the shadow prices s_1 and s_2, as functions of C_1 and C_2 and hence of k_1 and k_2 (since N_1 and N_2 are constant) never go below zero. Hence, the providers, who are trying to maximize the total revenue earned by the coalition, always have an incentive to share more and they will share as much of their resources as they can - resulting in full sharing. Note that this proof method is valid only when considering continuous C_i, not discrete. [1]

To prove that s_1 and s_2 are always positive, we start by assuming that for optimal $k_1 \leq N_1$ and $k_2 \leq N_2$:

$$\begin{aligned} s_1 &= 0, \\ s_2 &= 0. \end{aligned} \tag{28}$$

We will prove by contradiction that (28) can never be true for any value of k_1 and k_2, and hence, that shadow prices s_1 and s_2 will always be positive. (Note that since $\eta_i > 0$ and $\lambda_i > 0$ and by (27a, b), if we assume s_1 to be zero, s_2 must also be zero, and vice versa. Hence, it is sufficient to prove that (28) can never be true.)
Also note from the definition of the Erlang-B function in (21) that the value of the Erlang-B function is always less than 1, except for when $C = 0$ or λ goes to infinity. Hence, from the definition of b_3 (in (23)), we can conclude that $b_3 < 1$ unless:

$$N_1 + N_2 = 0 \text{ or } \lambda_1(1-b_1) + \lambda_2(1-b_2) \to \infty.$$

Since we have $N_1 > 0$, and $N_2 > 0$, and assuming finite λ_1 and λ_2, we get:

$$b_3 < 1, \tag{29}$$

and hence, from (27), (28) and (29),

$$s_3 = p. \tag{30}$$

[1] Kelly ([23]) however assumes continuous link capacities in his shadow price analysis, and our proof doesn't assume discrete capacities anywhere.

Substituting for $s_3 = p$ in (27),

$$\begin{aligned} & p = p\eta_3(\lambda_1(1-b_1)+\lambda_2(1-b_2)), \\ & \implies 1 = \eta_3(\lambda_1(1-b_1)+\lambda_2(1-b_2)), \\ & \implies \rho_3\eta_3 = 1. \end{aligned} \tag{31}$$

We will now prove that (31) can never be a valid equation, hence establishing our contradiction.

Lemma V.1. *$\rho\eta(\rho, C) < 1$, where $\eta(\rho, C) = E(\rho, C-1) - E(\rho, C), \forall$ finite ρ, C.*

Proof. For finite ρ,

$$\rho\eta(\rho, 1) = \rho(E(\rho, 0) - E(\rho, 1)) = \frac{\rho}{\rho+1} < 1, \tag{32}$$

From [24], we know that the Erlang-B function $E(\rho, C)$ is convex in C. Hence, we have,

$$E(\rho, C-1) - E(\rho, C) > E(\rho, C) - E(\rho, C+1), \tag{33}$$

and it follows that $\eta(\rho, C) = E(\rho, C-1) - E(\rho, C)$ is a decreasing function of C. Moreover, it also follows that $\rho\eta(\rho, C)$ is a decreasing function of C. Combining this fact with (32), we get that $\forall$ finite ρ and C, $\rho\eta < 1$. ■

Thus, we have arrived at a contradiction (i.e., that (31) cannot be true), and hence our initial assumption (28) is wrong. Hence, the shadow prices s_1 and s_2 can never be zero for any $k_1 \leq N_1$ and $k_2 \leq N_2$, and it is always in the best interest of the providers to dimension their loss network to the fullest, i.e ${k_1}^* = N_1, {k_2}^* = N_2$. ■

It is noteworthy that this full sharing strategy is optimal for any price $p_1 = p_2 = p$, and is independent of the total number of resources with the providers (N_1 and N_2). It is also independent of the traffic intensities (λ_1 and λ_2) that the two providers experience. This independence is surprising. For example, if provider 1 was experiencing a much higher request intensity than provider 2 (i.e, $\lambda_1 >> \lambda_2$) and the providers had equal resources ($N_1 = N_2$), intuitively it seems that the coalition would be benefited by provider 1 withholding some of its resources to service its high request volume, instead of allowing any of its resources to be blocked by customers of the less congested provider 2. Our analysis, however, shows that regardless of traffic intensity, full sharing is always optimal.

B. Unequal Prices $p_1 \neq p_2$, where $T = \frac{p_2}{p_1} > 1$.

Proof of Theorem IV.2. We begin by considering that $({k_1}^*, {k_2}^*) \neq (0, 0)$. Our proof method is as follows: assume initially that the prices are equal, i.e., $p_1 = p_2$ and hence $T = 1$, and that the network is optimally dimensioned, i.e., ${k_1}^* = N_1$ and ${k_2}^* = N_2$. We will show that as T increases (and holding link dimensioning constant), s_2 remains positive for any value of $T > 1$. This means that the optimal dimensioning for C_2 (and hence k_1) is to maximize capacity C_2 (19) and the optimal ${k_1}^* = N_1$, same as in when $T = 1$. On the other hand, we will show that s_1 decreases in magnitude, crosses zero and becomes increasingly negative as T increases. For values of T such that $s_1 \geq 0$, similar to Theorem IV.1, we get ${k_2}^* = N_2$. For values of T such that $s_1 < 0$, it follows from (24) that in order to maximize total revenue, C_1 (and hence k_2) must be decreased which implies that ${k_2}^* < N_2$, and we will describe the equation whose solution yields k_2^*.

Writing the shadow price equations (from (25)) again:

$$\begin{aligned} & s_1 = \eta_1\lambda_1(1-b_3)(p_1 - s_3), \\ & s_2 = \eta_2\lambda_2(1-b_3)(p_2 - s_3), \\ & s_3 = \eta_3(\lambda_1(1-b_1)(p_1 - s_1) + \lambda_2(1-b_2)(p_2 - s_2)). \end{aligned} \tag{34}$$

Upon inspection of these equations, it is clear that dividing all the three equations in (34) by a constant value will result in a set of three equations with appropriately scaled shadow prices. We hence divide all 3 equations by p_1, and in a slight abuse of notation, continue to refer to the scaled shadow prices as s_i. From (34), the shadow price equations are:

$$\begin{aligned} & s_1 = \eta_1\lambda_1(1-b_3)(1 - s_3), \\ & s_2 = \eta_2\lambda_2(1-b_3)(T - s_3), \\ & s_3 = \eta_3(\lambda_1(1-b_1)(1 - s_1) + \lambda_2(1-b_2)(T - s_2)). \end{aligned} \tag{35}$$

We can treat (35) as a set of 3 linear equations in $\{s_1, s_2, s_3\}$. Solving for s_1 and s_2, we get:

$$s_1 = \frac{B(1-CD) + B(-A - TC + TCD)}{-(AB + CD - 1)}, \tag{36a}$$

$$s_2 = \frac{D(1-AB)T + D(-A + AB - CT)}{-(AB + CD - 1)} \tag{36b}$$

where $A = \eta_3\lambda_1(1-b_1), B = \eta_1\lambda_1(1-b_3) = \eta_1\rho_1, C = \eta_3\lambda_2(1-b_2), D = \eta_2\lambda_2(1-b_3) = \eta_2\rho_2$. The inequality established in Lemma V.1 can be directly used to obtain the following 3 inequalities: $A + C = \eta_3\rho_3 < 1, B < 1, D < 1$, which yields

$$AB + CD < 1. \tag{37}$$

Hence, the denominators of the expressions for s_1 and s_2 in (36a, b) are positive. Consider now the numerators of the expressions for:

- s_1:
  $B(1-CD) + B(-A - TC + TCD)$
  $= B(1 - A - CD) + T(C(D-1))$.
  When $T = 1$, i.e, equal prices, this simplifies to $B(1 - A - C)$, which is greater than 0 (since $A + C < 1$). Hence, s_1 is positive initially.
  Now, note that $D < 1$. Hence, $C(D-1) < 0$. Hence, as T increases, s_1 decreases until it crosses 0 at some value of T, and continues to increase in magnitude while remaining negative.
- s_2:
  $D\{A(B-1) + T(1 - C - AB)\}$.
  When $T = 1$, this simplifies to $1 - A - C > 0$.
  Since $A + C < 1$ and $B < 1$, it follows that $AB + C < 1$. Hence, as T increases, s_2 becomes increasingly positive (since $1 - C - AB > 0$).

■

An equation describing ${k_2}^*$ (optimal k_2) for a given T can be obtained by equating s_1 to zero (36a). Via a simple rearrangement of terms, we get:

$$1 + (T - 1)CD = TC + A, \tag{38}$$

which is equivalent to

$$1 + (T - 1)\eta_2\eta_3{\lambda_2}^2(1 - b_2)(1 - b_3) = T\eta_3\lambda_2(1 - b_2) + \eta_3\lambda_1(1 - b_1), \tag{39}$$

which together with (23) and ${k_1}^* = N_1$ can be used to recursively find ${k_2}^*$.

VI. Conclusion

In this paper, we considered the problem of finding how many resources to commit to the common pool in a cooperative pooling game. First using the concept of Shapley value, we showed that there is a 'fair' way of splitting the total revenue such that both the providers have an incentive over acting alone. This particular revenue split also resulted in a single objective function for both the 'selfish' providers of maximizing the total revenue as a function of their individual resources. We showed that when both the providers charge their clients the same price per request, both providers should completely share their resources in order to maximally utilize the power of multiplexing. When the charges are unequal, we show that the provider that charges less should contribute all its resources to the common pool, while for the provider that charges more, the optimal resources to commit is given by a recursive equation. The results derived in this paper are basic in nature and can be used to tackle more complicated problems, such as how to find the optimal prices for each provider etc.

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