Optimal Joint Partitioning and Licensing of Spectrum Bands in Tiered Spectrum Access under Stochastic Market Models

Gourav Saha, and Alhussein A. Abouzeid

***Abstract*—We consider the problem of partitioning an entire band into M channels of equal bandwidth, and then further assigning these M channels into $P \leq M$ licensed channels and $M - P$ unlicensed channels. Licensed channels can be accessed both for licensed use and opportunistic use while unlicensed channels can be accessed only for opportunistic use. The access to licensed channels follows a tiered structure, where licensed use has a higher priority than opportunistic use. We address the following question in this paper. Given a market setup, what values of M and P maximize the net spectrum utilization of the entire bandwidth? This abstract problem is highly relevant in practical scenarios, e.g., in the context of partitioning the recently proposed Citizens Broadband Radio Service band. If M is too high (low), it may decrease (increase) spectrum utilization due to limited (wastage of) channel capacity. If P is too high (low), it will not incentivize the wireless operators who are primarily interested in licensed (unlicensed) channels to join the market. These trade-offs are captured in the optimization problem which is modeled as a two-stage Stackelberg game consisting of the regulator and the wireless operators. We design an algorithm to solve the Stackelberg game in order to find the optimal M and P. We use this algorithm to obtain interesting numerical results that suggest how the optimal values of M and P change with different market settings.**

I. Introduction

To support the ever growing wireless data traffic, the Federal Communication Commission (FCC) released the underutilized Citizens Broadband Radio Service (CBRS) band for shared use in 2015 [1]. CBRS band is a 150 MHz federal spectrum band from 3.55 GHz to 3.7 GHz. The 150 MHz band is divided into 15 channels of 10 MHz each. The shared use of the CBRS band follows an order of priority. Federal users have the highest priority access to the channels. Out of the 15 channels, 7 are Priority Access Licenses (PALs). PAL licenses are sold through auctions and the lease duration of a PAL license may range between $1-10$ years [1], [2], [3]. A PAL license holder can use their channel only if federal users are not using it. The remaining 8 out of the 15 channels are reserved only for opportunistic use by General Authorized Access (GAA) users. Opportunistic channel allocation to GAA users can happen at a time scale of minutes to days. GAA users can use these 8 channels if federal users are not using the channels. GAA users can also use the 7 PAL channels provided that neither federal users nor PAL license holders are using it.

As mentioned in the previous paragraph, the CBRS band is divided into $M = 15$ channels out of which there are $P = 7$ PAL licenses. But does $M = 15$ and $P = 7$ maximize the utilization of the CBRS band? In this paper, we are interested

G. Saha and A. A. Abouzeid are with the Department of Electrical, Computer and Systems Engineering, Rensselaer Polytechnic Institute, Troy, NY 12180, USA; Email: sahag@rpi.edu, abouzeid@ecse.rpi.edu.

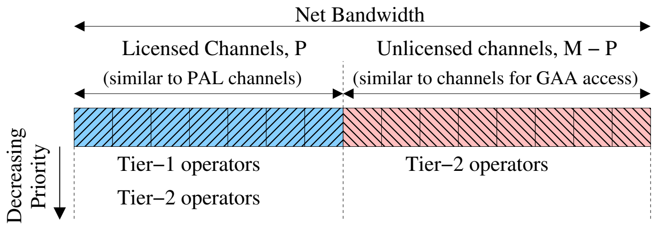

Fig. 1. Figure showing the tiered spectrum model used in this paper.

in the following abstraction of this question whose application is not limited to CBRS band. A net bandwidth is partitioned into M channels of equal bandwidth. These M channels are further partitioned into P licensed channels (similar to PAL channels) and $M - P$ unlicensed channels (similar to channels reserved for GAA users). Licensed channels are sold through periodic auctions. Wireless operators who are allocated these channels are called Tier-1 operators and they have highest priority access to licensed channels. Those wireless operators who use channels opportunistically are called Tier-2 operators. Licensed channels can also be used by Tier-2 operators, but the priority is given to Tier-1 operators. Unlicensed channels are reserved exclusively for Tier-2 operators. This spectrum access model is shown in Figure 1. The wireless operators earn revenue by serving customer demands. A wireless operator is incentivized to join the market only if the revenue which it can earn is above a desired threshold. For the given setup, what value of M and P maximizes spectrum utilization where spectrum utilization is defined as the net amount of customer demand served by the entire bandwidth?

There are various factors that decide the optimal values of M and P. Some of these factors are as follows. If the number of channels M increases, the bandwidth, and hence capacity, of each channel decreases. The capacity of each channel should be large enough to accommodate a good portion of the customer demand of a wireless operator but not so large that most of the capacity of the channel is not utilized for majority of the time. This suggests that M should not be too small or too large. If the number of licensed channels P is too high, there is a small number of unlicensed channels. Therefore, those operators who primarily rely on unlicensed channels to serve customer demands will not be able to generate enough revenue and hence will not be incentivized to join the market. Similarly, if the P is too low, wireless operators who primarily rely on licensed channels to serve customer demands will not be incentivized to join the market. P should be set such that enough operators join the market to ensure that the customer demands served over the entire bandwidth is as high as possible. There may be other

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qualitative factors governing optimal M and P. Given that the problem is quite complex, it may not be possible to qualitatively account for all the factors while deciding the optimal M and P. Therefore, in this paper, we design an algorithm to jointly optimize M and P such that spectrum utilization is maximized.

A. Related Work

Variations of the spectrum partitioning and spectrum licensing problems considered in this paper have been studied separately, but not jointly, in the spectrum sharing and related fields. There are a few works that have addressed problems similar to partitioning of a fixed bandwidth into optimal number of channels. In [4] the authors derive an analytical expression for the optimal number of channels such that the spatial density of transmission is maximized subject to a fixed link transmission rate and packet error rate. Partitioning of bandwidth in the presence of guard bands has been considered in [5] where the authors used a Stackelberg game formulation to analyze how a spectrum holder should partition its bandwidth in order to maximize its revenue in spectrum auctions.

The second problem studied in this paper deals with allocating a fixed bandwidth for licensed and unlicensed use. This has been studied in the literature from various perspectives. Some works concentrated on minimizing the amount of bandwidth allocated to backup channels (unlicensed channels in our case) while providing a certain level of guarantee to secondary users against channel preemption [6]. A few research efforts on overlay D2D and cellular devices, like [7], studied optimal partitioning of orthogonal in-band spectrum to maximize the average throughout rates of cellular and D2D devices. In [8], the authors investigated whether to allocate an additional spectrum band for licensed or unlicensed use and concluded that the licensed use is more favourable for maximizing the social surplus. Authors in [9] studied the CBRS band for a market setup which consists of Environmental Sensing Capability operators (ESCs) whose sole job is to monitor and report spectrum occupancy to the wireless operators. The authors analyzed how the ratio of the licensed and unlicensed band effects the market competition between the ESC operators, the wireless operators and the end users of the CBRS band. There is a line of work which studies spectrum partitioning for topics similar to licensed and unlicensed use using Stackelberg games; macro cells and small cells [10], [11], long-term leasing market and short-term rental market [12], and 4G cellular and Super WiFi services [13].

Such a diverse literature just on spectrum partitioning and licensing is justified because individual problem setups have their own salient features and hence require their own analysis. Our problem setup considers jointly optimizing spectrum partitioning and spectrum licensing, which has not been considered in the existing literature. Furthermore, the following reasons make the problem more challenging. *First*, the licensed channels can be used for opportunistic use by Tier-2 operators (similar to CBRS band). *Second*, the decision variables (M and P) are discrete. *Third*, M and P have to be chosen in order to incentivize the optimal set of operators to join the market.

B. Contribution and Paper Organization

We now present an overall outline of the paper and, in the process, discuss its main contributions. In Section II, we present a system model which can mathematically capture the effect of the number of channels, M, and number of licensed channels, P, on the spectrum utilization. The proposed system model captures spectrum auctions using a simple stochastic model without going into complex game-theoretic formulations. Our system model can, not only capture spectrum sharing in CBRS band, but also a variant of CBRS by simply changing a model parameter. The formulation of a simple yet generalized system model constitutes the first contribution of the paper.

It is possible that a choice of values of M and P that incentivizes one group of wireless operators may not incentivize another group. Therefore, it may not be possible to choose M and P that incentivize all the wireless operators. Even if it is possible to satisfy all the operators, it may not be optimal to do so in terms of maximizing the spectrum utilization. We capture this idea using a Stackelberg game in Section III-A which forms the second contribution of the paper. The Stackelberg game consists of the regulator (leader) and the wireless operators (followers). In the first stage, the regulator sets M and P to maximize spectrum utilization. In the second stage, the wireless operators decide whether or not to join the market based on the M and P set by the regulator in the first stage.

The solve the Stackelberg game, we have to calculate the expected revenue of an operator and expected spectrum utilization. The complex nature of the problem does not allow simple analytical formulas of these expected values. Therefore, we develop a Monte Carlo integrator to compute these expected values. Our choice of using a Monte Carlo integrator over deterministic numerical integration techniques is because the solution involves evaluation of high-dimensional integrals. Unlike deterministic numerical integration techniques, the computation time of Monte carlo integration does not scale with dimension. One of the main bottlenecks of Monte Carlo integration is random sampling. While designing our Monte Carlo integrator, we reduced random sampling as much as possible to make it more time efficient. Designing an efficient Monte Carlo integrator is the third contribution of the paper.

Finally, in Section III-B, we design an algorithm to solve the Stackelberg game. We approach this in steps. *First*, we discuss few properties of the expected revenue of an operator which lead to a simple solution of the second stage of the Stackelberg game. *Second*, to solve the first stage of the Stackelberg game, we use grid search to find the optimal M and P which maximize spectrum utilization. To the best of our knowledge, joint optimization of the two spectrum partitioning problems has not been considered in the existing related literature. Hence, designing an algorithm for joint optimization of M and P is the fourth contribution of the paper. We use this algorithm to obtain important numerical results in Section IV which constitutes the final contribution of the paper. Our numerical results show how optimal values of M and P vary with market parameters.

TABLE I
A TABLE OF IMPORTANT NOTATIONS.

Notation	Description
t , γ	t^{th} time slot and epoch γ resp.
M , P	Number of channels and number of licensed channels resp.
D	Maximum units of customer demand that can be served by the entire bandwidth if used by Tier-1 operators.
α_L , α_U	Interference parameter associated with licensed and unlicensed channels resp.
$\mathcal{S}_L^C$, $\mathcal{S}_U^C$	Set of candidate licensed and unlicensed operators resp.
$\mathcal{S}_L$, $\mathcal{S}_U$	Set of interested licensed and unlicensed operators resp.
$\mathcal{T}_1(\gamma)$	Set of Tier-1 operators in epoch γ, i.e. set of interested licensed operators who won licensed channels in epoch γ.
$\overline{\mathcal{T}}_1(\gamma)$	Set of interested licensed operators who did not win licensed channels in epoch γ.
$\mathcal{T}_2(\gamma)$	Set of Tier-2 operators in epoch γ; $\mathcal{T}_2(\gamma) = \overline{\mathcal{T}_1}(\gamma) \bigcup \mathcal{S}_U$.
$x_k(t)$	Customer demand of the k^{th} operator in t^{th} time slot.
μ_k^θ , σ_k^θ	Mean and standard deviation resp. of gaussian random variable $\theta_k(t)$ where, $x_k(t) = \max(0, \theta_k(t))$.
$\widetilde{x}_{k,i}(t)$	Amount of customer demand served by the k^{th} operator in the t^{th} time slot if it is a Tier-i operator where $i \in \{1, 2\}$.
$X_{k,i}(\gamma)$	Net demand served by the k^{th} operator in epoch γ if it is a Tier-i operator.
$R_{k,i}(\gamma)$	Revenue of k^{th} operator in epoch γ if it is a Tier-i operator.
$h_k(\cdot)$	A function associated with k^{th} operator which maps the mean of $X_{k,i}(\gamma)$ to the mean of $R_{k,i}(\gamma)$.
$\sigma_{k,i}^R$	Standard deviation of $R_{k,i}(\gamma)$.
ρ_k	Correlation coefficent between $X_{k,i}(\gamma)$ and $R_{k,i}(\gamma)$.
λ_k	Minimum revenue requirement of the k^{th} operator.
ξ_k	The tuple $\left(\mu_k^\theta, \sigma_k^\theta, h_k(\cdot), \sigma_{k,i}^R, \rho_k, \lambda_k\right)$ associated with the k^{th} operator.

II. SYSTEM MODEL

The list of important notations is included in Table I. Other notations used in the paper are standard.

A. Channel Model

A net bandwidth of $W\, Hz$ is partitioned into M channels of equal bandwidth $\frac{W}{M}$. Out of the M channels, P channels are assigned as licensed channels and the remaining $M-P$ channels are unlicensed channels. In our model, time is divided into slots where $t \in \mathbb{Z}^+$ denotes the t^{th} time slot. Licensed channels are allocated both for licensed use and opportunistic use while unlicensed channels are allocated only for opportunistic use. Allocation of licensed channels for licensed use happens through auctions and the operators who are allocated the channel are called Tier-1 operators. These auctions occur every $T \geq 1$ time slots where T is the lease duration. An entire lease duration is called an "epoch". Epoch γ is from time slot $(\gamma-1)T+1$ to γT (including the bounds). In our model, an operator can be allocated at most one licensed channel, i.e. spectrum cap is one. Allocation of licensed channels and unlicensed channels for opportunistic use occur every time slot. Wireless operators who use channels opportunistically are called Tier-2 operators. According to our model, an operator can either be a Tier-1 or a Tier-2 operator in a given epoch, i.e. an operator cannot serve customer demand using both licensed and unlicensed channel simultaneously.

The capacity of a channel/band is the maximum units of customer demand that can be served using that channel/band in a time slot. Let D denote the capacity of the entire bandwidth of $W\, hz$ if used by Tier-1 operators. As the entire bandwidth is partitioned into M channels, each licensed channel has a capacity $\frac{D}{M}$ when used by Tier-1 operators while the unlicensed channels has a capacity $\frac{\alpha_U D}{M}$ where $\alpha_U \in [0, 1]$ is the interference parameter of Tier-2 operators associated with unlicensed channels. Licensed channels can also be used by Tier-2 operators following the priority hierarchy. Let a Tier-1 operator use a licensed channel to serve d units of customer demand where $d \leq \frac{D}{M}$. Then, according to our model, a Tier-2 operator can serve up to $\alpha_L\left(\frac{D}{M} - d\right)$ units of customer demand per time slot using the licensed channel where $\alpha_L \in [0, 1]$ is the interference parameter of Tier-2 operators associated with licensed channels. The interference parameters α_L and α_U capture the lower efficiency of opportunistic use compared to licensed use [1]. In general, we expect $\alpha_L \leq \alpha_U$. This may happen because the maximum transmission power cap for opportunistic use may be lower for licensed channels compared to unlicensed channels in order to protect Tier-1 operators from harmful interference.

B. Operators, Demand, and Revenue Model

The market consists of the *candidate licensed operators* denoted by $\mathcal{S}_L^C$ and the *candidate unlicensed operators* denoted by $\mathcal{S}_U^C$ where $\mathcal{S}_L^C$ and $\mathcal{S}_U^C$ are disjoint sets. A candidate licensed operator is a Tier-1 operator in those epochs in which it is allocated a licensed channel in the auction and a Tier-2 operator in those epochs in which it is not allocated a licensed channel. A candidate unlicensed operator is always a Tier-2 operator. A candidate operator has to invest in infrastructure development if it wants to join the market.

In order to generate return on infrastructure cost, a candidate operator wants to earn a minimum expected revenue in an epoch. Let λ_k be the minimum expected revenue (MER) of the k^{th} operator. A subset of candidate licensed and unlicensed operators are interested in joining the market if the value of M and P set by the regulator is such that the expected revenue of the operator in an epoch is greater than its MER. The set of *interested licensed operators* and *interested unlicensed operators* are denoted by $\mathcal{S}_L$ and $\mathcal{S}_U$ respectively. We have $\mathcal{S}_L \subseteq \mathcal{S}_L^C$ and $\mathcal{S}_U \subseteq \mathcal{S}_U^C$. The set of operators, $\left(\mathcal{S}_L^C - \mathcal{S}_L\right) \bigcup \left(\mathcal{S}_U^C - \mathcal{S}_U\right)$, does not join the market. A candidate licensed/unlicensed operator gets to decide whether to join or not join the market only once. An operator gets to participate in auctions for licensed channels or to use channels opportunistically only if it decides join the market.

In our model, every operator has a separate pool of customers each with its own stochastic demands, i.e. we do not model price competition between operators to attract a common pool of customers. Consider the t^{th} time slot of epoch γ. The customer demand, or simply demand, of the k^{th} operator in the t^{th} time slot is $x_k(t)$ which is a stochastic process. In our model, $x_k(t) = \max(0, \theta_k(t))$ where $\theta_k(t)$ is a Gaussian random variable with mean μ_k^θ and standard deviation σ_k^θ, i.e. $\theta_k(t) \sim \mathcal{N}\left(\mu_k^\theta, \left(\sigma_k^\theta\right)^2\right)$, $\forall t$. The k^{th} operator may be able to serve only a fraction of the customer demand. Let $\widetilde{x}_{k,1}(t)$ and $\widetilde{x}_{k,2}(t)$ denote the amount of customer demand served by the k^{th} operator if it is a Tier-1 operator and Tier-2 operator respectively in epoch γ. We have $\widetilde{x}_{k,1}(t) \le x_k(t)$ and $\widetilde{x}_{k,2}(t) \le x_k(t)$ where,

$$\widetilde{x}_{k,1}(t) = \min\left(x_k(t), \frac{D}{M}\right) \quad (1)$$

$$\widetilde{x}_{k,2}(t) = G_k(Y(t)) \quad (2)$$

Equations 1 and 2 can be explained as follows. If the k^{th} operator is allocated a licensed channel, it is a Tier-1 operator. As mentioned in Section II-A, the capacity of a licensed channel is $\frac{D}{M}$ when used by Tier-1 operators. Therefore, if the k^{th} operator is allocated a licensed channel, it serves $\min\left(x_k(t), \frac{D}{M}\right)$ units of customer demand in the t^{th} time slot as shown in (1). In (1), for Tier-1 operators, $\widetilde{x}_{k,1}(t)$ is a time-invariant function of i.i.d. random variable $x_k(t)$. Similarly, in (2), the demand served by Tier-2 operators, $\widetilde{x}_{k,2}(t)$, is expressed as a function $G_k(\cdot)$ of a vector of random variables $Y(t)$. It will be shown in Section II-C that the function $G_k(\cdot)$ is time-invariant and random variables $Y(t)$ are i.i.d. In short, $\widetilde{x}_{k,1}(t)$ and $\widetilde{x}_{k,2}(t)$ can be expressed as a time invariant function of i.i.d. random variables. Therefore, $\widetilde{x}_{k,1}(t)$ and $\widetilde{x}_{k,2}(t)$ are i.i.d. random variables as well. Throughout the rest of the paper we will use the subscript k, i, where $i \in \{1, 2\}$, to denote variables associated with k^{th} operator when its is a Tier i operator.

Let $\widetilde{\mu}^x_{k,i}$ and $\widetilde{\sigma}^x_{k,i}$ denote the mean and standard deviation of $\widetilde{x}_{k,i}(t)$ respectively. $\widetilde{\mu}^x_{k,1}$ and $\widetilde{\sigma}^x_{k,1}$ can be expressed as follows

$$\widetilde{\mu}^x_{k,1} = \int_0^{\frac{D}{M}} \vartheta f^\theta_k(\vartheta)\, d\vartheta + \frac{D}{M}\int_{\frac{D}{M}}^{\infty} f^\theta_k(\vartheta)\, d\vartheta \quad (3)$$

$$\widetilde{\sigma}^x_{k,1} = \sqrt{\int_0^{\frac{D}{M}} \vartheta^2 f^\theta_k(\vartheta)\, d\vartheta + \left(\frac{D}{M}\right)^2 \int_{\frac{D}{M}}^{\infty} f^\theta_k(\vartheta)\, d\vartheta - \left(\widetilde{\mu}^x_{k,1}\right)^2} \quad (4)$$

where $f^\theta_k(\vartheta)$ is the probability density function (pdf) of $\theta_k(t)$. Unlike $\widetilde{\mu}^x_{k,1}$ and $\widetilde{\sigma}^x_{k,1}$, analytical expressions for $\widetilde{\mu}^x_{k,2}$ and $\widetilde{\sigma}^x_{k,2}$ are generally not possible because of the complex nature of the function $G_k(\cdot)$ in (2). We have designed a Monte Carlo integrator which can compute $\widetilde{\mu}^x_{k,2}$ and $\widetilde{\sigma}^x_{k,2}$. The explanation of the Monte Carlo integrator can be found in [14].

Let the k^{th} operator be in Tier i in epoch γ. The net demand served by the k^{th} operator in epoch γ is

$$X_{k,i}(\gamma) = \sum_{t=(\gamma-1)T+1}^{\gamma T} \widetilde{x}_{k,i}(t) \quad (5)$$

Since $\widetilde{x}_{k,i}(t)$ is i.i.d. random variable and the lease duration T is quite large in practice, $X_{k,i}(\gamma)$ can be approximated as a Gaussian random variable using *Central Limit Theorem* whose mean $\mu^X_{k,i}$ and standard deviation $\sigma^X_{k,i}$ are given by

$$\mu^X_{k,i} = \widetilde{\mu}^x_{k,i} T \quad ; \quad \sigma^X_{k,i} = \widetilde{\sigma}^x_{k,i}\sqrt{T} \quad (6)$$

To this end we have, $X_{k,i}(\gamma) \sim \mathcal{N}\left(\mu^X_{k,i}, \left(\sigma^X_{k,i}\right)^2\right)$, $\forall\gamma$. An operator generates revenue by serving customer demand. If

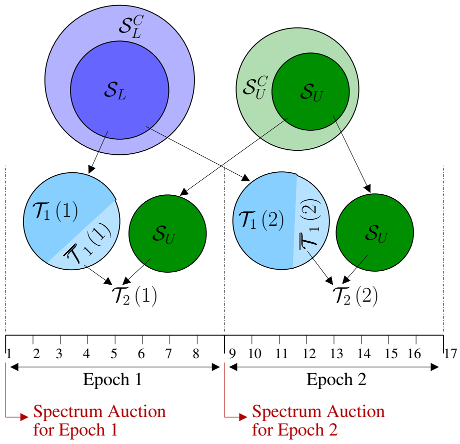

Fig. 2. Pictorial representation of the set of candidate licensed operators $\mathcal{S}_L^C$, candidate unlicensed operators $\mathcal{S}_U^C$, interested licensed operators $\mathcal{S}_L$, interested unlicensed operators $\mathcal{S}_U$, the set of interested licensed operators who are allocated (not allocated) licensed channels in an epoch $\mathcal{T}_1(\gamma)$ ($\overline{\mathcal{T}}_1(\gamma)$) and the set of Tier-2 operators in an epoch $\mathcal{T}_2(\gamma)$. Note that $\mathcal{T}_1(\gamma)$, $\overline{\mathcal{T}}_1(\gamma)$ and $\mathcal{T}_2(\gamma)$ are not the same for epochs 1 and 2.

the k^{th} operator is in Tier i in epoch γ, it generates a revenue $R_{k,i}(\gamma)$ in epoch γ. We model $R_{k,i}(\gamma)$ as a random variable which follows the stochastic model

$$\begin{bmatrix} X_{k,i}(\gamma) \\ R_{k,i}(\gamma) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu^X_{k,i} \\ h_k\left(\mu^X_{k,i}\right) \end{bmatrix}, \begin{bmatrix} \left(\sigma^X_{k,i}\right)^2 & \rho_k \sigma^X_{k,i}\sigma^R_{k,i} \\ \rho_k \sigma^X_{k,i}\sigma^R_{k,i} & \left(\sigma^R_{k,i}\right)^2 \end{bmatrix}\right) \quad (7)$$

for all γ. According to (7), the net demand served and the net revenue earned in epoch γ are jointly Gaussian. The mean of $R_{k,i}(\gamma)$ is $h_k\left(\mu^X_{k,i}\right)$ which is a monotonic increasing function of the mean demand served by the k^{th} operator in an epoch, $\mu^X_{k,i}$. The standard deviation of $R_{k,i}(\gamma)$ is $\sigma^R_{k,i}$ which can be used to capture the effect of exogeneous stochastic processes like market dynamics on $R_{k,i}(\gamma)$. The relative change between $R_{k,i}(\gamma)$ and $X_{k,i}(\gamma)$ is captured with correlation coefficent $\rho_k \in [0, 1)$. It captures how much a deviation of $X_{k,i}(\gamma)$ around its mean $\mu^X_{k,i}$ will effect the deviation of $R_{k,i}(\gamma)$ around its mean $h_k\left(\mu^X_{k,i}\right)$. A monotonic increasing function, $h_k(\cdot)$, and a positive correlation coefficient, ρ_k, implies that from statistical standpoint, if the k^{th} operator serves more customer demand, it will generate higher revenue.

C. Spectrum Allocation Model

Licensed channels are allocated to the set of interested licensed operators, $\mathcal{S}_L$, through spectrum auctions. The spectrum auction for epoch γ happens at time slot $(\gamma-1)T+1$. The set of interested licensed operators bids for licensed channels. Let $V_k(\gamma)$ be the bid of the k^{th} operator in epoch γ. Our model assumes truthful spectrum auctions. For such auctions, the operators always bid their true valuations of a licensed channel. The true value of a licensed channel to the k^{th} operator is the revenue it can generate using the licensed channel in an epoch, i.e., $V_k(\gamma)$ is equal to $R_{k,1}(\gamma)$. According to (7), the marginal distribution of $R_{k,1}(\gamma)$ and hence $V_k(\gamma)$ is $\mathcal{N}\left(h_k\left(\mu^X_{k,1}\right), \left(\sigma^R_{k,1}\right)^2\right)$. Given that there are P licensed channels and the spectrum cap is one, the interested licensed operators with the P highest bids $V_k(\gamma)$ are allocated one licensed channel each in epoch γ. Let $\mathcal{T}_1(\gamma) \subseteq \mathcal{S}_L$ denote the set of interested licensed operators who are allocated licensed channels in epoch γ. Similarly, $\overline{\mathcal{T}}_1(\gamma) = \mathcal{S}_L \backslash \mathcal{T}_1(\gamma)$ are the set of interested licensed operators who are not allocated licensed channels in epoch γ. The operators in $\mathcal{T}_1(\gamma)$ serve their customer demand as Tier-1 operators in epoch γ. On the other hand, operators in $\overline{\mathcal{T}}_1(\gamma)$ serve their customer demand as Tier-2 operators in epoch γ. It is to be noted that $\mathcal{T}_1(\gamma)$ and $\overline{\mathcal{T}}_1(\gamma)$ are random sets as they get decided by the bids $V_k(\gamma)$ which are random variables. Also, unlike the set $\mathcal{S}_L$ which is decided once, sets $\mathcal{T}_1(\gamma)$ and $\overline{\mathcal{T}}_1(\gamma)$ are decided in the beginning of every epoch. A pictorial representation of all the important sets discussed till now is shown in Figure 2. Figure 2 also shows the variation of sets $\mathcal{T}_1(\gamma)$ and $\overline{\mathcal{T}}_1(\gamma)$ with epoch γ.

Opportunistic spectrum allocation happens in every time slot to all the Tier-2 operators. The set of Tier-2 operators in epoch γ is $\mathcal{T}_2(\gamma) = \overline{\mathcal{T}_1}(\gamma) \bigcup \mathcal{S}_U$, i.e. interested unlicensed operators and interested licensed operators who are not allocated a licensed channel in epoch γ. This is shown in Figure 2. The net opportunistic channel capacity in time slot t of epoch γ is

$$D_O(t) = \alpha_U \left(M - \widetilde{P}\right) \frac{D}{M} + \alpha_L \sum_{k \in \overline{\mathcal{T}}_1(\gamma)} \max\left(0, \frac{D}{M} - x_k(t)\right) \tag{8}$$

where $\widetilde{P} = \min(|\mathcal{S}_L|, P)$. In (8), the first term is the net channel capacity of unlicensed channels and the second term is the net remaining channel capacity of the licensed channels. The variable $\widetilde{P}$ is used to capture edge cases where the number of licensed channels is more than number of interested licensed operators. In such cases, the remaining $P - |\mathcal{S}_L|$ channels which are not allocated to licensed operators are used as unlicensed channels. As our model is inspired by the CBRS band, it may be desirable to ensure that opportunistic spectrum allocation is fair [15]. One simple approach to do this is to divide the net opportunistic channel capacity equally among the operators in $\mathcal{T}_2(\gamma)$. So each operator gets to use a maximum capacity of $\frac{D_O(t)}{|\mathcal{T}_2(\gamma)|}$. So the amount of customer demand served by the k^{th} operator, where $k \in \mathcal{T}_2(\gamma)$, in time slot t of epoch γ is

$$\widetilde{x}_{k,2}(t) = \min\left(x_k(t), \frac{D_O(t)}{|\mathcal{T}_2(\gamma)|}\right) \tag{9}$$

In (9), $x_k(t)$ and $D_O(t)$ are random variables. Note that even though $\mathcal{T}_2(\gamma)$ is a random variable, $|\mathcal{T}_2(\gamma)|$ is deterministic and is equal to $|\mathcal{S}_U| + \left|\overline{\mathcal{T}_1}(\gamma)\right| = |\mathcal{S}_U| + \max(0, |\mathcal{S}_L| - P)$. $x_k(t)$ is an i.i.d. random variable. Refering to (8) we can see that $D_O(t)$ is a time-invariant function of i.i.d. random variables $x_j(t)$, where $j \in \mathcal{T}_1(\gamma)$, and is hence an i.i.d. random variable itself. Since the function $\min(\cdot)$ in (9) is time-invariant, $\widetilde{x}_{k,2}(t)$ is a time-invariant function of i.i.d. random variables $x_k(t)$ and $D_O(t)$ as claimed in Section II-B (refer to (2)).

Throughout the rest of this paper, we will use this simple opportunistic channel allocation algorithm. However, we want to stress that our solution approach in Section III holds for any opportunistic channel allocation algorithm which satisfies (2).

III. Optimization Problem

A. Stackelberg Game Formulation

In this subsection, we formulate the optimal spectrum partitioning problem as a two-stage Stackelberg game between the regulator and the wireless operators. The k^{th} operator can be completely characterised by six parameters which can be represented as a tuple $\xi_k = \left(\mu_k^\theta, \sigma_k^\theta, h_k(\cdot), \sigma_{k,i}^R, \rho_k, \lambda_k\right)$. We consider complete information games, i.e. an operator and the regulator knows ξ_k of all the operators. This is done for mathematical tractability and similar assumptions has been used in prior works like [16] an references therein. These works also suggests that our overall approach for optimal spectrum partitioning remains valid for incomplete information games as well but it will lead to sub-optimal spectrum utilization. In other words, our work provides an useful upper bound on spectrum utilization when the operators and the regulator have incomplete market information. The player in Stage-1 of the Stackelberg game is the regulator whose decision variables are M and P. The payoff of the regulator is the expected spectrum utilization over a period of $\Gamma \geq 1$ epochs which is given by

$$Q = E\left[\sum_{\gamma=1}^{\Gamma} \sum_{t=(\gamma-1)T+1}^{\gamma T} \left(Q_1(\gamma, t) + Q_2(\gamma, t)\right)\right] \text{ where,} \tag{10}$$

$$Q_1(\gamma, t) = \sum_{k \in \mathcal{T}_1(\gamma)} \min\left(x_k(t), \frac{D}{M}\right) \tag{11}$$

$$Q_2(\gamma, t) = \sum_{k \in \mathcal{T}_2(\gamma)} \min\left(x_k(t), \frac{D_O(t)}{|\mathcal{T}_2(\gamma)|}\right) \tag{12}$$

In (10), the inner summation is to calculate spectrum utilization over all time slots in an epoch while the outer summation is to calculate spectrum utilization over all the epochs. $Q_1(\gamma, t)$ and $Q_2(\gamma, t)$ are the net spectrum utilization in the t^{th} time slot by all the Tier-1 and Tier-2 operators of epoch γ respectively. The regulator wants to maximize Q. Using *linearity of expectation*, we can rewrite (10) as

$$Q = \sum_{\gamma=1}^{\Gamma} \sum_{t=(\gamma-1)T+1}^{\gamma T} E\left[Q_1(\gamma, t) + Q_2(\gamma, t)\right] \tag{13}$$

We now prove that $E[Q_1(\gamma, t) + Q_2(\gamma, t)]$ is not a function of γ and t. Recall that $\mathcal{T}_2(\gamma) = (\mathcal{S}_L \backslash \mathcal{T}_1(\gamma)) \bigcup \mathcal{S}_U$ and $D_O(t)$ is given by (8). But $\mathcal{T}_1(\gamma)$ depends on the number of licensed channels, P, and the valuations, $V_k(\gamma)$, of the operators in $\mathcal{S}_L$. This shows that $x_k(t)$ and $V_k(\gamma)$ are the only random variables in (11) and (12). But the statistical properties of $x_k(t)$ and $V_k(\gamma)$ are independent of t and γ respectively. Hence, the expectation $E[Q_2(\gamma, t) + Q_3(\gamma, t)]$ is not a function if γ and t. In fact, it is a function of M, P, $\mathcal{S}_L$ and $\mathcal{S}_U$. Let,

$$U(M, P, \mathcal{S}_L, \mathcal{S}_U) = E\left[Q_1(\gamma, t) + Q_2(\gamma, t)\right]. \tag{14}$$

Substituting (14) in (13) we get,

$$\begin{aligned} Q &= \sum_{\gamma=1}^{\Gamma} \sum_{t=(\gamma-1)T+1}^{\gamma T} U(M, P, \mathcal{S}_L, \mathcal{S}_U) \\ &= \Gamma T U(M, P, \mathcal{S}_L, \mathcal{S}_U) \end{aligned} \tag{15}$$

Equation 15 shows that maximizing Q is the same as maximizing $U(M, P, \mathcal{S}_L, \mathcal{S}_U)$. Therefore, we will use $U(M, P, \mathcal{S}_L, \mathcal{S}_U)$ as the payoff function of the regulator in the rest of the paper. $U(M, P, \mathcal{S}_L, \mathcal{S}_U)$ is also called the *objective function* as it is a direct measure of spectrum utilization which we are trying to maximize in this paper.

The players in Stage-2 of the Stackelberg game are the candidate licensed operators, $\mathcal{S}_L^C$, and candidate unlicensed operators, $\mathcal{S}_U^C$. The decision variables of the Stage-2 game are the set of interested licensed operators, $\mathcal{S}_L$, and the set of interested unlicensed operators, $\mathcal{S}_U$. As mentioned in Section II-B, the k^{th} operator is interested in joining the market only if the expected revenue in an epoch is greater than λ_k. The expected revenue in an epoch of an interested licensed or unlicensed operator is given by the *revenue function*. The formula for revenue function is different for interested licensed operators versus interested unlicensed operators. The revenue function of an interested licensed operator, i.e. $k \in \mathcal{S}_L$, is

$$\begin{aligned}\mathcal{R}_k(M, P, \mathcal{S}_L, \mathcal{S}_U) = E[R_{k,1}(\gamma) \mid \mathcal{E}_k(\gamma) = 1] \, P[\mathcal{E}_k(\gamma) = 1] \\ + E[R_{k,2}(\gamma) \mid \mathcal{E}_k(\gamma) = 0] \, P[\mathcal{E}_k(\gamma) = 0] \quad (16)\end{aligned}$$

where $P[Z]$ denotes the probability of event Z and $\mathcal{E}_k(\gamma) = 1$ if $k \in \mathcal{T}_1(\gamma)$ and 0 otherwise. In other words, $\mathcal{E}_k(\gamma) = 1$ is the event that k^{th} operator is allocated a licensed channel in epoch γ. In (16), $E[R_{k,1}(\gamma) \mid \mathcal{E}_k(\gamma) = 1]$ ($E[R_{k,2}(\gamma) \mid \mathcal{E}_k(\gamma) = 0]$) is the expected revenue of the k^{th} operator in epoch γ if it is allocated (not allocated) a licensed channel. If the k^{th} operator is allocated (not allocated) a licensed channel, it is a Tier-1 operator (Tier-2 operator) and hence generates a revenue of $R_{k,1}(\gamma)$ ($R_{k,2}(\gamma)$). Finally, (16) is obtained using the law of total expectation. Similar to the objective function, the revenue function of an interested licensed operator is also not a function of epoch γ. This is because the statistical properties of the involved random variables $R_{k,1}(\gamma)$ and $R_{k,2}(\gamma)$ are independent of γ.

If the k^{th} operator is an interested unlicensed operator, i.e. $k \in \mathcal{S}_U$, its expected revenue in an epoch is

$$\mathcal{R}_k(M, P, \mathcal{S}_L, \mathcal{S}_U) = E[R_{k,2}(\gamma)] \quad (17)$$

The payoff function of an operator that is interested in joining the market either as a licensed or an unlicensed operator is

$$\pi_k(M, P, \mathcal{S}_L, \mathcal{S}_U) = \mathcal{R}_k(M, P, \mathcal{S}_L, \mathcal{S}_U) - \lambda_k \quad (18)$$

where $\mathcal{R}_k(M, P, \mathcal{S}_L, \mathcal{S}_U)$ is given by (16) if $k \in \mathcal{S}_L$ and by (17) if $k \in \mathcal{S}_U$. If an operator does not join the market, its payoff is zero. An operator decides to join the market only if its payoff $\pi_k(M, P, \mathcal{S}_L, \mathcal{S}_U)$ is *strictly* greater than zero. With (18) as the payoff function, the Stage-2 game can have multiple Nash Equilibria which complicates the analysis. This can be simplified if we assume that the operators are *pessimistic*. Therefore, their decisions to join the market are governed by a *max-min strategy*, i.e. an operator decides to join the market only if its worst case payoff over all possible $\mathcal{S}_L$ and $\mathcal{S}_U$ is strictly greater than zero. Pessimistic models to address the issue of multiple Nash Equilibria have been considered in prior works like [17], [18].

B. Solution of the Stackelberg Game

In order to solve Stage-1 and Stage-2 of the Stackelberg game, we have to compute the objective functions (as given by (14)) and the revenue functions (as given by (16) and (17)). There are no closed form analytical expressions to compute the revenue and the objective function because of the complex nature of the problem. The complexity mainly arises because of the tiered sharing rule that includes both licensed and opportunistic spectrum allocation. We thus design a Monte Carlo integrator to compute the revenue and the objective functions. Our Monte Carlo integrator is computationally efficient. It is also generic enough to accommodate other opportunistic spectrum allocation rules, not just the one given by (9). The explanation of the Monte Carlo integrator has been deferred to [14] due to lack of space.

The revenue function given by (16) and (17) have the following properties:

Property 1: $\mathcal{R}_k(M, P, \mathcal{S}_L, \mathcal{S}_U)$ is monotonic decreasing in $\mathcal{S}_L$, i.e. $\mathcal{R}_k(M, P, \mathcal{S}_L, \mathcal{S}_U) \geq \mathcal{R}_k(M, P, \mathcal{S}_L \bigcup \{a\}, \mathcal{S}_U)$ where $a \notin \mathcal{S}_L$ and $a \in \mathcal{S}_L^C$.

Property 2: $\mathcal{R}_k(M, P, \mathcal{S}_L, \mathcal{S}_U)$ is monotonic decreasing in $\mathcal{S}_U$, i.e. $\mathcal{R}_k(M, P, \mathcal{S}_L, \mathcal{S}_U) \geq \mathcal{R}_k(M, P, \mathcal{S}_L, \mathcal{S}_U \bigcup \{a\})$ where $a \notin \mathcal{S}_U$ and $a \in \mathcal{S}_U^C$.

We have verified these properties numerically using the Monte Carlo integrator. These properties can be intuitively justified as follows. Property 1 states that as the set of interested licensed operators, $\mathcal{S}_L$, increases, the revenue function of both the licensed and the unlicensed operators decreases. The revenue function of a licensed operator decreases with increase in $\mathcal{S}_L$ because the operator has to compete with more operators in the spectrum auctions to get a channel. This reduces the operator's probability of winning spectrum auctions which in turn decreases its revenue function as it can effectively serve fewer customer demand. The revenue function of an unlicensed operator also decreases with increase in $\mathcal{S}_L$. This happens because with increase in $\mathcal{S}_L$, there is an increase in $\overline{\mathcal{T}}_1(\gamma)$, the set of interested licensed operators who are not allocated licensed channels in epoch γ. The operators in $\overline{\mathcal{T}}_1(\gamma)$ uses channels opportunistically in epoch γ. This reduces the share of opportunistic channels for the unlicensed operators. Therefore, its revenue decreases as it can serve fewer customer demand.

Property 2 states that as the set of interested unlicensed operators, $\mathcal{S}_U$, increases, the revenue function of both the licensed and the unlicensed operators decreases. The revenue function of an unlicensed operator decreases with increase in $\mathcal{S}_U$ because the operator's share of the opportunistic channels decreases with increase in $\mathcal{S}_U$. This in turn decreases its revenue function. The revenue function of a licensed operator also decreases with increase in $\mathcal{S}_U$. This happens because in some epoch, the operator may not be allocated a licensed channel in which case it has to use channels opportunistically along with operators in $\mathcal{S}_U$. As $\mathcal{S}_U$ increases, the operator's share of the opportunistic channels decreases which in turn decreases its revenue function.

Algorithm 1: Optimization algorithm for joint spectrum partitioning of tiered spectrum access.

Input: T, D, α_L, α_U, $\mathcal{S}_L^C$, $\mathcal{S}_U^C$, and $\xi_k; \forall k \in \mathcal{S}_L^C \bigcup \mathcal{S}_U^C$
Output: M^*, P^*, U^*, $\mathcal{S}_L^*$, and $\mathcal{S}_U^*$

1 Set $U^* = -\infty$
2 **for** $M \leftarrow 1$ **to** M_{max} **do**
3   **for** $P \leftarrow 0$ **to** $\min\left(\left|\mathcal{S}_L^C\right|, M\right)$ **do**
4   Set $\mathcal{S}_L = \emptyset$ and $\mathcal{S}_U = \emptyset$
5   **for** k ***in*** $\mathcal{S}_L^C$ **do**
6   **if** $\mathcal{R}_k\left(M, P, \mathcal{S}_L^C, \mathcal{S}_U^C\right) > \lambda_k$ **then**
7   Set $\mathcal{S}_L = \mathcal{S}_L \bigcup \{k\}$
8   **for** k ***in*** $\mathcal{S}_U^C$ **do**
9   **if** $\mathcal{R}_k\left(M, P, \mathcal{S}_L^C, \mathcal{S}_U^C\right) > \lambda_k$ **then**
10   Set $\mathcal{S}_U = \mathcal{S}_U \bigcup \{k\}$
11   Set $\widetilde{U} = U\left(M, P, \mathcal{S}_L, \mathcal{S}_U\right)$
12   **if** $\widetilde{U} > U^*$ **then**
13   Set $M^* = M$, $P^* = M$, $U^* = \widetilde{U}$, $\mathcal{S}_L^* = \mathcal{S}_L$, and $\mathcal{S}_U^* = \mathcal{S}_U$

Let $\mathcal{S}_L\left(M, P\right)$ and $\mathcal{S}_U\left(M, P\right)$ denote the set of interested licensed and unlicensed operators if the entire bandwidth is divided into M channels out of which P are licensed channels. $\mathcal{S}_L\left(M, P\right)$ and $\mathcal{S}_U\left(M, P\right)$ constitutes the solution of the Stage-2 game. Properties 1 and 2 suggest that the revenue function of the k^{th} operator is minimum when $\mathcal{S}_L$ and $\mathcal{S}_U$ are the largest, i.e. $\mathcal{S}_L = \mathcal{S}_L^C$ and $\mathcal{S}_U = \mathcal{S}_U^C$. Therefore, minimum payoff of the k^{th} operator is $\mathcal{R}_k\left(M, P, \mathcal{S}_L^C, \mathcal{S}_U^C\right) - \lambda_k$ if it decides to join the market. Since the decision of the wireless operators to join the market is based on max-min strategy, the solution of the Stage-2 game is

$$\mathcal{S}_L\left(M, P\right) = \left\{k \in \mathcal{S}_L^C \,:\, \mathcal{R}_k\left(M, P, \mathcal{S}_L^C, \mathcal{S}_U^C\right) > \lambda_k\right\} \quad (19)$$

$$\mathcal{S}_U\left(M, P\right) = \left\{k \in \mathcal{S}_U^C \,:\, \mathcal{R}_k\left(M, P, \mathcal{S}_L^C, \mathcal{S}_U^C\right) > \lambda_k\right\} \quad (20)$$

where the revenue function in (19) is given by (16) while the revenue function in (20) is given by (17). Given that $\mathcal{S}_L\left(M, P\right)$ and $\mathcal{S}_U\left(M, P\right)$ are the solutions of the Stage-2 game, the objective function in (14) can be re-written as

$$\widetilde{U}\left(M, P\right) = U\left(M, P, \mathcal{S}_L\left(M, P\right), \mathcal{S}_U\left(M, P\right)\right) \quad (21)$$

In Stage-1, the regulator chooses M and P to maximize $\widetilde{U}\left(M, P\right)$. Let the optimal solution be M^* and P^*, the optimal value of the objective function be U^*, where $U^*=\widetilde{U}\left(M^*, P^*\right)$, and the optimal set of interested licensed and unlicensed operators be $\mathcal{S}_L^*$ and $\mathcal{S}_U^*$, where $\mathcal{S}_L^* = \mathcal{S}_L\left(M^*, P^*\right)$ and $\mathcal{S}_U^* = \mathcal{S}_U\left(M^*, P^*\right)$. M^* and P^* are found by performing a grid-search from $M = 1$ to a certain M_{max} and from $P = 0$ to $\min\left(\left|\mathcal{S}_L^C\right|, M\right)$. Note that since the spectrum cap is one, the number of licensed channels should be less than the number of candidate licensed operators, $\left|\mathcal{S}_L^C\right|$. The grid search is detailed in Algorithm 1 (refer to Table I for notations). In Algorithm 1, the for loop in *lines 5-7* computes $\mathcal{S}_L\left(M, P\right)$ of (19) and that in *lines 8-10* computes $\mathcal{S}_U\left(M, P\right)$ of (20).

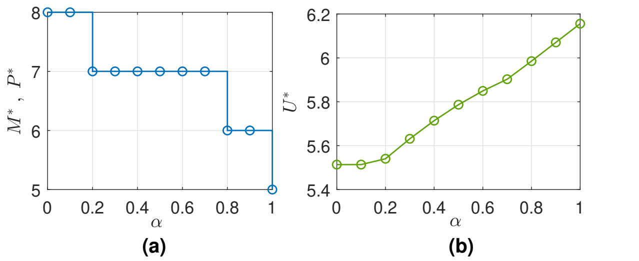

Fig. 3. Plots showing the effect of interference parameter α for a market containing only candidate licensed operators: (a) the optimal number of channels M^* and optimal number of licensed channels P^*, (b) optimal value of the objective function U^*.

IV. Numerical Results

In this section, we use Algorithm 1 to numerically explore the trends of optimal solution M^* and P^* as a function of various market paramaters. Throughout this section, we use $h_k\left(\mu_{k,i}^X\right) = a_k\mu_{k,i}^X$, where $a_k > 0$, and $\sigma_{k,i}^R = \eta_k h_k\left(\mu_{k,i}^X\right)$ where $\eta_k > 0$ is the coefficient of variation associated with the revenue $R_{k,i}\left(\gamma\right)$ of the k^{th} operator. Also, each time slot is one week and lease duration of licensed channels is one year, i.e. $T = 52$ (there are 52 weeks in a year).

Our first numerical result is to study the effect of interference parameter on optimal solution. The simulation setup is as follows. The interference parameters associated with the licensed and the unlicensed channels are equal, i.e. $\alpha_L = \alpha_U = \alpha$. There are 8 candidate licensed operators and no candidate unlicensed operators. The candidate licensed operators are homogeneous in ξ_k, i.e. $\xi_k = \xi\,; \forall k$. The minimum revenue requirement λ_k is set to zero for all the operators which ensures that all the operators join the market. D is set equal to 80% of the sum of μ_k^θ of all the 8 operators. We study how M^*, P^* and U^* varies with α. The simulation result is shown in Figure 3. As shown in Figure 3.b, optimal value of objective function U^*, which captures optimal spectrum utilization, increases with increase in α. This is expected because with increase in α, the opportunistic channel access becomes more efficient and hence it leads to a better spectrum utilization. Since there are no candidate unlicensed operators, it is intuitive that there are no unlicensed channels, i.e. $M^* = P^*$. This is shown in Figure 3.a. Figure 3.a also shows that M^* decreases with increase in α. This can be explained as follows. If M is low, the bandwidth, and hence the capacity of each licensed channel is high. Therefore, a licensed operator can serve more customer demand using the allocated licensed channel thereby increasing spectrum utilization. But if M is too low, only few of the 8 licensed operators are allocated the licensed channels in an epoch. The remaining operators who uses channels opportunistically as Tier-2 operators. The efficiency of opportunistic access is decided by α. If α is low, it is better to have fewer Tier-2 operators in an epoch because opportunistic spectrum access is inefficient. This can be ensured with a higher M so that there are more Tier-1 operators in every epoch. This explains the trend observed in Figure 3.a.

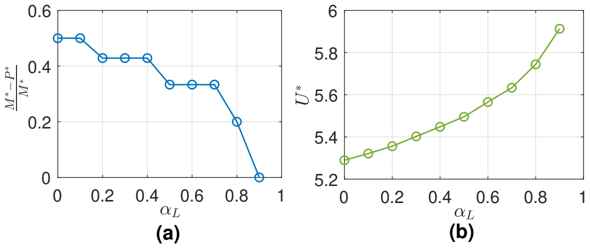

Fig. 4. Plots showing the effect of interference parameter of licensed channel α_L for a market containing both candidate licensed operators and candidate unlicensed operators: (a) the optimal ratio of the bandwidth allocated for unlicensed channels, $\frac{M^*-P^*}{M^*}$, (b) optimal value of the objective function U^*.

For our next simulation, we include candidate unlicensed operators in our simulation setup. The simulation setup of our next simulation is similar to our first simulation except in the following ways. *First*, out of the 8 operators, 4 are candidate licensed operators and 4 are candidate unlicensed operators. *Second*, the interference parameters α_L and α_U are not same. We set $\alpha_U = 0.9$ and vary α_L from 0 to 0.9. We study how the ratio of the bandwidth allocated for unlicensed channels characterized by the ratio $\frac{M^*-P^*}{M^*}$ and the optimal value of objective function U^* changes with α_L. This is shown in Figure 4. As expected, with increase in α_L, U^* increases. This is because with increase in α_L, the the opportunistic access of licensed channels become more efficient and hence it leads to a better spectrum utilization. This is shown in Figure 4.a. Unlike the previous simulation setup, the current simulation setup has candidate unlicensed operators. Therefore, we expect that there will be unlicensed channels dedicated for the candidate unlicensed operators. But the question is: what portion of the bandwidth should be allocated for unlicensed channels? If α_L is high, most of the bandwidth can be reserved for licensed channels because even if the Tier-1 operators are not using the licesnsed channels, the Tier-2 operators can use the remaining capacity of the licensed channels efficiently. But as α_L decreases, the opportunistic access of licensed channels becomes inefficient. Therefore, it is better to reserve higher portion of the bandwidth for unlicensed channels.

V. Conclusion and Future Work

In this paper, we designed an optimization algorithm to partition a bandwidth into channels and further decide the number of licensed channels in order to maximize spectrum utilization. The access to this bandwidth is governed by a tiered spectrum access model inspired by the CBRS band. We first propose a system model which accurately captures various aspects of the tiered spectrum access model under consideration. Based on this model, we formulate our optimization problem as a two-staged Stackelberg game and then designed a grid-search based algorithm to solve the Stackelberg game. Using this algorithm we obtain numerical results which gives us intuitions as to how various market parameters effect optimal partitioning of the bandwidth.

An interesting direction of future research would be to explore various robustness related issues related to our problem setup. For example, how sensitive is the optimal solution of our algorithm with respect to market parameters. This question is relevant because the knowledge of market parameters is bound to have some errors. Along similar lines, it will be interesting to extend our work to include incomplete information games. Another interesting direction for future research is to generalize our system model to capture various degrees of risk-averse nature of wireless operators in heterogeneous markets.

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