The reflection-absorption model for directed lattice paths

Cyril Banderier[a], Michael Wallner[b]

[a]Laboratoire d'Informatique de Paris Nord, UMR CNRS 7030, Université Paris Nord, France
[b]E104 - Institute of Discrete Mathematics and Geometry, TU Wien

Lattice paths

- **Step set:** $\mathcal{S} = \{(1, b_1), \dots, (1, b_m)\} \subset \mathbb{Z}^2$
- **n-step lattice path:** Sequence of vectors $(v_1, \dots, v_n) \in \mathcal{S}^n$

Probabilistic weights

- For $\mathcal{S} = \{-c, \dots, d\}$ define $\Pi = \{p_{-c}, \dots, p_d\}$, $p_i \in [0, 1]$ and $\sum_i p_i = 1$
- **Jump polynomial** $P(u) = \sum_{i=-c}^{d} p_i u^i$
- $P(1) = 1$

The reflection-absorption model

- Lattice: $\mathbb{Z}_+^2$
- Altitude $k \neq 0$
  - Weighted step set $\mathcal{S}$
  - $P(u) = \sum_{i=-c}^{d} p_i u^i$
- Altitude $k = 0$
  - Weighted step set $\mathcal{S}_0$
  - $P_0(u) = \sum_{i=0}^{d_0} p_{0,i} u^i$

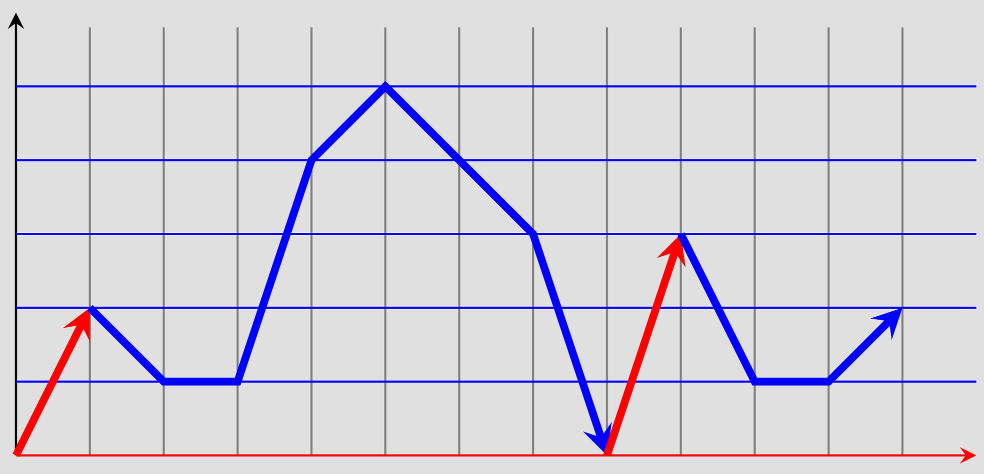
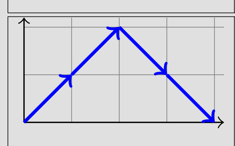
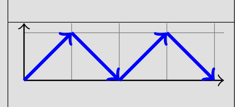
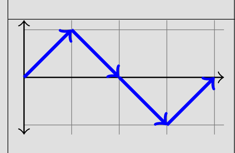
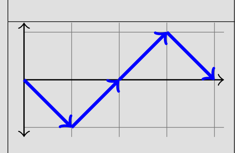
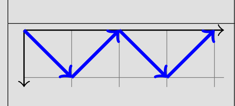
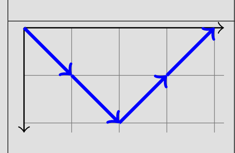


Figure : Chosen steps depend on altitude

Reflection model: **No loss** of mass at 0: $P_0(1) = 1$
Absorption model: **Loss** of mass at 0: $P_0(1) < 1$

Different constraints on the boundary

- **Characteristic polynomial:** $P(u) = pu + qu^{-1}$
- **Reflection model:** $P_0(u) = u$
- **Absorption model:** $P_0(u) = p_0 u$ with $p_0 < 1$

Dyck path	bridges, uniform model	absolute value of bridges	excursions, reflection m.	excursions, absorption m.
	$\frac{1}{6}$	$\frac{1}{1+p_0/p+q_0/q}$	$\frac{p}{1+p}$	$\frac{p}{p+p_0}$
	$\frac{1}{6}$	$\frac{p_0/p+q_0/q}{1+p_0/p+q_0/q}$	$\frac{1}{1+p}$	$\frac{p_0}{p+p_0}$
	$\frac{1}{6}$	0	0	0
	$\frac{1}{6}$	0	0	0
	$\frac{1}{6}$	0	0	0
	$\frac{1}{6}$	0	0	0

Relevant constants

Let τ be the *structural constant* given by $P'(\tau) = 0$, $\tau > 0$, and let $\rho = 1/P(\tau)$ be the *structural radius.*

Let $u_1(z)$ be the unique solution of the *kernel equation*

$$1 - zP(u) = 0,$$

with $\lim_{z\to 0} u_1(z) = 0$. Then, the equation $1 - zP_0(u_1(z)) = 0$ has at most one solution in $z \in (0, \rho]$, which we denote by ρ_1.

Additionally, we define the constants

$$\alpha = (P_0(u_1(z)))'|_{z=\rho_1}, \qquad \alpha_2 = (P_0(u_1(z)))''|_{z=\rho_1},$$

$$\gamma = \frac{1}{\alpha\rho_1^2 + 1}, \qquad \kappa = \rho\sqrt{2\frac{P(\tau)}{P''(\tau)}}P_0'(\tau),$$

$$\lambda = \frac{P_0(\tau)}{P(\tau)}.$$

Asymptotic number of excursions

Let e_n be number of excursions of length n. Then, the generating function of excursions is of the kind

$$E(z) := \sum_{n\geq 0} e_n z^n = \frac{1}{1 - zP_0(u_1(z))}.$$

The asymptotic number of excursions is given by

$$e_n \sim \begin{cases} \gamma\rho_1^{-n}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), & \text{supercritical case: } \lambda > 1, \\ \frac{1}{\kappa\sqrt{\pi}}\frac{\rho^{-n}}{n^{1/2}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), & \text{critical case: } \lambda = 1, \\ \frac{\kappa}{2\sqrt{\pi}(1-\rho P_0(\tau))^2}\frac{\rho^{-n}}{n^{3/2}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), & \text{subcritical case: } \lambda < 1. \end{cases}$$

Returns to zero

Definition

- An *arch* is an excursion of size > 0 whose only contact with the x-axis is at its end point.
- A *return to zero* is a vertex of a path of altitude 0 whose abscissa is positive.

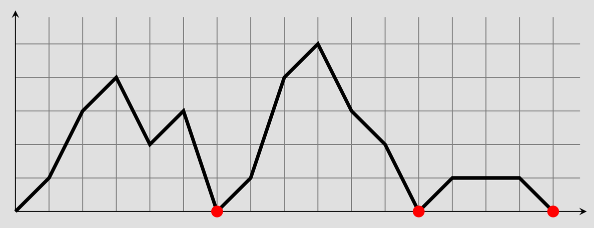


Figure : An excursion with 3 returns to zero

Corresponding generating function

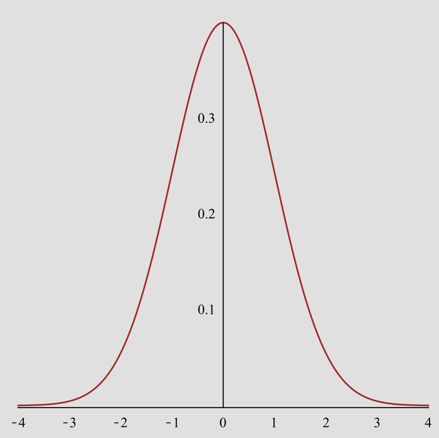
$$E(z, u) := \sum_{n,k\geq 0} e_{n,k} z^n u^k = \frac{1}{1 - uzP_0(u_1(z))}$$

Excursion of length n having k returns to zero

$$\mathbb{P}(X_n = k) := \mathbb{P}(\text{size} = n,\ \#\text{returns to zero} = k) = \frac{e_{n,k}}{e_n}$$

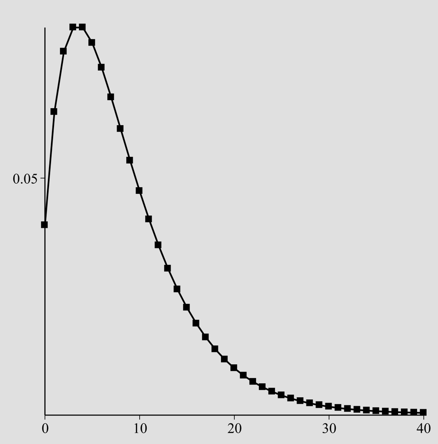
Limit law for returns to zero of excursions

1. In the supercritical case, i.e. $\lambda > 1$,
$$\frac{X_n - \mu n}{\sigma\sqrt{n}}, \quad \text{with}$$
$$\mu = \gamma, \quad \sigma = \gamma^3(\alpha_2\rho_1^3 - 2) + \gamma^2(\rho_1 + 2) - \gamma,$$
converges in law to a **Gaussian variable** $N(0, 1)$.



2. In the critical case, i.e. $\lambda = 1$, the normalized random variable $\frac{\kappa}{\sqrt{2n}}X_n$, converges in law to a **Rayleigh distribution** (density: $xe^{-x^2/2}$):
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\kappa}{\sqrt{2n}}X_n \leq x\right) = 1 - e^{-x^2/2}.$$

3. In the subcritical case, i.e. $\lambda < 1$, the limit distribution of $X_n - 1$ is the **negative binomial distribution** $\mathrm{NegBin}(2, 1-\lambda)$:
$$\mathbb{P}(X_n - 1 = k) \sim (k + 1)\lambda^k(1 - \lambda)^2.$$



Conclusions

- Similar results hold for the asymptotics of bridges and meanders,
- Limit laws for other parameters like final altitude of meanders, or returns to zero of bridges exist,
- Extensions to more general lattice path models are possible.

References

[1] Banderier, C.; Flajolet, P.: *"Basic analytic combinatorics of directed lattice paths"*. Theoretical Computer Science, 281, p. 37-80, 2002.

[2] Banderier, C.; Wallner, M.: *"Some reflections on lattice paths"*. Proceedings of the 25th Int. Conf. on Prob., Comb. and Asymptotic Methods for the Analysis of Algorithms (AofA'14), p. 25-36, 2014.

[3] Flajolet, P.; Sedgewick, R.: *"Analytic Combinatorics"*. Cambridge University Press, 2009.