This Maple file accompanies the paper "Asymptotics of minimal deterministic finite automata recognizing a finite binary language"
by Andrew Elvey Price, Wenjie Fang and Michael Wallner.

##

Abstract:

We show that the number of minimal discrete finite automata with n transient states recognizing a finite binary language grows asymptotically for n to infinity like
Theta(n! \, 4^n e^(3a1 n^(1/3)) n^(7/8)),
where a1 is approximately -2.338 and the largest root of the Airy function.
For this purpose, we derive a new two parameter recurrence relation which yields an algorithm of quadratic arithmetic complexity for computing the number of such automata up to a given size.
Using this result, we prove by induction asymptotically tight bounds that are sufficiently accurate for large n to determine the asymptotic form using adapted Netwon polygons.

##

In this worksheet
*) we compute the initial terms of the recurrences
*) compute the Newton polygons in the proof so Lemmas 6 and 7

```
> restart;
  with(plots):
```

Recurrences

In some computations at the end we use the gfun-package.
It is just needed to guess simple functional equations for generating functions and manipulate their coefficient.
This is not needed in the remainder of this worksheet.

For the latest version see Salvy's homepage http://perso.ens-lyon.fr/bruno.salvy/software/the-gfun-package/.
For these computations gfun verison 3.76 was used.

```
> #libname := "<path>\\gfun.mla", libname;    #set the path to gfun.mla
  libname := "D:\\lib\\gfun.mla", "D:\\lib", "/home/michaelwallner/lib/gfun.mla", "home/michaelwallner/lib", libname;
  with(gfun): gfun:-version;
```

libname := "D:\lib\gfun.mla", "D:\lib", "/home/michaelwallner/lib/gfun.mla", "home/michaelwallner/lib", "/usr/local/Maple2017/lib"

3.76 **(1.1)**

Maximal number of computed terms

```
> NN := 100;
```

$NN := 100$ **(1.2)**

Minimal binary DFAs or B-paths

Compute the minimal binary DFAs of size up to NN

(Recall: size is the number of transient states, i.e., not counting the unique recurrent state)

```
> for n from 0 to NN do
  for m from 0 to NN do
     bb[n,m] := 0:
  end:
  end:

  #initial conditions
  bb[0,0] := 1:
  for n from 1 to NN do
     bb[n,0] := 1:
     bb[n,1] := bb[n,0] + 2*bb[n-1,1]:
  end:

  for n from 1 to NN do
     for m from 2 to n do
        bb[n,m] := 2*bb[n,m-1]+(m+1)*bb[n-1,m]-m*bb[n-2,m-1]
  ;
     end do:
  end do:
```

print the array

```
> for m from 7 to 0 by -1 do
  for n from 0 to 10 do
  printf("%10.0f ",bb[n,m]);
  end;
  printf("\n");
  end;
```

```
         0          0          0          0          0
    0          0   15824880  306252120 4211946600          0
49545344760
         0          0          0          0          0
    0     487560    7912440   91533000  908658360
8245251480
         0          0          0          0          0
 18480     243780    2305200   18804300  140879280
998734380
         0          0          0          0        900
  9240      68700     444360    2658300   15137640
83375100
         0          0          0         60        450
  2490      12150      55410     242550    1033890
4328550
         0          0          6         30        114
   390       1266       3990      12354      37830
115026
         0          1          3          7         15
    31         63        127        255        511
1023
         1          1          1          1          1
    1          1          1          1          1
 1
```

check

According to [Domaratzki, Kisman, Shallit 2002] page 16 this should be equal to f'_2(n)

1, 1, 6, 60, 900, 18480, 487560

```
> seq(bb[n,n],n=0..min(NN,12));
```

1, 1, 6, 60, 900, 18480, 487560, 15824880, 612504240, 27619664640, **(1.1.1)**

1425084870240, 82937356685760, 5381249970008640

Minimal binary DFAs or B-paths divided by 2^(n-1)

Compute the relaxed binary trees of size up to NN

```
> for n from 0 to NN do
  for m from 0 to NN do
     cc[n,m] := 0:
  end:
  end:

  #initial conditions
  cc[0,0] := 2:
  for n from 1 to NN do
     cc[n,0] := 1:
     cc[n,1] := cc[n,0] + 2*cc[n-1,1]:
  end:

  for n from 1 to NN do
     for m from 2 to n do
        cc[n,m] := cc[n,m-1]+(m+1)*cc[n-1,m]-m/2*cc[n-2,m-1]
  ;
     end do:
  end do:
```

print the array

```
> for m from 7 to 0 by -1 do
  for n from 0 to 10 do
  printf("%10.0f ",cc[n,m]*2^(n-1));
  end;
  printf("\n");
  end;
```

```
         0          0          0          0          0
    0          0   15824880  612504240 16847786400
396362758080
         0          0          0          0          0
    0     487560   15824880  366132000 7269266880
131924023680
         0          0          0          0          0
 18480     487560    9220800  150434400 2254068480
31959500160
         0          0          0          0        900
 18480     274800    3554880   42532800  484404480
5336006400
         0          0          0         60        900
  9960      97200     886560    7761600   66168960
554054400
         0          0          6         60        456
  3120      20256     127680     790656    4842240
29446656
         0          1          6         28        120
   496       2016       8128      32640     130816
523776
         1          1          2          4          8
    16         32         64        128        256
512
```

check

According to [Domaratzki, Kisman, Shallit 2002] page 16 this should be equal to f'_2(n)
1, 1, 6, 60, 900, 18480, 487560

```
> seq(cc[n,n]*2^(n-1),n=0..min(NN,12));
```

1, 1, 6, 60, 900, 18480, 487560, 15824880, 612504240, 27619664640, 1425084870240, 82937356685760, 5381249970008640 **(1.2.1)**

The array without rescaling

```
> (*
  for m from 6 to 0 by -1 do
  for n from 0 to 8 do
  printf("%10.2f ",cc[n,m]);
  end;
  printf("\n");
  end;
  *)
```

Weighted minimal binary DFAs

Maximal number of meanders computed

```
> NN:=100:
```

compute the meanders of size up to NN

```
> for n from -1 to NN do
  for m from -1 to NN do
     ee[n,m] := 0:
  end:
  end:

  ee[0,0] := 2:
  ee[1,1] := 1:
  for n from 2 to NN do
     ee[n,0] := ee[n-1,1];
     for m from 1 to n do
        ee[n,m] := (n-m+2)/(n+m)*ee[n-1,m-1]+ee[n-1,m+1]-(n-
  m)/(n+m)/(n+m-2)*ee[n-3,m-1];
     end:
     #special case for bb[n,1]
     ee[n,n-2] := 2/(n-1)*ee[n-1,n-3]+ee[n-1,n-1];
  end:
```

print the array

```
> for m from 6 to 0 by -1 do
  for n from 0 to 11 do
  printf("%8.0f ",ee[n,m]*factorial(floor((n+m)/2))*2^((n+m)
  /2-1));
  end;
  printf("\n");
  end;
```

```
         0         0         0         0         0         0
  32        0      8128        0    790656        0
         0         0         0         0         0        16
   0      2016        0    127680        0   7761600
         0         0         0         0         8         0
 496        0     20256        0    886560        0
         0         0         0         4         0       120
   0      3120        0     97200        0   3554880
         0         0         2         0        28         0
 456        0      9960        0    274800        0
         0         1         0         6         0        60
```

```
  0        900          0     18480          0    487560
       1           0          1          0          6          0
 60          0        900          0     18480          0
```

check

```
> seq(factorial(n/2)*ee[n,0]*2^((n)/2-1),n=0..min(NN,22),2);
  seq(bb[n,n],n=0..min(NN,11));
```

1, 1, 6, 60, 900, 18480, 487560, 15824880, 612504240, 27619664640, 1425084870240, 82937356685760

1, 1, 6, 60, 900, 18480, 487560, 15824880, 612504240, 27619664640, 1425084870240, 82937356685760 **(1.3.1)**

We guess a few simple formulas for the numbers ending close to the upper end.

There is just 1 path ending on the very top

```
> seq(ee[n,n]*factorial(n),n=2..20);
```

1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 **(1.3.2)**

```
> n:='n':
  seq(ee[n,n-2]*factorial(n-1),n=2..20);
  listtorec([%],a(n));
  rsolve(op(1,%),a(n));
  simplify(subs(n=n-2,%));
```

1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, 32767, 65535, 131071, 262143, 524287

$$[\{2\,a(n) - 3\,a(n+1) + a(n+2), a(0) = 1, a(1) = 3\}, ogf]$$

$$2\,2^n - 1$$

$$2^{n-1} - 1 \qquad \textbf{(1.3.3)}$$

```
> n:='n':
  seq(ee[n,n-4]*factorial(n-2),n=4..20);
  listtorec([%],a(n));
  rsolve(op(1,%),a(n));
  map(simplify,subs(n=n-4,%)) assuming n::posint;
```

3, 15, 57, 195, 633, 1995, 6177, 18915, 57513, 174075, 525297, 1582035, 4758393, 14299755, 42948417, 128943555, 387027273

$$[\{6\,a(n) - 5\,a(n+1) + a(n+2), a(0) = 3, a(1) = 15\}, ogf]$$

$$-6\,2^n + 9\,3^n$$

$$-\frac{3\,2^n}{8} + \frac{3^n}{9} \qquad \textbf{(1.3.4)}$$

```
> n:='n':
  seq(ee[n,n-6]*factorial(n-3),n=6..20);
  listtorec([%],a(n));
  rsolve(op(1,%),a(n));
  simplify(subs(n=n-6,%));
```

$$15, \frac{225}{2}, \frac{1245}{2}, \frac{6075}{2}, \frac{27705}{2}, \frac{121275}{2}, \frac{516945}{2}, \frac{2164275}{2}, \frac{8948505}{2}, \frac{36672075}{2}, \frac{149330145}{2}, \frac{605261475}{2}, \frac{2444899305}{2}, \frac{9851218875}{2}, \frac{39619863345}{2}$$

$$\Bigg[\Bigg\{-24\,a(n) + 26\,a(n+1) - 9\,a(n+2) + a(n+3), a(0) = 15, a(1) = \frac{225}{2}, a(2)$$

$$= \frac{1245}{2} \Big\}, ogf \Big]$$

$$75\ 4^n - \frac{135\ 3^n}{2} + \frac{15\ 2^n}{2}$$

$$\frac{75\ 4^n}{4096} - \frac{5\ 3^n}{54} + \frac{15\ 2^n}{128} \tag{1.3.5}$$

```
> (*
for m from 6 to 0 by -1 do
for n from 0 to 11 do
printf("%8.2f ",ee[n,m]);
end;
printf("\n");
end;
*)
```

Check lower and upper bounds

```
> NN;
```

$$100 \tag{1.3.1.1}$$

```
> (*
m:=10;
seq(mysign(-ee[n,m] + (n-m+2)/(n+m)*ee[n-1,m-1] + (n-
m-1)/(n-m)*ee[n-1,m+1]),n=max(3,m+1)..30);
seq(mysign(-1/(n-m)*ee[n-1,m+1] + 1/(n+m)*ee[n-2,m] +
(n-m-3)/(n-m)/(n-m-2)*ee[n-2,m+2]),n=max(m+3,5)..30);
*)
```

This should always be negative

```
> for n from -1 to NN do
  for m from -1 to NN do
     eelow[n,m] := 0:
  end:
  end:

  for n from 3 to NN do
     for m from 1 to n-3 do
        eelow[n,m] := -ee[n,m] + (n-m+2)/(n+m)*ee[n-1,m-1]
  + (n-m-1)/(n-m)*ee[n-1,m+1] + (n-m-3)/(n-m-2)*(1/(n-m)*
  ee[n-2,m+2] + 1/(n+m)*ee[n-3,m+1]);
     end:
  end:
> mysign := proc(i) if i=0 then return 0 else return sign
  (i) end end:
```

looks good

```
> for m from 16 to 0 by -1 do
  for n from 5 to 20 do
  printf("%3.0f ",mysign(eelow[n,m]*factorial(floor((n+m)
  /2))));
  end;
  printf("\n");
  end;
  0   0   0   0   0   0   0   0   0   0   0   0   0   0
  0  -1
  0   0   0   0   0   0   0   0   0   0   0   0   0   0
 -1   0
  0   0   0   0   0   0   0   0   0   0   0   0   0  -1
  0  -1
  0   0   0   0   0   0   0   0   0   0   0   0  -1   0
```

```
 -1   0
  0   0   0   0   0   0   0   0   0   0   0  -1   0  -1
  0  -1
  0   0   0   0   0   0   0   0   0   0  -1   0  -1   0
 -1   0
  0   0   0   0   0   0   0   0   0  -1   0  -1   0  -1
  0  -1
  0   0   0   0   0   0   0   0  -1   0  -1   0  -1   0
 -1   0
  0   0   0   0   0   0   0  -1   0  -1   0  -1   0  -1
  0  -1
  0   0   0   0   0   0  -1   0  -1   0  -1   0  -1   0
 -1   0
  0   0   0   0   0  -1   0  -1   0  -1   0  -1   0  -1
  0  -1
  0   0   0   0  -1   0  -1   0  -1   0  -1   0  -1   0
 -1   0
  0   0   0  -1   0  -1   0  -1   0  -1   0  -1   0  -1
  0  -1
  0   0  -1   0  -1   0  -1   0  -1   0  -1   0  -1   0
 -1   0
  0  -1   0  -1   0  -1   0  -1   0  -1   0  -1   0  -1
  0  -1
 -1   0  -1   0  -1   0  -1   0  -1   0  -1   0  -1   0
 -1   0
  0   0   0   0   0   0   0   0   0   0   0   0   0   0
  0   0
```

```
> n0:=51:
  seq(mysign(eelow[n0,m]),m=n0 mod 2..n0,2);
```

$-1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, 0, 0$ **(1.3.1.2)**

This should always be non-negative

```
> for n from -1 to NN do
  for m from -1 to NN do
     eeup[n,m] := 0:
  end:
  end:

  for n from 3 to NN do
     for m from 1 to n-3 do
        eeup[n,m] := -ee[n,m] + (n-m+2)/(n+m)*ee[n-1,m-1]
  + (n-m-1)/(n-m)*ee[n-1,m+1] + 1/(n-m)*ee[n-2,m+2] + 1/
  (n+m)*ee[n-3,m+1];
     end:
  end:
```

```
> n0:=51:
  seq(mysign(eeup[n0,m]),m=n0 mod 2..n0,2);
```

$1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0$ **(1.3.1.3)**

looks good

```
> for m from 16 to 0 by -1 do
  for n from 5 to 20 do
  printf("%3.0f ",mysign(eeup[n,m]*factorial(floor((n+m)
  /2))));
  end;
  printf("\n");
```

```
end;
```

```
0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0
0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0
0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  1
0  0  0  0  0  0  0  0  0  0  0  0  0  0
1  0
0  0  0  0  0  0  0  0  0  0  0  0  0  1
0  1
0  0  0  0  0  0  0  0  0  0  0  0  1  0
1  0
0  0  0  0  0  0  0  0  0  0  0  1  0  1
0  1
0  0  0  0  0  0  0  0  0  0  1  0  1  0
1  0
0  0  0  0  0  0  0  0  0  1  0  1  0  1
0  1
0  0  0  0  0  0  0  0  1  0  1  0  1  0
1  0
0  0  0  0  0  0  0  1  0  1  0  1  0  1
0  1
0  0  0  0  0  0  1  0  1  0  1  0  1  0
1  0
0  0  0  0  0  1  0  1  0  1  0  1  0  1
0  1
0  0  0  0  1  0  1  0  1  0  1  0  1  0
1  0
0  0  0  1  0  1  0  1  0  1  0  1  0  1
0  1
0  0  1  0  1  0  1  0  1  0  1  0  1  0
1  0
0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0
```

```
> m := 'm':
  n := 'n':
```

Newton polygons

Programs

look at possible degrees of each term in n and m
x-axis m degree
y-axis n degree

```
> mynewt := proc(F,m,n)
     local ss,newt,tt,mdeg,ndeg;
     ss := 0;
     newt := {}:
     for tt in expand(collect(F,[m,n],simplify)) do
        mdeg := simplify(m*diff(tt,m)/tt);
        ndeg := simplify(n*diff(tt,n)/tt);
        newt := {op(newt), [mdeg,ndeg]};
     end:
     return newt;
```

```
end:
```

Get the term of order Theta(m^a*n^b) from the Newton polygon computed using mynewt

```
> getel := proc(tmp,a,b)
     local tt,mdeg,ndeg,ret:
     ret := 0:
     for tt in expand(tmp) do
        mdeg := m*diff(tt,m)/tt;
        ndeg := n*diff(tt,n)/tt;
        if mdeg=a and ndeg=b then ret := ret + tt end;
     end:
     return ret;
  end:
```

Get the maximal power of n^b for each m^a for a=0..M

```
> getMaxNewt := proc(M::posint, newt)
     local i, el, mnmax;

     for i from 0 to M do mnmax[i]:=-infinity end:
     for el in newt do
        if el[1]<=M then
        if mnmax[el[1]]<el[2] then mnmax[el[1]]:=el[2] end:
     end;
     end;

     return mnmax;
  end:
```

Compute the slopes and corners of the convex hull

```
> maxslope := proc(ll,M,i)
     local j,sl,tmp,sj;
     sl := ll[i+1]-ll[i]:   #initial slope between first 2
  points
     #is this the one of the hull? find max starting from 0
     sj:=M;
     for j from i+1 to M do
        tmp := (ll[j]-ll[i])/(j-i);
        if tmp >= sl then sl:=tmp;sj:=j; end;
     end:
     return sl,sj;
  end:
```

Compute the slopes and corners of the convex hull

```
> getslopes := proc(ll,M)
     local sl,sj,li,ls;
     sj:=0;
     li:=sj:
     ls:=0:
     #go on and find other slopes of convex hull
     while sj<M do
        sl,sj := maxslope(ll,M,sj):
        #save it
        li:=li,sj;
        ls:=ls,sl;
     end:

     li:=[li];
     ls:=[seq(ls[i],i=2..nops(li))];
     return ls,li;
  end:
```

Shorthands

The largest root of the Airy funciton Ai(z)

```
> A1  := AiryAiZeros(1);
  evalf(%);
```

$$A1 := \mathrm{AiryAiZeros}(1)$$
$$-2.338107410 \quad \textbf{(2.2.1)}$$

This is the constant c of Equation (6):
s(n) = 2 + c*n^(-2/3) + ...
(Also called sigma_2 in the ansatz used in the proof of Lemma 4.2)

```
> csubs := isolate((1/2)*2^(1/3)*(c) = a1,c);
```

$$csubs := c = a1\, 2^{2/3} \quad \textbf{(2.2.2)}$$

Labeling options to produce nice plots.

```
> myoptionsLo := labels=["i", "j"],symbolsize=25, symbol=
  diamond,axesfont = ["HELVETICA", "ROMAN", 15],labelfont =
  ["HELVETICA", 18]:
  myoptionsUp := labels=["i", "j"],symbolsize=20, symbol=
  diamond,axesfont = ["HELVETICA", "ROMAN", 15],labelfont =
  ["HELVETICA", 18]:
```

We introduce the following shorthands for the Airy function and its derivative

```
> kaplam := AiryAi(AiryAiZeros(1)+2^(1/3)*m/n^(1/3))=kappa,
  AiryAi(1, AiryAiZeros(1)+2^(1/3)*m/n^(1/3))=lambda;
```

$$kaplam := \mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1) + \frac{2^{1/3} m}{n^{1/3}}\right) = \kappa, \mathrm{AiryAi}\left(1, \mathrm{AiryAiZeros}(1) + \frac{2^{1/3} m}{n^{1/3}}\right) = \lambda \quad \textbf{(2.2.3)}$$

Expansions

We expand the Airy function around a1+2^(1/3)*m/n^(1/3) up to chosen order ordAi

```
> FFy := AiryAi(AiryAiZeros(1)+2^(1/3)*m/n^(1/3)+y);
```

$$FFy := \mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1) + \frac{2^{1/3} m}{n^{1/3}} + y\right) \quad \textbf{(2.3.1)}$$

We use two different expansion orders for the upper and lower bound
(to speed up the computations, and to produce the best pictures)
Here, we start with the **lower bound**, which needs less terms

```
> ordAiLo := 13;
  FFyserLo := map(expand,series(FFy,y,ordAiLo)):
```

$$ordAiLo := 13 \quad \textbf{(2.3.2)}$$

For y = x-(a1+2^(1/3)*m/n^(1/3)) we have that FF(x)=Ai(x),
i.e. an expansion of the Airy function

```
> FFxserLo := subs(y=x-(AiryAiZeros(1)+2^(1/3)*m/n^(1/3)),
  FFyserLo):
```

Replace the appearing Airy functions by our shorthands kappa and lambda

```
> indets(FFxserLo);
  FFxserLoKL := subs(kaplam,AiryAiZeros(1)=a1,FFxserLo):
  indets(%);
```

$$\left\{m, n, x, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, \mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1) + \frac{2^{1/3} m}{n^{1/3}}\right),\right.$$

$$\mathrm{AiryAi}\left(1, \mathrm{AiryAiZeros}(1) + \frac{2^{1/3}\, m}{n^{1/3}}\right)\Bigg\}$$

$$\left\{a1, \kappa, \lambda, m, n, x, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}\right\} \qquad \textbf{(2.3.3)}$$

Then, we use the generic ansatz for factor of FFy, i.e. the Airy function

```
> facAiryLo := (1+(add(q[i]*m^i,i=0..2))/(n));
```

$$facAiryLo := 1 + \frac{q_2\, m^2 + q_1\, m + q_0}{n} \qquad \textbf{(2.3.4)}$$

This is our ansatz
(only the substitution is influenced by the parameters, i.e. not the ms and ns that are already in FFxser;
that is what we want, as all of them should be expanded at a1+2^(1/3)*m/n(1/3))
Note that due to the replacement with kappa and lambda, the expansions are fixed already at this point and the ms and ns in the arguments of Ai and Ai' are not influenced.
Note that we substitute now a1+2^(1/3)*(m+1)/n^(1/3), i.e. m+1 instead of m around which we expanded above.

```
> XFL := (n0,m0) -> subs(n=n0,m=m0,facAiryLo)*subs(x=a1+2^
  (1/3)*(m0+1)/n0^(1/3),FFxserLoKL);
```

$$XFL := (n0, m0) \mapsto subs(n = n0, m = m0, facAiryLo)\ subs\left(x = a1 + \frac{2^{1/3}\,(m0+1)}{n0^{1/3}}, FFxserLoKL\right) \qquad \textbf{(2.3.5)}$$

Then, we do the same for the **upper bound**, with a few more terms

```
> ordAiUp := 19;
  FFyserUp := map(expand,series(FFy,y,ordAiUp)):
```

$$ordAiUp := 19 \qquad \textbf{(2.3.6)}$$

For y = x-(a1+2^(1/3)*m/n^(1/3)) we have that FF(x)=Ai(x),
i.e. an expansion of the Airy function

```
> FFxserUp := subs(y=x-(AiryAiZeros(1)+2^(1/3)*m/n^(1/3)),
  FFyserUp):
```

Replace the appearing Airy functions by our shorthands kappa and lambda

```
> indets(FFxserUp);
  FFxserUpKL := subs(kaplam,AiryAiZeros(1)=a1,FFxserUp):
  indets(%);
```

$$\left\{m, n, x, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, \mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1) + \frac{2^{1/3}\, m}{n^{1/3}}\right), \mathrm{AiryAi}\left(1, \mathrm{AiryAiZeros}(1) + \frac{2^{1/3}\, m}{n^{1/3}}\right)\right\}$$

$$\left\{a1, \kappa, \lambda, m, n, x, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}\right\} \qquad \textbf{(2.3.7)}$$

In the upper bound, we use a more generic factor of FFy, there are the additional terms p[i] which are divided by n^2

```
> facAiryUp := (1+(add(p[i]*m^i,i=0..4))/(n^2)+(add(q[i]*
  m^i,i=0..2))/(n));
```

$$facAiryUp := 1 + \frac{p_4\, m^4 + p_3\, m^3 + p_2\, m^2 + p_1\, m + p_0}{n^2} + \frac{q_2\, m^2 + q_1\, m + q_0}{n} \qquad \textbf{(2.3.8)}$$

This is our ansatz for the upper bound (comparable to XFL)

(only the substitution is influenced by the parameters, i.e. not the ms and ns that are already in FFxser;
that is what we want, as all of them should be expanded at a1+2^(1/3)*m/n(1/3))

```
> XFU := (n0,m0) -> subs(n=n0,m=m0,facAiryUp)*subs(x=a1+2^(1/3)*(m0+1)/n0^(1/3),FFxserUpKL);
```

$$XFU := (n0, m0) \mapsto subs(n = n0, m = m0, facAiryUp)\, subs\left(x = a1 + \frac{2^{1/3}(m0+1)}{n0^{1/3}}, FFxserUpKL\right) \quad \textbf{(2.3.9)}$$

Finally, the ansatz for the **quotient** of h(n)/h(n-1).
Note that pterm is a mnemonic for "polynomial term", as this value influences the polynomial term n^{alpha};
The other values have similar interpretations:
a .... exponantial growth, i.e. a^n
b .... b will be zero
c .... stretched exponential
d .... technical choice, to simplify the proofs

```
> SF := n -> a+b/n^(1/3)+c/n^(2/3)+pterm/n + d/n^(7/6);
```

$$SF := n \mapsto a + \frac{b}{n^{1/3}} + \frac{c}{n^{2/3}} + \frac{pterm}{n} + \frac{d}{n^{7/6}} \quad \textbf{(2.3.10)}$$

Minimal DFAs recoginizing a finite binary language - Lower bound

Xtilde

This is Y tilde

```
> Xansatz := (n,m) -> (1-2*(m)^2/(3*n)+3*m/(8*n))* AiryAi(a1+2^(1/3)*(m+1)/n^(1/3));
```

$$Xansatz := (n, m) \rightarrow \left(1 - \frac{2}{3}\frac{m^2}{n} + \frac{3}{8}\frac{m}{n}\right) \mathrm{AiryAi}\left(a1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right) \quad \textbf{(2.4.1.1)}$$

```
> Sansatz := n -> 2 + c*n^(-2/3) + pterm/n - 1/(n^(7/6));
```

$$Sansatz := n \rightarrow 2 + \frac{c}{n^{2/3}} + \frac{pterm}{n} - \frac{1}{n^{7/6}} \quad \textbf{(2.4.1.2)}$$

lower bound

```
> posansatz := -XX(n,m)*SS(n)*SS(n-1)*SS(n-2)
                 + (n-m+2)/(n+m)*XX(n-1,m-1)*SS(n-1)*SS(n-2)
                 + (n-m-1)/(n-m)*XX(n-1,m+1)*SS(n-1)*SS(n-2)
                 + (n-m-3)/(n-m-2)*( 1/(n-m)*XX(n-2,m+2)*SS(n-2) + 1/(n+m)*XX(n-3,m+1) )
                 ;
```

$$posansatz := -XX(n, m)\, SS(n)\, SS(n-1)\, SS(n-2) + \frac{(n-m+2)\, XX(n-1, m-1)\, SS(n-1)\, SS(n-2)}{n+m} + \frac{(n-m-1)\, XX(n-1, m+1)\, SS(n-1)\, SS(n-2)}{n-m} \quad \textbf{(2.4.1.3)}$$

$$+\frac{(n-m-3)\left(\frac{XX(n-2,m+2)\ SS(n-2)}{n-m}+\frac{XX(n-3,m+1)}{n+m}\right)}{n-m-2}$$

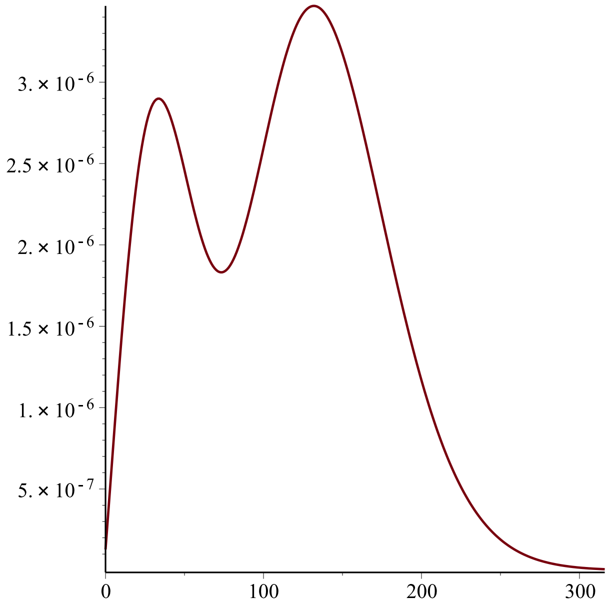
```
> posXS := map(simplify,subs(XX=Xansatz,SS=Sansatz,
  posansatz)):
```

For a large n this function of m seems to be positive

```
> Digits:=20:
  e1 := subs(csubs,pterm=29/12,a1=A1,posXS):
  N := 100000;
  M := floor(N^(1/2));
  P1 := plot([seq([mm,(subs(n=N,m=mm,e1))],mm=0..M)])
  :display(P1);
```

$$N := 100000$$

$$M := 316$$



Prove it

We start with the ansatz of Ytilde in Lemma 5.2.

Recall the general ansatz

```
> facAiryLo*Airy(a1+2^(1/3)*(m+1)/n^(1/3));
  SF(n);
```

$$\left(1+\frac{m^2 q_2+m q_1+q_0}{n}\right) Airy\left(a1+\frac{2^{1/3}(m+1)}{n^{1/3}}\right)$$

$$a + \frac{b}{n^{1/3}} + \frac{c}{n^{2/3}} + \frac{pterm}{n} + \frac{d}{n^{7/6}} \quad \textbf{(2.4.2.1)}$$

Substitute ansatz into the sequence we want to be positive for large n and all m

```
> posF := map(expand, subs(XX=XFL,SS=SF,posansatz)):indets
  (%);
```

$$\left\{a, a1, b, c, d, \kappa, \lambda, m, n, pterm, q_0, q_1, q_2, \frac{1}{n^{7/6}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, \frac{1}{(n-3)^{1/3}}, \frac{1}{(n-2)^{7/6}}, \frac{1}{(n-2)^{2/3}}, \frac{1}{(n-2)^{1/3}}, \frac{1}{(n-1)^{7/6}}, \frac{1}{(n-1)^{2/3}}, \frac{1}{(n-1)^{1/3}}\right\} \quad \textbf{(2.4.2.2)}$$

The error terms are (to check, expand posF)

```
> simplify(O((2^(1/3)*(m+1)/n^(1/3)-2^(1/3)*m/n^(1/3))
  ^ordAiLo));
  simplify(O((2^(1/3)*(m)/(n-1)^(1/3)-2^(1/3)*m/n^(1/3))
  ^ordAiLo));
  simplify(O((2^(1/3)*(m+2)/(n-1)^(1/3)-2^(1/3)*m/n^(1/3))
  ^ordAiLo));
```

$$\mathrm{O}\left(\frac{16\, 2^{1/3}}{n^{13/3}}\right)$$

$$\mathrm{O}\left(-\frac{16\, 2^{1/3}\, m^{13} \left((n-1)^{1/3} - n^{1/3}\right)^{13}}{(n-1)^{13/3}\, n^{13/3}}\right)$$

$$\mathrm{O}\left(-\frac{16\, 2^{1/3} \left(m\, (n-1)^{1/3} - n^{1/3}\, m - 2\, n^{1/3}\right)^{13}}{(n-1)^{13/3}\, n^{13/3}}\right) \quad \textbf{(2.4.2.3)}$$

remove error terms

```
> posFd := convert(posF,polynom):
```

Next we fit the display size of the plots in the n (Nord) and m (Mord) direction.
(Note that everything up to ordAi is computed, but possibly not shown)

```
> Nord := -ordAiLo/3;
  Mord := floor(ordAiLo/3)+1;
  myview := view=[0..Mord,Nord..0]:
```

$$Nord := -\frac{13}{3}$$

$$Mord := 5 \quad \textbf{(2.4.2.4)}$$

Expand again with respect to n,
these are then our unknowns

```
> posFe := series(posFd,n=infinity,ceil(-Nord)+1):indets
  (%);
  posFf := convert(%%,polynom):
```

$$\left\{a, a1, b, c, d, \kappa, \lambda, m, n, pterm, q_0, q_1, q_2, \left(\frac{1}{n}\right)^{1/3}, \left(\frac{1}{n}\right)^{2/3}, \left(\frac{1}{n}\right)^{3/2}, \left(\frac{1}{n}\right)^{4/3}, \left(\frac{1}{n}\right)^{5/2}, \left(\frac{1}{n}\right)^{5/3}, \left(\frac{1}{n}\right)^{7/2}, \left(\frac{1}{n}\right)^{7/3}, \left(\frac{1}{n}\right)^{7/6}, \left(\frac{1}{n}\right)^{8/3}, \left(\frac{1}{n}\right)^{9/2}, \left(\frac{1}{n}\right)^{10/3}, \left(\frac{1}{n}\right)^{11/3}, \left(\frac{1}{n}\right)^{11/6}, \left(\frac{1}{n}\right)^{13/3}, \left(\frac{1}{n}\right)^{13/6}, \left(\frac{1}{n}\right)^{14/3}, \right. \quad \textbf{(2.4.2.5)}$$

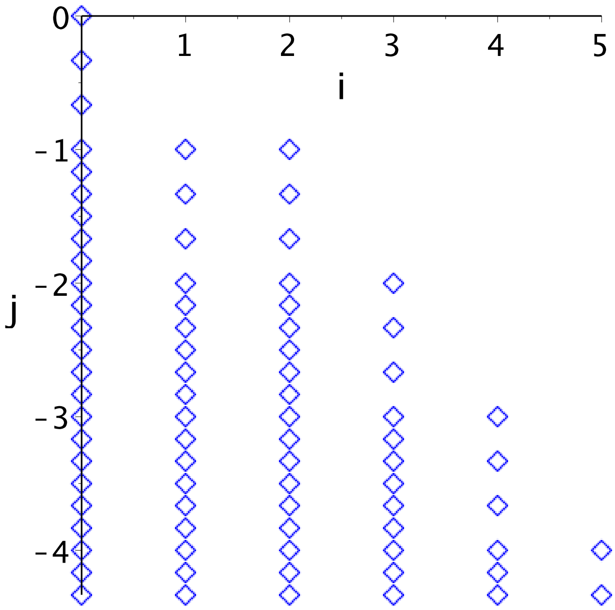
$$\left(\frac{1}{n}\right)^{17/6}, \left(\frac{1}{n}\right)^{19/6}, \left(\frac{1}{n}\right)^{23/6}, \left(\frac{1}{n}\right)^{25/6}, \left(\frac{1}{n}\right)^{29/6}, \mathrm{O}\left(\frac{1}{n^5}\right)\}$$

The mynewt function computes the Newton polygon of posFf

```
> newt1 := mynewt(posFf,m,n):
```

First Newton polygon, where no unknowns have been fixed

```
> P1 := pointplot(newt1,myoptionsLo,color=blue):
  display(P1,myview);
```



Here, we want to kill the element (0,0)

```
> getel(posFf,0,0);
```

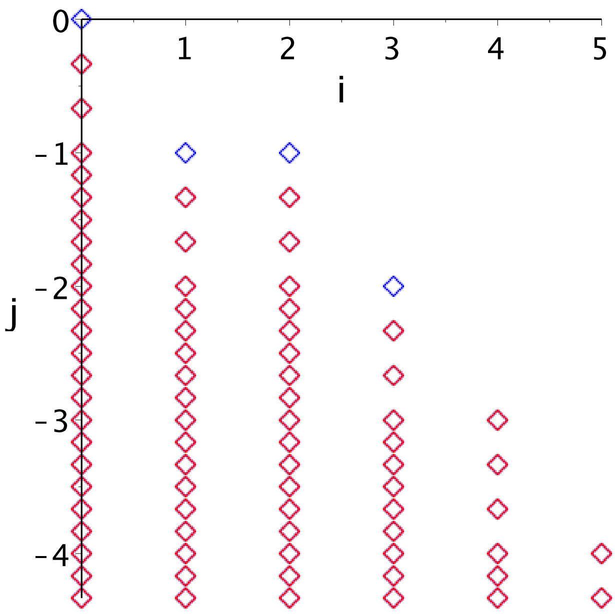
$$-a^3\,\kappa + 2\,a^2\,\kappa \qquad \textbf{(2.4.2.6)}$$

Set a=2

```
> posFfa := expand(simplify(subs(a=2,posFf))) assuming
  n::posint,m::posint:
> newta := mynewt(posFfa,m,n):
```

All blue points have been eliminated, and only the red ones remain

```
> P1a := pointplot(newta,myoptionsLo,color=red):
  display(P1,P1a,myview);
```



b=0 is forced due to the term n^{-1/3)

```
> getel(posFfa,0,-1/3);
```

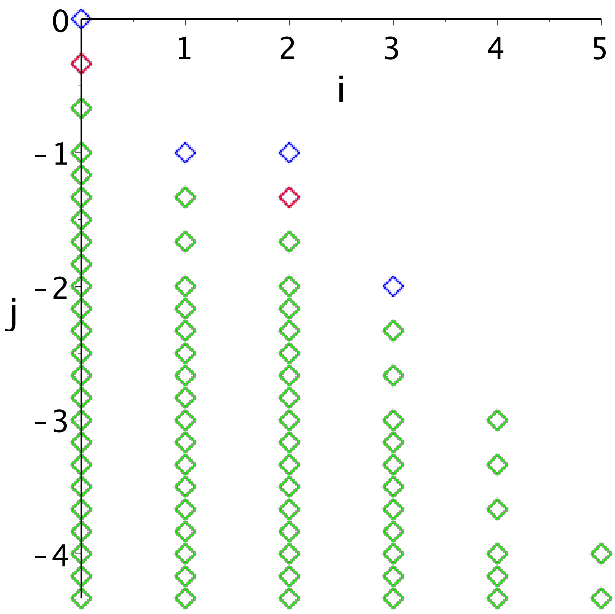
$$-\frac{4\,\kappa\,b}{n^{1/3}} \tag{2.4.2.7}$$

set a=2, b=0

```
> posFfab := expand(simplify(subs(b=0,posFfa))) assuming
  n::posint,m::posint:
> newtab := mynewt(posFfab,m,n):
```

Now only the green points remain.

```
> P1ab := pointplot(newtab,myoptionsLo,color=green):
  display(P1,P1a,P1ab,myview);
```



at this point we find our choice for c,
should we also set q[2]=-2/3??? (Check later if this is really necessary)

```
> csubs;
  factor(getel(posFfab,0,-2/3));factor(subs(csubs,%));
  factor(getel(posFfab,1,-4/3));factor(subs(csubs,%));
  factor(getel(posFfab,2,-5/3));factor(subs(csubs,%));
```

$$c = a1\, 2^{2/3}$$

$$\frac{4\,\kappa\,\left(a1\, 2^{2/3} - c\right)}{n^{2/3}}$$

$$0$$

$$\frac{16}{3}\,\frac{2^{1/3}\,\lambda\, m\,\left(3\, q_2 + 2\right)}{n^{4/3}}$$

$$\frac{16}{3}\,\frac{2^{1/3}\,\lambda\, m\,\left(3\, q_2 + 2\right)}{n^{4/3}}$$

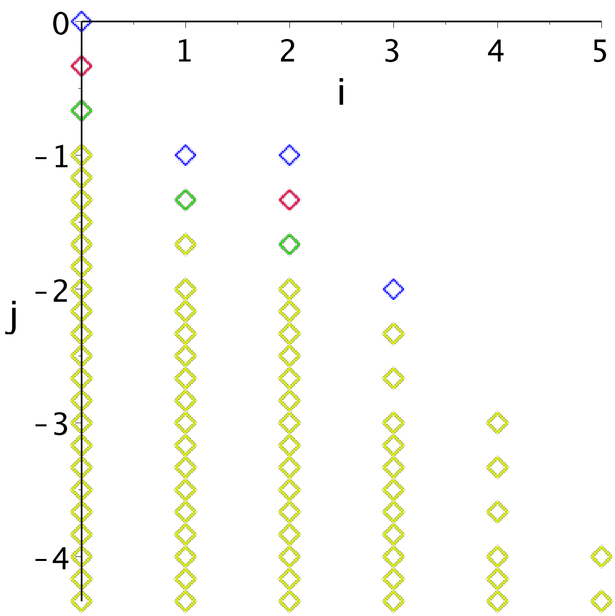
$$\frac{4\,\kappa\, m^2\, q_2\,\left(a1\, 2^{2/3} - c\right)}{n^{5/3}}$$

$$0 \qquad \textbf{(2.4.2.8)}$$

set a=2, b=0, c=a1*2^(2/3),
q[2]=-2/3

```
> posFfabc := expand(simplify(subs(csubs,q[2]=-2/3,
```

```
    posFfab))) assuming n::posint,m::posint:
> newtabc := mynewt(posFfabc,m,n):
> P1abc := pointplot(newtabc,myoptionsLo,color=yellow):
  display(P1,P1a,P1ab,P1abc,myview);
```



Here we get pterm

```
> factor(getel(posFfabc,0,-1));
  factor(getel(posFfabc,1,-4/3));
  solve({%%,%},{q[2],pterm});
```

$$-\frac{1}{3}\frac{\kappa\left(-29+12\,pterm\right)}{n}$$

$$0$$

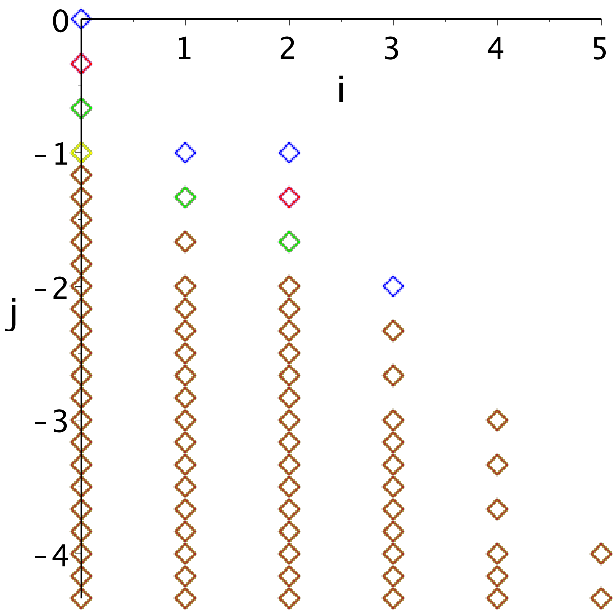
$$\left\{pterm=\frac{29}{12},\ q_2=q_2\right\} \qquad \textbf{(2.4.2.9)}$$

set a=2, b=0,c,pterm=29/12,q[2]=-2/3

```
> posFfabcd := expand(simplify(subs(pterm=29/12,q[2]=-2/3,
  posFfabc))) assuming n::posint,m::posint:
> newtabcd := mynewt(posFfabcd,m,n):
```

only the brown points remain
Now all points are strictly below n^{-1}

```
> P1abcd := pointplot(newtabcd,myoptionsLo,color=brown):
  display(P1,P1a,P1ab,P1abc,P1abcd,myview);
```



Here are the dominating corners and we see that we have to choose d=-1 to have a positive term;
note that we will see that the second term should be negative, as lambda=Ai' is negative for large m

```
> getel(posFfabcd,0,-7/6);
  getel(posFfabcd,3,-14/6);
  (14/6-7/6)/3;   #slope
```

$$-\frac{4\,\kappa\,d}{n^{7/6}}$$

$$-\frac{64}{9}\,\frac{2^{1/3}\,\lambda\,m^3}{n^{7/3}}$$

$$\frac{7}{18} \tag{2.4.2.10}$$

and continuing

```
> getel(posFfabcd,4,-3);
  (3-14/6)/1;   #slope
```

$$-\frac{136}{9}\,\frac{\kappa\,m^4}{n^3}$$

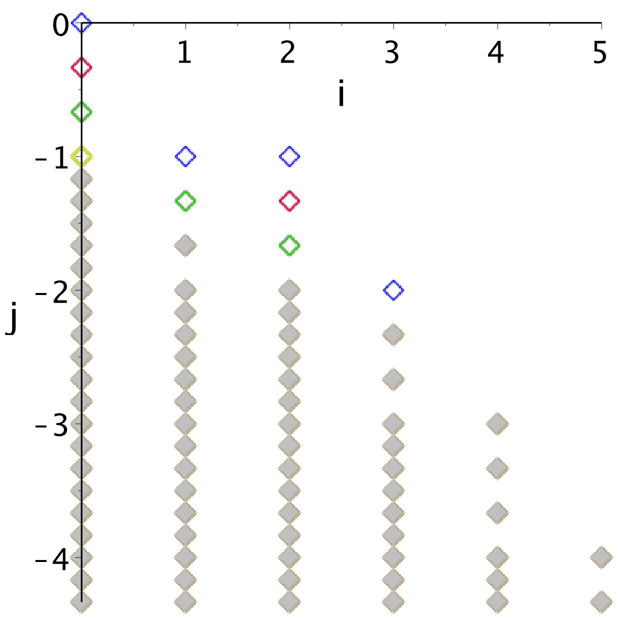
$$\frac{2}{3} \tag{2.4.2.11}$$

set a=2, b=0,c,pterm=13/6,q[2]=-2/3 and d=-1

```
> posFfabcde := expand(simplify(subs(d=-1,posFfabcd)))
  assuming n::posint,m::posint:
> newtabcde := mynewt(posFfabcde,m,n):
```

This is the final result, where only the solid diamonds are non-zero

```
> P1abcde := pointplot(newtabcde,myoptionsLo,symbol=
  soliddiamond,color=gray):
  display(P1,P1a,P1ab,P1abc,P1abcd,P1abcde,myview);
```

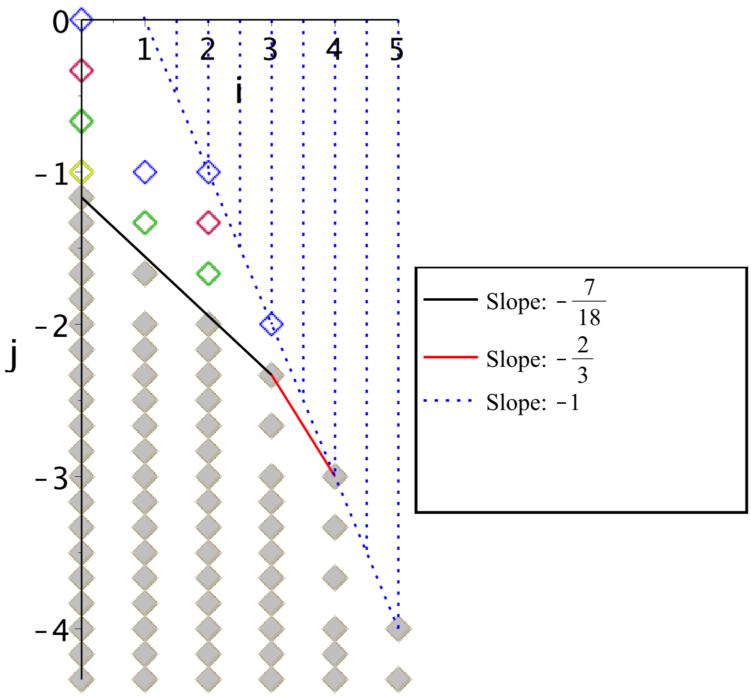


Plot the boundary and the slopes of the Newton polygon;
Note that we have already proved that there are now points above the blue dotted line

```
> LegendSize := size=[600,450]:
  P1dom1 := plot(-7/6-(7/18)*m,m=0..3,color=black,legend=
  [typeset("Slope: ", -7/18)],legendstyle=[location=right]
  ):
  P1dom2 := plot(-1/3-(2/3)*m,m=3..4,color=red,legend=
  [typeset("Slope: ", -2/3)],legendstyle=[location=right])
  :
  P1dom3a := plot(1-m,m=0..5,color=blue,linestyle=dot,
  legend=[typeset("Slope: ", -1)],legendstyle=[location=
  right]):
  P1all := display(P1,P1a,P1ab,P1abc,P1abcd,P1abcde,
  P1dom1,P1dom2,P1dom3a,myview,LegendSize):

  for i from 1 to 8 do
     P1dom3[i] := plot([[1+i/2,0],[1+i/2,-i/2]],color=
  blue,linestyle=dot):
```

```
end:
display(P1all,seq(P1dom3[i],i=1..8));
```



This is the choice for SF

```
> SF(n);
  subs(a=2,b=0,csubs,pterm=29/12,d=-1,%);
```

$$a + \frac{b}{n^{1/3}} + \frac{c}{n^{2/3}} + \frac{pterm}{n} + \frac{d}{n^{7/6}}$$

$$2 + \frac{a1\ 2^{2/3}}{n^{2/3}} + \frac{29}{12\ n} - \frac{1}{n^{7/6}} \qquad \textbf{(2.4.2.12)}$$

recall

```
> kaplam;
```

$$\mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1) + \frac{2^{1/3}\ m}{n^{1/3}}\right) = \kappa, \mathrm{AiryAi}\left(1, \mathrm{AiryAiZeros}(1) + \frac{2^{1/3}\ m}{n^{1/3}}\right) = \lambda \qquad \textbf{(2.4.2.13)}$$

Look at the corners

Here we see the necessary choice: p[4]>2/9 (note that for m>n^(1/3) lambda is negative)

```
> getel(posFfabcde,0,-7/6);
```

```
simplify(getel(posFfabcde,3,-14/6));
simplify(getel(posFfabcd,4,-3));
simplify(getel(posFfabcd,5,-4));
#simplify(getel(posFfabcd,6,-5));
```

$$\frac{4\,\kappa}{n^{7/6}}$$

$$-\frac{64}{9}\,\frac{2^{1/3}\,\lambda\,m^3}{n^{7/3}}$$

$$-\frac{136}{9}\,\frac{\kappa\,m^4}{n^3}$$

$$-\frac{248}{135}\,\frac{\kappa\,m^5}{n^4} \qquad \textbf{(2.4.2.14)}$$

Now split the black dots into the contributions from Ai and Ai'

```
> indets(posFfabcd);
```

$$\left\{a1, d, \kappa, \lambda, m, n, q_0, q_1, \frac{1}{n^{29/6}}, \frac{1}{n^{25/6}}, \frac{1}{n^{23/6}}, \frac{1}{n^{19/6}}, \frac{1}{n^{17/6}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{13/6}}, \frac{1}{n^{11/3}}, \frac{1}{n^{11/6}}, \frac{1}{n^{10/3}}, \frac{1}{n^{9/2}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/2}}, \frac{1}{n^{7/3}}, \frac{1}{n^{7/6}}, \frac{1}{n^{5/2}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{3/2}}\right\} \qquad \textbf{(2.4.2.15)}$$

Sanity check that there are no other contributions

```
> subs(kappa=0,lambda=0,posFfabcd);
```

$$0 \qquad \textbf{(2.4.2.16)}$$

Extract the coefficients of kappa=Ai and lambda=Ai'
and treat then seperately

```
> posFk  := coeff(posFfabcd,kappa):indets(%);
  posFl  := coeff(posFfabcd,lambda):indets(%);
```

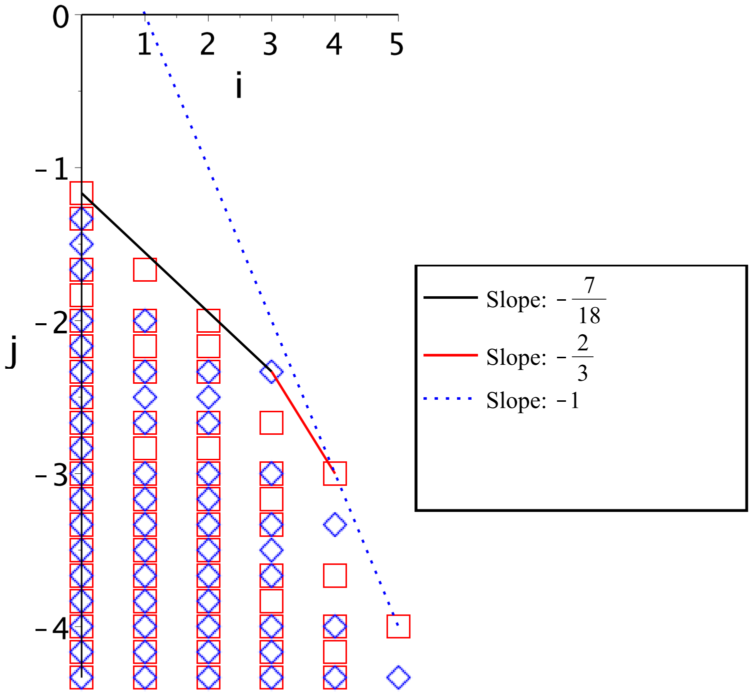
$$\left\{a1, d, m, n, q_0, q_1, \frac{1}{n^{29/6}}, \frac{1}{n^{25/6}}, \frac{1}{n^{23/6}}, \frac{1}{n^{19/6}}, \frac{1}{n^{17/6}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{13/6}}, \frac{1}{n^{11/3}}, \frac{1}{n^{11/6}}, \frac{1}{n^{10/3}}, \frac{1}{n^{9/2}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/2}}, \frac{1}{n^{7/3}}, \frac{1}{n^{7/6}}, \frac{1}{n^{5/2}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}\right\}$$

$$\left\{a1, d, m, n, q_0, q_1, \frac{1}{n^{29/6}}, \frac{1}{n^{25/6}}, \frac{1}{n^{23/6}}, \frac{1}{n^{19/6}}, \frac{1}{n^{17/6}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{13/6}}, \frac{1}{n^{11/3}}, \frac{1}{n^{10/3}}, \frac{1}{n^{9/2}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/2}}, \frac{1}{n^{7/3}}, \frac{1}{n^{5/2}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{3/2}}\right\} \qquad \textbf{(2.4.2.17)}$$

We color the non-zero nodes of the lastNewton polygon into
red squared .... coefficients of kappa=Ai
blue diamonds ...... coefficients of lambda=Ai'

```
> newt4a  := mynewt(posFk,m,n):
  newt4b  := mynewt(posFl,m,n):
```

```
> P4a := pointplot(newt4a,labels=["m deg", "n deg"],
  symbolsize=25, symbol=box,color=red):
  P4b := pointplot(newt4b,labels=["m deg", "n deg"],
  symbolsize=25, symbol=diamond, color=blue):
  P1dom3s := plot(1-m,m=4..5,color=black):
  display(P4a,P4b,P1dom1,P1dom2,P1dom3a,myview,
  myoptionsLo,LegendSize);
```



red extremes of Newton polygon

```
> mnmaxR := getMaxNewt(Mord,newt4a):
  seq([i,mnmaxR[i]],i=0..Mord);
```

$$\left[0, -\frac{7}{6}\right], \left[1, -\frac{5}{3}\right], [2, -2], \left[3, -\frac{8}{3}\right], [4, -3], [5, -4] \qquad \textbf{(2.4.2.18)}$$

These are the specific values at these points;
we see that we still have some degree in freedom: d and p[4]

```
> for i from 0 to Mord do
     i,factor(getel(posFk,i,mnmaxR[i]));
  end;
```

$$0, -\frac{4\, d}{n^{7/6}}$$

$$1, -\frac{8}{3}\frac{a1\,2^{2/3}\,m}{n^{5/3}}$$

$$2, -\frac{164}{9}\frac{m^2}{n^2}$$

$$3, -\frac{16}{3}\frac{2^{2/3}\,a1\,m^3}{n^{8/3}}$$

$$4, -\frac{136}{9}\frac{m^4}{n^3}$$

$$5, -\frac{248}{135}\frac{m^5}{n^4} \tag{2.4.2.19}$$

These are the slopes of the convex hull where the corners are given by the second sequence;
hence, in order to be positive when the slope > -1, we need to choose d>0, e.g. d=1;
note that p[4] is not important here, as the slope first slope -5/12 is less then -7/18;
and in the later regimes it will be dominated by the blue points.

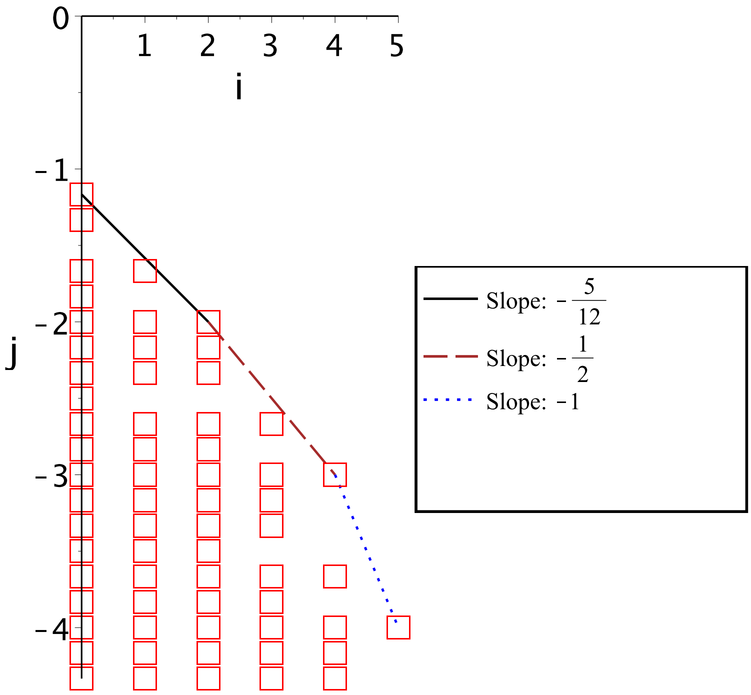
```
> ls,li := getslopes(mnmaxR,Mord);
```

$$ls, li := \left[-\frac{5}{12}, -\frac{1}{2}, -1\right], [0, 2, 4, 5] \tag{2.4.2.20}$$

```
> colors := [green,black,brown,blue,olive,red]:
  styles := [spacedot,solid, dash, dot, dashdot, longdash,
  spacedash]:
```

Draw the conveux hull in red

```
> for i from 1 to nops(ls) do
     ls[i];li[i];
     tt[i] := plot((mnmaxR[li[i]]-ls[i]*li[i])+ls[i]*m,m=
  li[i]..li[i+1],color=colors[i mod nops(colors)+1],
  linestyle=styles[i mod nops(styles)+1],legend=[typeset
  ("Slope: ", ls[i])],legendstyle=[location=right]):
  end:
  Pconvred := seq(tt[i],i=1..nops(ls)):
  #display(%);
> display(Pconvred,P4a,myview,myoptionsLo,LegendSize);
```



We continue with the blue diamonds, i.e., the coefficients of Ai'

```
> mnmaxB := getMaxNewt(Mord,newt4b):
  seq([i,mnmaxB[i]],i=0..Mord);
```

$$\left[0, -\frac{4}{3}\right], [1, -2], \left[2, -\frac{7}{3}\right], \left[3, -\frac{7}{3}\right], \left[4, -\frac{10}{3}\right], \left[5, -\frac{13}{3}\right] \tag{2.4.2.21}$$

```
> for i from 0 to Mord do
     i,factor(getel(posFl,i,mnmaxB[i]));
  end;
```

$$0, \frac{2^{1/3}\left(8 q_1 - 3\right)}{n^{4/3}}$$

$$1, -\frac{32}{9} \frac{a1\ m}{n^2}$$

$$2, \frac{2}{9} \frac{2^{1/3} m^2 \left(48 q_1 - 65\right)}{n^{7/3}}$$

$$3, -\frac{64}{9} \frac{2^{1/3} m^3}{n^{7/3}}$$

$$4, -\frac{40}{9}\frac{2^{1/3}m^4}{n^{10/3}}$$

$$5, -\frac{712}{135}\frac{2^{1/3}m^5}{n^{13/3}} \quad \textbf{(2.4.2.22)}$$

Again we derive the slopes of the convex hull and its corners;
note that if we choose q[1]=3/8 we eliminate the first term and decrease the slope.

TODO:
This will be useful in the proof of Lemma 5.3, in the same way as it was used in the proof of Lemma 4.4.

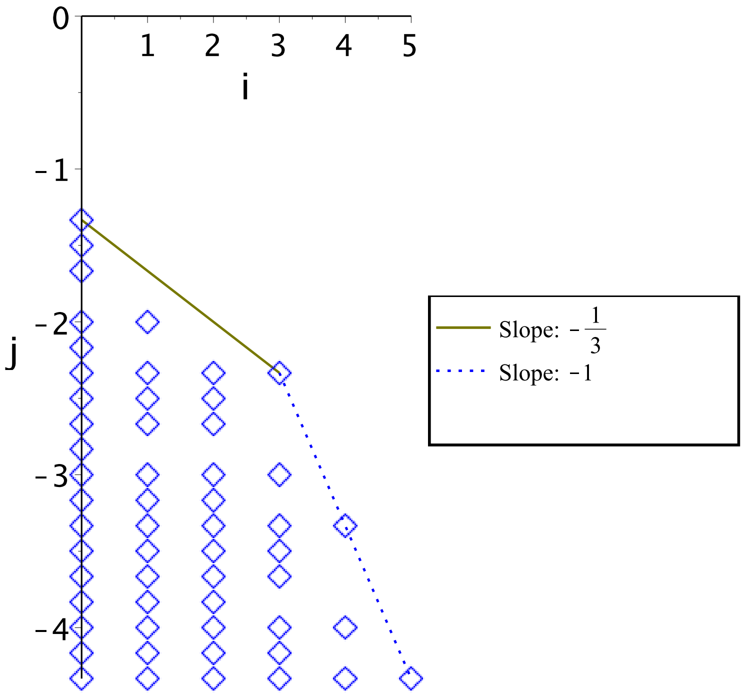
```
> ls,li := getslopes(mnmaxB,Mord);
```

$$ls, li := \left[-\frac{1}{3}, -1\right], [0, 3, 5] \quad \textbf{(2.4.2.23)}$$

```
> colors := [green,olive,blue,black,brown,blue,red]:
  styles := [spacedot,solid, dot, dash, dashdot, longdash,
  spacedash]:
```

Draw the conveux hull in blue which still includes the term of order Theta(n^(-4/3))

```
> for i from 1 to nops(ls) do
     ls[i];li[i];
     tt[i] := plot((mnmaxB[li[i]]-ls[i]*li[i])+ls[i]*m,m=
  li[i]..li[i+1],color=colors[i mod nops(colors)+1],
  linestyle=styles[i mod nops(styles)+1],legend=[typeset
  ("Slope: ", ls[i])],legendstyle=[location=right]):
  end:
  Pconvblue := seq(tt[i],i=1..nops(ls)):
  #display(%);
> display(Pconvblue,P4b,myview,myoptionsLo,LegendSize);
```

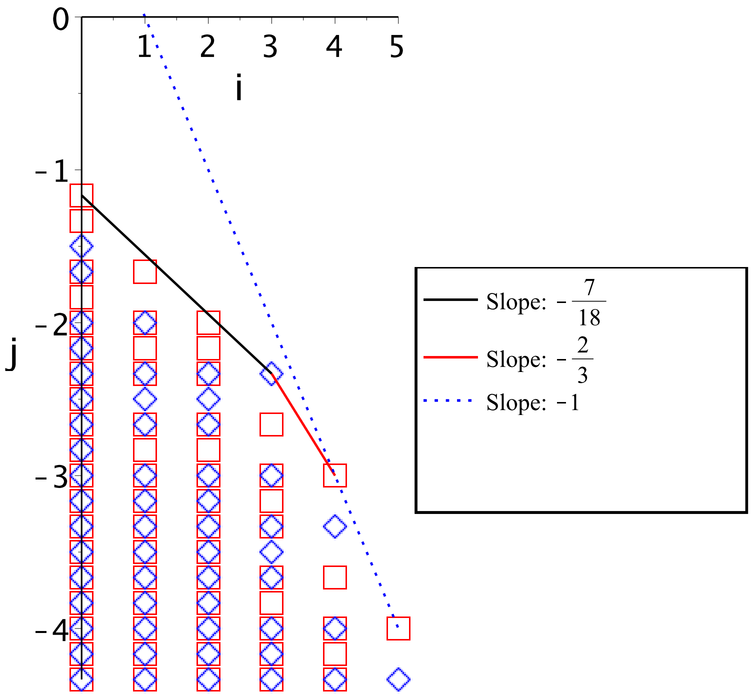


We kill this term by setting q[1] = 3/8 and recompute the Newton polygons.
In the next picture the left-top blue point disappeared.
(note that the coefficients of lower order terms of posFk change as well, which is why we recompute them; however the picture will not change as the convex hull is not influenced by this change.)

```
> posFk2 := coeff(subs(q[1]=3/8,posFfabcde),kappa):indets
  (%);
  newt4a2 := mynewt(posFk2,m,n):
  P4a2 := pointplot(newt4a2,labels=["m deg", "n deg"],
  symbolsize=25, symbol=box, color=red):
  posFl2 := coeff(subs(q[1]=3/8,posFfabcde),lambda):indets
  (%);
  newt4b2 := mynewt(posFl2,m,n):
  P4b2 := pointplot(newt4b2,labels=["m deg", "n deg"],
  symbolsize=25, symbol=diamond, color=blue):
  display(P4a2,P4b2,P1dom1,P1dom2,P1dom3a,myview,
  myoptionsLo,LegendSize);
```

$$\left\{a1, m, n, q_0, \frac{1}{n^{29/6}}, \frac{1}{n^{25/6}}, \frac{1}{n^{23/6}}, \frac{1}{n^{19/6}}, \frac{1}{n^{17/6}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{13/6}}, \frac{1}{n^{11/3}}, \frac{1}{n^{11/6}}, \frac{1}{n^{10/3}}, \frac{1}{n^{9/2}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/2}}, \frac{1}{n^{7/3}}, \frac{1}{n^{7/6}}, \frac{1}{n^{5/2}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}\right\}$$

$$\left\{a1, m, n, q_0, \frac{1}{n^{29/6}}, \frac{1}{n^{25/6}}, \frac{1}{n^{23/6}}, \frac{1}{n^{19/6}}, \frac{1}{n^{17/6}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{13/6}}, \frac{1}{n^{11/3}}, \frac{1}{n^{10/3}}, \frac{1}{n^{9/2}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/2}}, \frac{1}{n^{7/3}}, \frac{1}{n^{5/2}}, \frac{1}{n^{5/3}}, \frac{1}{n^{3/2}}\right\}$$



Recompute blue with q[1]=1/4

```
> mnmaxB  := getMaxNewt(Mord,newt4b2):
  seq([i,mnmaxB[i]],i=0..Mord);
```

$$\left[0, -\frac{3}{2}\right], [1, -2], \left[2, -\frac{7}{3}\right], \left[3, -\frac{7}{3}\right], \left[4, -\frac{10}{3}\right], \left[5, -\frac{13}{3}\right] \quad \textbf{(2.4.2.24)}$$

```
> for i from 0 to Mord do
     i,factor(getel(posFl2,i,mnmaxB[i]));
  end;
```

$$0, \frac{4\, 2^{1/3}}{n^{3/2}}$$

$$1, -\frac{32}{9} \frac{a1\, m}{n^2}$$

$$2, -\frac{94}{9} \frac{2^{1/3}\, m^2}{n^{7/3}}$$

$$3, -\frac{64}{9} \frac{2^{1/3} m^3}{n^{7/3}}$$

$$4, -\frac{40}{9} \frac{2^{1/3} m^4}{n^{10/3}}$$

$$5, -\frac{712}{135} \frac{2^{1/3} m^5}{n^{13/3}} \quad \textbf{(2.4.2.25)}$$

And we get new slopes, yet at the same m powers given in the second sequence;
here we need that the term m^3 is negative as it will dominate in the regime when Ai' is negative,

TODO:
therefore we have to choose p[4]>2/9 (note that this also makes m^5 and m^7 multiplied by Ai' positive)

```
> ls,li := getslopes(mnmaxB,Mord);
```

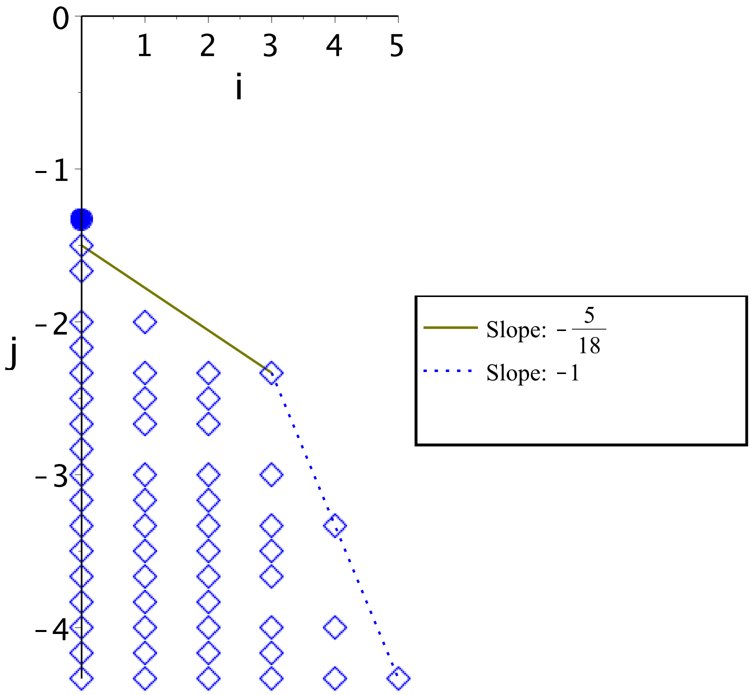
$$ls, li := \left[-\frac{5}{18}, -1\right], [0, 3, 5] \quad \textbf{(2.4.2.26)}$$

Draw the new conveux hull in blue

```
> for i from 1 to nops(ls) do
     ls[i];li[i];
     tt[i] := plot((mnmaxB[li[i]]-ls[i]*li[i])+ls[i]*m,m=
  li[i]..li[i+1],color=colors[i mod nops(colors)+1],
  linestyle=styles[i mod nops(styles)+1],legend=[typeset
  ("Slope: ", ls[i])],legendstyle=[location=right]):
  end:
  Pconvblue := seq(tt[i],i=1..nops(ls)):
  #display(%);
```

Plot the difference to before:
Only the solid circle on the top-left disappeared.

```
> P4bdiffshort := pointplot([0,-4/3],labels=["m deg", "n
  deg"], symbolsize=25, symbol=solidcircle, color=blue):
  display(Pconvblue,P4bdiffshort,P4b2,myview,myoptionsLo,
  LegendSize);
```



Minimal DFAs recoginizing a finite binary language - Upper bound

Xhat

This is Y hat

```
> Xansatz := (n,m) -> (1-2*m^2/(3*n)+3*m/(8*n)+1/3*
  m^4/n^2)* AiryAi(a1+2^(1/3)*(m+1)/n^(1/3));
```

$$Xansatz := (n, m) \rightarrow \left(1 - \frac{2}{3}\frac{m^2}{n} + \frac{3}{8}\frac{m}{n} + \frac{1}{3}\frac{m^4}{n^2}\right) \mathrm{AiryAi}\left(a1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right) \quad (2.5.1.1)$$

```
> Sansatz := n -> 2 + c*n^(-2/3) + pterm/n + 1/(n^(7/6));
```

$$Sansatz := n \rightarrow 2 + \frac{c}{n^{2/3}} + \frac{pterm}{n} + \frac{1}{n^{7/6}} \quad (2.5.1.2)$$

lower bound
(only difference to upper bound is missing factor (n-m-4)/(n-m-2) multiplied with the last two terms;
as it is a lower bound, and we still want to prove positivity, we multiply the full equation with -1)

```
> posansatz := -(-XX(n,m)*SS(n)*SS(n-1)*SS(n-2)
                    + (n-m+2)/(n+m)*XX(n-1,m-1)*SS(n-1)*SS
  (n-2)
                    + (n-m-1)/(n-m)*XX(n-1,m+1)*SS(n-1)*SS
  (n-2)
                    + 1/(n-m)*XX(n-2,m+2)*SS(n-2)
                    + 1/(n+m)*XX(n-3,m+1)
                    );
```

$$posansatz := XX(n, m)\ SS(n)\ SS(n-1)\ SS(n-2) - \frac{(n-m+2)\ XX(n-1, m-1)\ SS(n-1)\ SS(n-2)}{n+m} - \frac{(n-m-1)\ XX(n-1, m+1)\ SS(n-1)\ SS(n-2)}{n-m} - \frac{XX(n-2, m+2)\ SS(n-2)}{n-m} - \frac{XX(n-3, m+1)}{n+m} \quad \textbf{(2.5.1.3)}$$

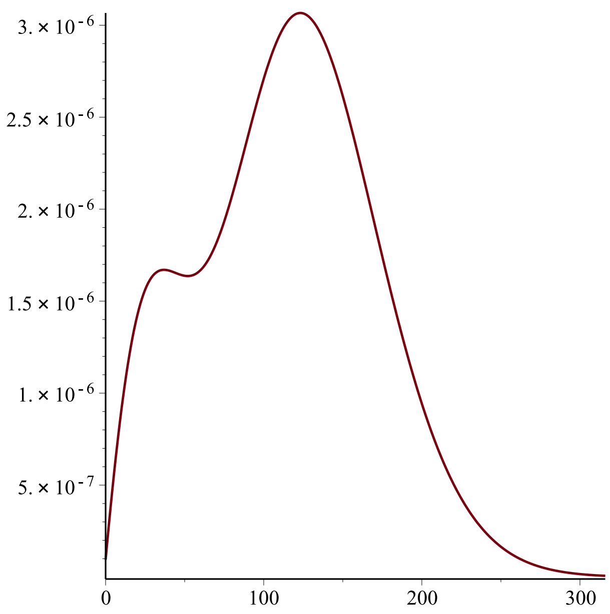
```
> posXS := map(simplify,subs(XX=Xansatz,SS=Sansatz,
  posansatz)):
```

For a large n this function of m seems to be positive

```
> Digits:=20:
  e1 := subs(csubs,pterm=29/12,a1=A1,posXS):
  N := 100000;
  M := floor(N^(1/2));
  P1 := plot([seq([mm,(subs(n=N,m=mm,e1))],mm=0..M)])
  :display(P1);
```

$$N := 100000$$

$$M := 316$$



Prove it

We start with the ansatz of Yhat in Lemma 5.3.

Recall the general ansatz

```
> facAiryUp*Airy(a1+2^(1/3)*(m+1)/n^(1/3));;
  SF(n);
```

$$\left(1+\frac{m^4 p_4+m^3 p_3+m^2 p_2+m p_1+p_0}{n^2}+\frac{m^2 q_2+m q_1+q_0}{n}\right) Airy\left(a1+\frac{2^{1/3}(m+1)}{n^{1/3}}\right)$$

$$a+\frac{b}{n^{1/3}}+\frac{c}{n^{2/3}}+\frac{pterm}{n}+\frac{d}{n^{7/6}} \quad \textbf{(2.5.2.1)}$$

Substitute ansatz into sequence we want to be positive for large n and all m

```
> posF := map(expand, subs(XX=XFU,SS=SF,posansatz)):indets
  (%);
```

$$\left\{a, a1, b, c, d, \kappa, \lambda, m, n, pterm, p_0, p_1, p_2, p_3, p_4, q_0, q_1, q_2, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{7/6}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, \frac{1}{(n-3)^{1/3}}, \frac{1}{(n-2)^{7/6}}, \frac{1}{(n-2)^{2/3}}, \right. \quad \textbf{(2.5.2.2)}$$

$$\frac{1}{(n-2)^{1/3}}, \frac{1}{(n-1)^{7/6}}, \frac{1}{(n-1)^{2/3}}, \frac{1}{(n-1)^{1/3}} \}$$

The error terms are (to check, look at posFc)

```
> simplify(O((2^(1/3)*(m+1)/n^(1/3)-2^(1/3)*m/n^(1/3))
  ^ordAiUp));
  simplify(O((2^(1/3)*(m)/(n-1)^(1/3)-2^(1/3)*m/n^(1/3))
  ^ordAiUp));
  simplify(O((2^(1/3)*(m+2)/(n-1)^(1/3)-2^(1/3)*m/n^(1/3))
  ^ordAiUp));
```

$$O\left(\frac{64\,2^{1/3}}{n^{19/3}}\right)$$

$$O\left(-\frac{64\,2^{1/3}\,m^{19}\left((n-1)^{1/3}-n^{1/3}\right)^{19}}{(n-1)^{19/3}\,n^{19/3}}\right)$$

$$O\left(-\frac{64\,2^{1/3}\left(m\,(n-1)^{1/3}-n^{1/3}\,m-2\,n^{1/3}\right)^{19}}{(n-1)^{19/3}\,n^{19/3}}\right) \qquad \textbf{(2.5.2.3)}$$

remove error terms

```
> posFd := convert(posF,polynom):
```

Next we fit the display size of the plots in the n (Nord) and m (Mord) direction. (Note that everything up to ordAi is computed, but possibly not shown)

```
> Nord := -ordAiUp/3;
  Mord := floor(ordAiUp/3)+1;
  myview := view=[0..Mord,Nord..0]:
```

$$Nord := -\frac{19}{3}$$

$$Mord := 7 \qquad \textbf{(2.5.2.4)}$$

Expand again with respect to n,
these are then our unknowns

```
> posFe := series(posFd,n=infinity,ceil(-Nord)+1):indets
  (%);
  posFf := convert(%%,polynom):
```

$$\left\{a, a1, b, c, d, \kappa, \lambda, m, n, pterm, p_0, p_1, p_2, p_3, p_4, q_0, q_1, q_2, \left(\frac{1}{n}\right)^{1/3}, \left(\frac{1}{n}\right)^{2/3}, \right. \qquad \textbf{(2.5.2.5)}$$

$$\left(\frac{1}{n}\right)^{3/2}, \left(\frac{1}{n}\right)^{4/3}, \left(\frac{1}{n}\right)^{5/2}, \left(\frac{1}{n}\right)^{5/3}, \left(\frac{1}{n}\right)^{7/2}, \left(\frac{1}{n}\right)^{7/3}, \left(\frac{1}{n}\right)^{7/6},$$

$$\left(\frac{1}{n}\right)^{8/3}, \left(\frac{1}{n}\right)^{9/2}, \left(\frac{1}{n}\right)^{10/3}, \left(\frac{1}{n}\right)^{11/2}, \left(\frac{1}{n}\right)^{11/3}, \left(\frac{1}{n}\right)^{11/6}, \left(\frac{1}{n}\right)^{13/2},$$

$$\left(\frac{1}{n}\right)^{13/3}, \left(\frac{1}{n}\right)^{13/6}, \left(\frac{1}{n}\right)^{14/3}, \left(\frac{1}{n}\right)^{16/3}, \left(\frac{1}{n}\right)^{17/3}, \left(\frac{1}{n}\right)^{17/6}, \left(\frac{1}{n}\right)^{19/3},$$

$$\left(\frac{1}{n}\right)^{19/6}, \left(\frac{1}{n}\right)^{20/3}, \left(\frac{1}{n}\right)^{23/6}, \left(\frac{1}{n}\right)^{25/6}, \left(\frac{1}{n}\right)^{29/6}, \left(\frac{1}{n}\right)^{31/6}, \left(\frac{1}{n}\right)^{35/6},$$

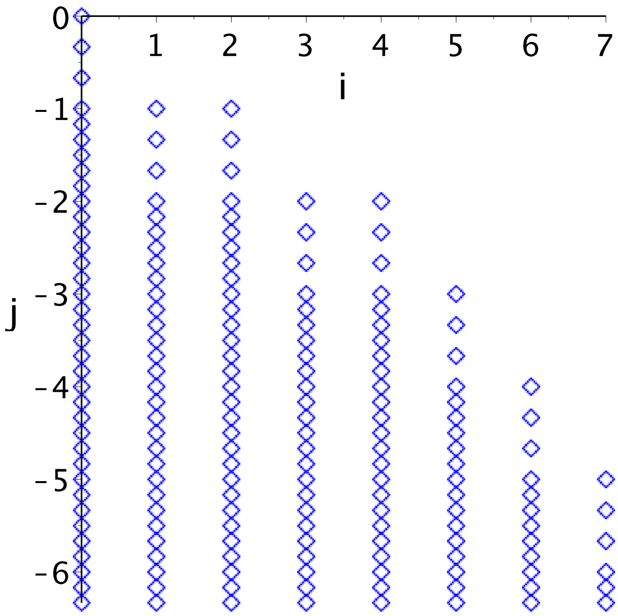
$$\left.\left(\frac{1}{n}\right)^{37/6}, \left(\frac{1}{n}\right)^{41/6}, O\left(\frac{1}{n^7}\right)\right\}$$

The mynewt function computes the Newton polygon of posFf

```
> newt1 := mynewt(posFf,m,n):
```

First Newton polygon, where no unknowns have been fixed

```
> P1 := pointplot(newt1,myoptionsUp,color=blue):
  display(P1,myview);
```



Here, we want to kill the element (0,0)

```
> getel(posFf,0,0);
```

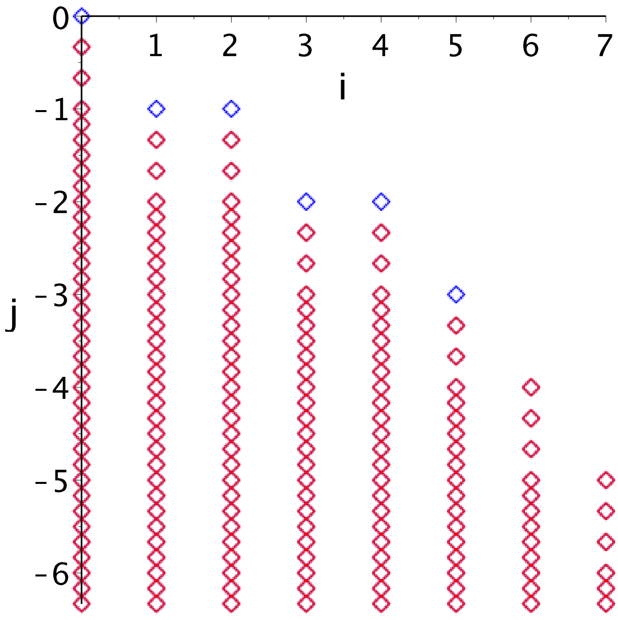
$$a^3 \kappa - 2\, a^2 \kappa \qquad \textbf{(2.5.2.6)}$$

Set a=2

```
> posFfa := expand(simplify(subs(a=2,posFf))) assuming
  n::posint,m::posint:
> newta := mynewt(posFfa,m,n):
```

All blue points have been eliminated, and only the red ones remain

```
> P1a := pointplot(newta,myoptionsUp,color=red):
  display(P1,P1a,myview);
```



b=0 is forced due to the term n^{-1/3)

```
> getel(posFfa,0,-1/3);
  #getel(posFfa,2,-4/3);
```

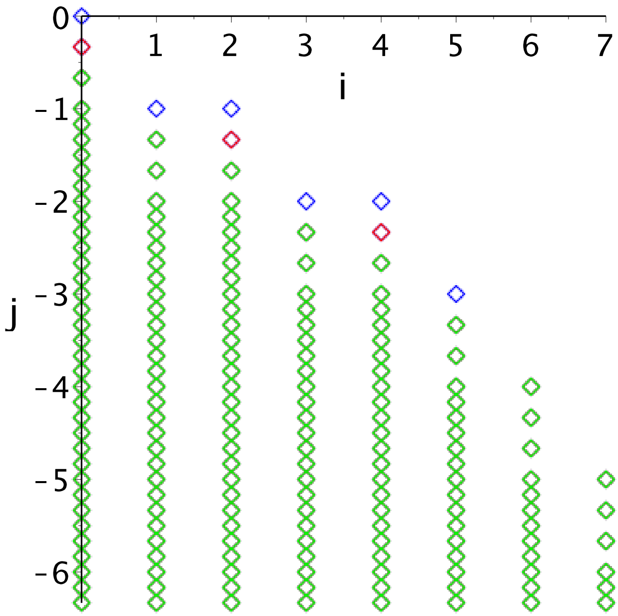
$$\frac{4\,\kappa\,b}{n^{1/3}} \tag{2.5.2.7}$$

set a=2, b=0

```
> posFfab := expand(simplify(subs(b=0,posFfa))) assuming
  n::posint,m::posint:
> newtab := mynewt(posFfab,m,n):
```

Now only the green points remain.

```
> P1ab := pointplot(newtab,myoptionsUp,color=green):
  display(P1,P1a,P1ab,myview);
```



at this point we find our choice for c, which we heuristically computed already before in Section 3

```
> csubs;
  factor(getel(posFfab,0,-2/3));factor(subs(csubs,%));
  factor(getel(posFfab,1,-4/3));factor(subs(csubs,%));
  factor(getel(posFfab,2,-5/3));factor(subs(csubs,%));
```

$$c = a1\ 2^{2/3}$$

$$-\frac{4\,\kappa\,(a1\ 2^{2/3} - c)}{n^{2/3}}$$

$$0$$

$$-\frac{16}{3}\,\frac{2^{1/3}\,\lambda\,m\,(3\,q_2 + 2)}{n^{4/3}}$$

$$-\frac{16}{3}\,\frac{2^{1/3}\,\lambda\,m\,(3\,q_2 + 2)}{n^{4/3}}$$

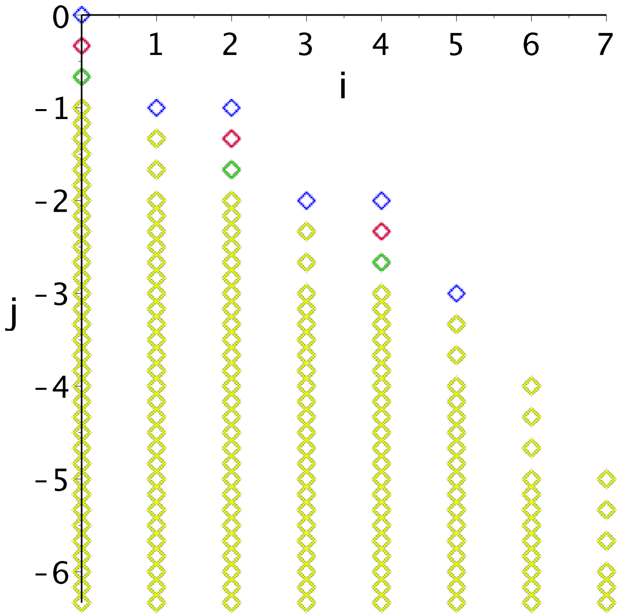
$$-\frac{4\,\kappa\,m^2\,q_2\,(a1\ 2^{2/3} - c)}{n^{5/3}}$$

$$0 \qquad \textbf{(2.5.2.8)}$$

set a=2, b=0, c=a1*2^(2/3)

```
> posFfabc := expand(simplify(subs(csubs,posFfab)))
  assuming n::posint,m::posint:
```

```
> newtabc := mynewt(posFfabc,m,n):
> P1abc := pointplot(newtabc,myoptionsUp,color=yellow):
  display(P1,P1a,P1ab,P1abc,myview);
```



Here we get q[2] and pterm

```
> factor(getel(posFfabc,0,-1));
  factor(getel(posFfabc,1,-4/3));
  solve({%%,%},{q[2],pterm});
```

$$\frac{\kappa\left(4\,pterm-8\,q_2-15\right)}{n}$$

$$-\frac{16}{3}\,\frac{2^{1/3}\,\lambda\,m\left(3\,q_2+2\right)}{n^{4/3}}$$

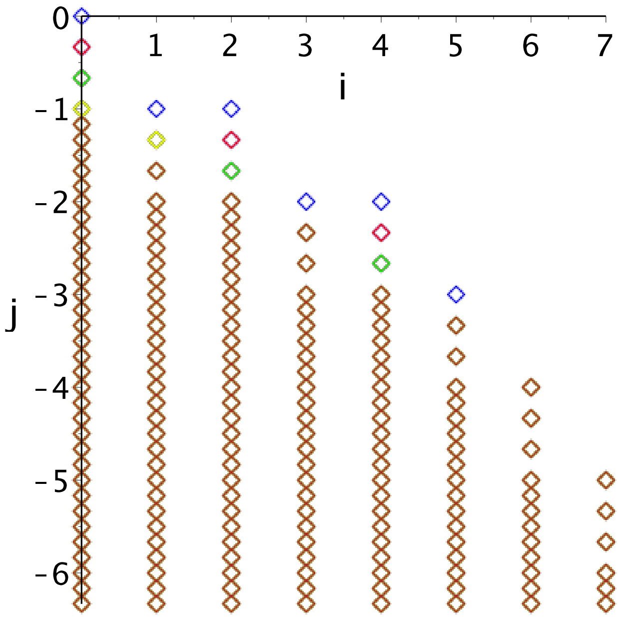
$$\left\{pterm=\frac{29}{12},\,q_2=-\frac{2}{3}\right\} \quad \textbf{(2.5.2.9)}$$

set a=2, b=0,c,pterm=29/12,q[2]=-2/3

```
> posFfabcd := expand(simplify(subs(pterm=29/12,q[2]=-2/3,
  posFfabc))) assuming n::posint,m::posint:
> newtabcd := mynewt(posFfabcd,m,n):
```

only the brown points remain
Now all points are strictly below n^{-1}

```
> P1abcd := pointplot(newtabcd,myoptionsUp,color=brown):
  display(P1,P1a,P1ab,P1abc,P1abcd,myview);
```



Here are the dominating corners and we see that we have to choose d=1 to have a positive term;
note that we will see that the second term should be negative, as lambda=Ai' is negative for large m,
hence here we will choose a p[4]>2/9 (see below in the decomposition into lambda and kappa contributions)

```
> getel(posFfabcd,0,-7/6);
  factor(getel(posFfabcd,3,-14/6));
  (14/6-7/6)/3;   #slope
```

$$\frac{4\,\kappa\,d}{n^{7/6}}$$

$$-\frac{32}{9}\,\frac{2^{1/3}\,\lambda\,m^3\,\left(-2+9\,p_4\right)}{n^{7/3}}$$

$$\frac{7}{18} \qquad \textbf{(2.5.2.10)}$$

and continuing

```
> getel(posFfabcd,5,-20/6);
  (20/6-14/6)/2;    #slope
```

$$-\frac{32}{3}\,\frac{2^{1/3}\,\lambda\,m^5\,p_4}{n^{10/3}}$$

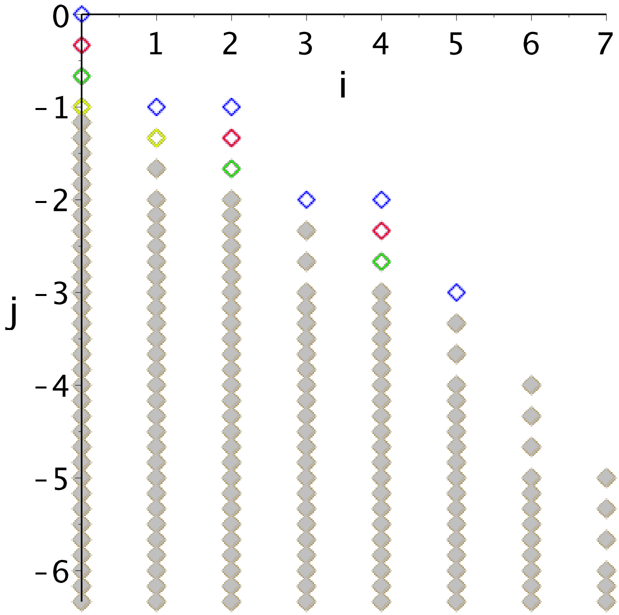
$$\frac{1}{2} \qquad \textbf{(2.5.2.11)}$$

set a=2, b=0,c,pterm=29/12,q[2]=-2/3 and d=1

```
> posFfabcde := expand(simplify(subs(d=1,posFfabcd)))
  assuming n::posint,m::posint:
> newtabcde := mynewt(posFfabcde,m,n):
```

This is the final result, where only the solid diamonds are non-zero

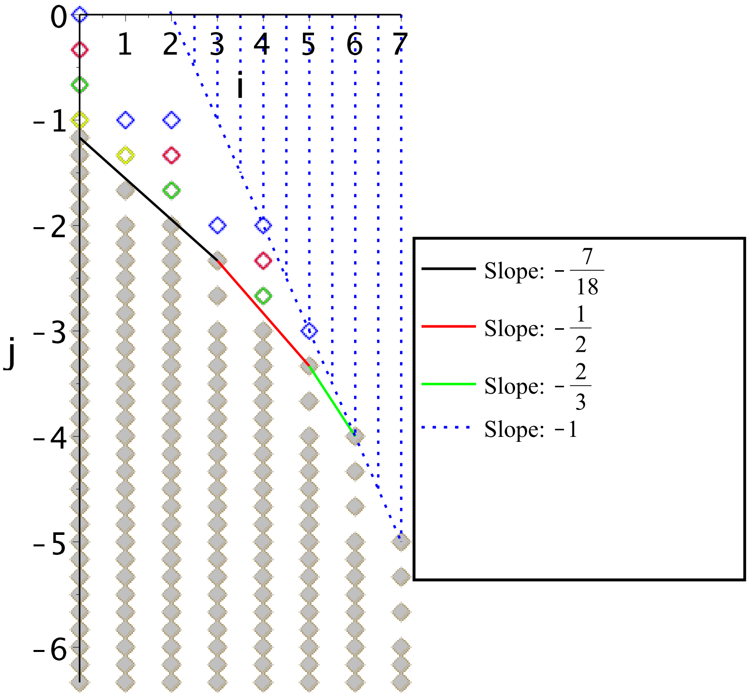
```
> P1abcde := pointplot(newtabcde,myoptionsUp,symbol=
  soliddiamond,color=gray):
  display(P1,P1a,P1ab,P1abc,P1abcd,P1abcde,myview);
```



Plot the boundary and the slopes of the Newton polygon;
Note that we have already proved that there are now points above the blue dotted line

```
> LegendSize := size=[600,450]:
  P1dom1 := plot(-7/6-(7/18)*m,m=0..3,color=black,legend=
  [typeset("Slope: ", -7/18)],legendstyle=[location=right]
  ):
  P1dom2 := plot(-5/6-(1/2)*m,m=3..5,color=red,legend=
  [typeset("Slope: ", -1/2)],legendstyle=[location=right])
  :
  P1dom2b := plot(0-(2/3)*m,m=5..6,color=green,legend=
  [typeset("Slope: ", -2/3)],legendstyle=[location=right])
  :
  P1dom3a := plot(2-m,m=0..7,color=blue,linestyle=dot,
  legend=[typeset("Slope: ", -1)],legendstyle=[location=
  right]):
  P1all := display(P1,P1a,P1ab,P1abc,P1abcd,P1abcde,
  P1dom1,P1dom2,P1dom2b,P1dom3a,myview,LegendSize):
```

```
for i from 1 to 10 do
   P1dom3[i] := plot([[2+i/2,0],[2+i/2,-i/2]],color=
blue,linestyle=dot):
end:
display(P1all,seq(P1dom3[i],i=1..10));
```



This is the choice for SF

```
> SF(n);
  subs(a=2,b=0,csubs,pterm=29/12,d=1,%);
```

$$a+\frac{b}{n^{1/3}}+\frac{c}{n^{2/3}}+\frac{pterm}{n}+\frac{d}{n^{7/6}}$$

$$2+\frac{a1\,2^{2/3}}{n^{2/3}}+\frac{29}{12\,n}+\frac{1}{n^{7/6}} \quad \textbf{(2.5.2.12)}$$

recall

```
> kaplam;
```

$$\mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1)+\frac{2^{1/3}\,m}{n^{1/3}}\right)=\kappa,\ \mathrm{AiryAi}\left(1,\ \mathrm{AiryAiZeros}(1)+\frac{2^{1/3}\,m}{n^{1/3}}\right)=\lambda \quad \textbf{(2.5.2.13)}$$

Look at the corners

Here we see the necessary choice: p[4]>2/9 (note that for m>n^(1/3) lambda is negative)

```
> getel(posFfabcde,0,-7/6);
  simplify(getel(posFfabcde,3,-14/6));
  getel(posFfabcde,5,-20/6);
  getel(posFfabcde,6,-4);
  getel(posFfabcde,7,-5);
```

$$\frac{4\,\kappa}{n^{7/6}}$$

$$-\frac{32}{9}\,\frac{2^{1/3}\,\lambda\,m^3\,(-2+9\,p_4)}{n^{7/3}}$$

$$-\frac{32}{3}\,\frac{2^{1/3}\,\lambda\,m^5\,p_4}{n^{10/3}}$$

$$-\frac{68}{3}\,\frac{\kappa\,m^6\,p_4}{n^4}$$

$$-\frac{124}{45}\,\frac{\kappa\,m^7\,p_4}{n^5} \qquad \textbf{(2.5.2.14)}$$

Now split the black dots into the contributions from Ai and Ai'

```
> indets(posFfabcd);
```

$$\left\{a1, d, \kappa, \lambda, m, n, p_0, p_1, p_2, p_3, p_4, q_0, q_1, \frac{1}{n^{41/6}}, \frac{1}{n^{37/6}}, \frac{1}{n^{35/6}}, \frac{1}{n^{31/6}}, \frac{1}{n^{29/6}}, \frac{1}{n^{25/6}}, \frac{1}{n^{23/6}}, \frac{1}{n^{20/3}}, \frac{1}{n^{19/3}}, \frac{1}{n^{19/6}}, \frac{1}{n^{17/3}}, \frac{1}{n^{17/6}}, \frac{1}{n^{16/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/2}}, \frac{1}{n^{13/3}}, \frac{1}{n^{13/6}}, \frac{1}{n^{11/2}}, \frac{1}{n^{11/3}}, \frac{1}{n^{11/6}}, \frac{1}{n^{10/3}}, \frac{1}{n^{9/2}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/2}}, \frac{1}{n^{7/3}}, \frac{1}{n^{7/6}}, \frac{1}{n^{5/2}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{3/2}}\right\} \qquad \textbf{(2.5.2.15)}$$

Sanity check that there are no other contributions

```
> subs(kappa=0,lambda=0,posFfabcd);
```

$$0 \qquad \textbf{(2.5.2.16)}$$

Extract the coefficients of kappa=Ai and lambda=Ai'

and treat then seperately

```
> posFk := coeff(posFfabcd,kappa):indets(%);
  posFl := coeff(posFfabcd,lambda):indets(%);
```

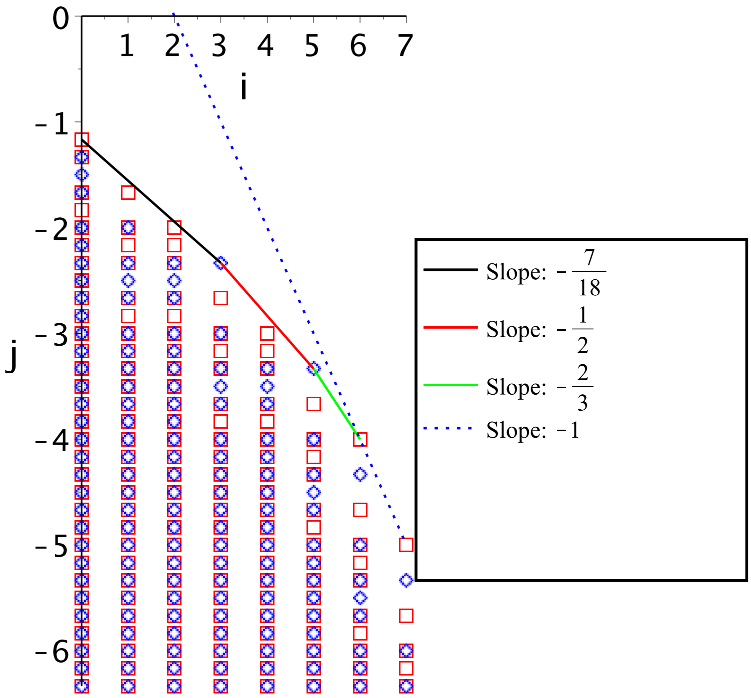
$$\left\{a1, d, m, n, p_0, p_1, p_2, p_3, p_4, q_0, q_1, \frac{1}{n^{41/6}}, \frac{1}{n^{37/6}}, \frac{1}{n^{35/6}}, \frac{1}{n^{31/6}}, \frac{1}{n^{29/6}}, \frac{1}{n^{25/6}}, \frac{1}{n^{23/6}}, \frac{1}{n^{20/3}}, \frac{1}{n^{19/3}}, \frac{1}{n^{19/6}}, \frac{1}{n^{17/3}}, \frac{1}{n^{17/6}}, \frac{1}{n^{16/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/2}}, \frac{1}{n^{13/3}}, \frac{1}{n^{13/6}}, \frac{1}{n^{11/2}}, \frac{1}{n^{11/3}}, \frac{1}{n^{11/6}}, \frac{1}{n^{10/3}}, \frac{1}{n^{9/2}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/2}}, \frac{1}{n^{7/3}}, \frac{1}{n^{7/6}}, \frac{1}{n^{5/2}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}\right\}$$

(2.5.2.17)

$$\left\{a1, d, m, n, p_0, p_1, p_2, p_3, p_4, q_0, q_1, \frac{1}{n^{41/6}}, \frac{1}{n^{37/6}}, \frac{1}{n^{35/6}}, \frac{1}{n^{31/6}}, \frac{1}{n^{29/6}}, \frac{1}{n^{25/6}}, \frac{1}{n^{23/6}}, \frac{1}{n^{20/3}}, \frac{1}{n^{19/3}}, \frac{1}{n^{19/6}}, \frac{1}{n^{17/3}}, \frac{1}{n^{17/6}}, \frac{1}{n^{16/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/2}}, \frac{1}{n^{13/3}}, \frac{1}{n^{13/6}}, \frac{1}{n^{11/2}}, \frac{1}{n^{11/3}}, \frac{1}{n^{10/3}}, \frac{1}{n^{9/2}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/2}}, \frac{1}{n^{7/3}}, \frac{1}{n^{5/2}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{3/2}}\right\} \quad \textbf{(2.5.2.17)}$$

We color the non-zero nodes of the lastNewton polygon into
red squared .... coefficients of kappa=Ai
blue diamonds ...... coefficients of lambda=Ai'

```
> newt4a := mynewt(posFk,m,n):
  newt4b := mynewt(posFl,m,n):
> P4a := pointplot(newt4a,labels=["m deg", "n deg"],
  symbolsize=15, symbol=box,color=red):
  P4b := pointplot(newt4b,labels=["m deg", "n deg"],
  symbolsize=15, symbol=diamond, color=blue):
  P1dom3s := plot(1-m,m=4..5,color=black):
  display(P4a,P4b,P1dom1,P1dom2,P1dom2b,P1dom3a,myview,
  myoptionsUp,LegendSize);
```



red extremes of Newton polygon

```
> mnmaxR := getMaxNewt(Mord,newt4a):
  seq([i,mnmaxR[i]],i=0..Mord);
```

$$\left[0, -\frac{7}{6}\right], \left[1, -\frac{5}{3}\right], [2, -2], \left[3, -\frac{8}{3}\right], [4, -3], \left[5, -\frac{11}{3}\right], [6, -4], [7, -5] \quad \textbf{(2.5.2.18)}$$

These are the specific values at these points;
we see that we still have some degree in freedom: d and p[4]

```
> for i from 0 to Mord do
     i,factor(getel(posFk,i,mnmaxR[i]));
  end;
```

$$0, \frac{4\,d}{n^{7/6}}$$

$$1, \frac{8}{3}\,\frac{a1\,2^{2/3}\,m}{n^{5/3}}$$

$$2, -\frac{4}{9}\,\frac{m^2\,(108\,p_4 - 41)}{n^2}$$

$$3, -\frac{16}{3}\frac{2^{2/3}\,a1\,m^3\,(6\,p_4-1)}{n^{8/3}}$$

$$4, -\frac{8}{9}\frac{m^4\,(-17+132\,p_4)}{n^3}$$

$$5, -\frac{8\,2^{2/3}\,a1\,m^5\,p_4}{n^{11/3}}$$

$$6, -\frac{68}{3}\frac{m^6\,p_4}{n^4}$$

$$7, -\frac{124}{45}\frac{m^7\,p_4}{n^5} \qquad \textbf{(2.5.2.19)}$$

These are the slopes of the convex hull where the corners are given by the second sequence;
hence, in order to be positive when the slope > -1, we need to choose d>0, e.g. d=1;
note that p[4] is not important here, as the slope first slope -5/12 is less then -7/18;
and in the later regimes it will be dominated by the blue points.

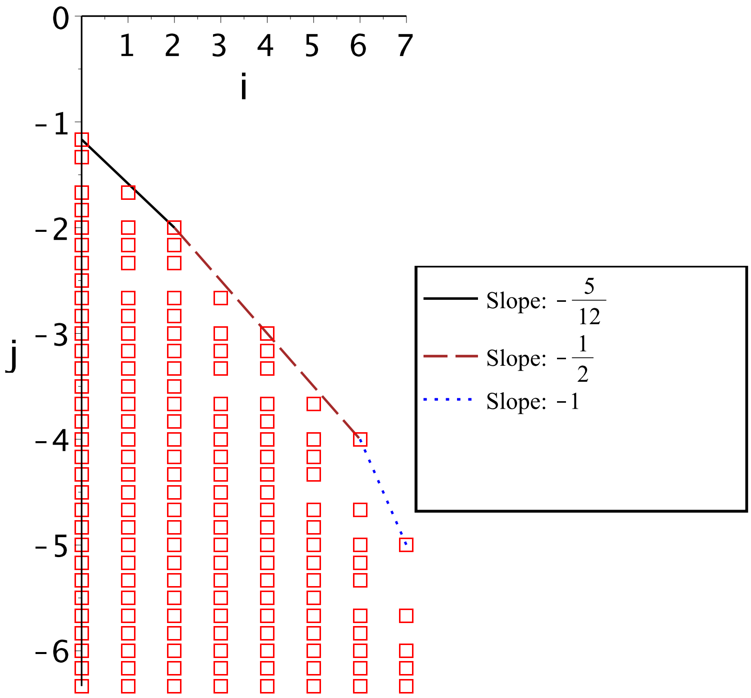
```
> ls,li := getslopes(mnmaxR,Mord);
```

$$ls, li := \left[-\frac{5}{12}, -\frac{1}{2}, -1\right], [0, 2, 6, 7] \qquad \textbf{(2.5.2.20)}$$

```
> colors := [green,black,brown,blue,olive,red]:
  styles := [spacedot,solid, dash, dot, dashdot, longdash,
  spacedash]:
```

Draw the conveux hull in red

```
> for i from 1 to nops(ls) do
     ls[i];li[i];
     tt[i] := plot((mnmaxR[li[i]]-ls[i]*li[i])+ls[i]*m,m=
  li[i]..li[i+1],color=colors[i mod nops(colors)+1],
  linestyle=styles[i mod nops(styles)+1],legend=[typeset
  ("Slope: ", ls[i])],legendstyle=[location=right]):
  end:
  Pconvred := seq(tt[i],i=1..nops(ls)):
  #display(%);
> display(Pconvred,P4a,myview,myoptionsUp,LegendSize);
```



We continue with the blue diamonds, i.e., the coefficients of Ai'

```
> mnmaxB := getMaxNewt(Mord,newt4b):
  seq([i,mnmaxB[i]],i=0..Mord);
```

$$\left[0, -\frac{4}{3}\right], [1, -2], \left[2, -\frac{7}{3}\right], \left[3, -\frac{7}{3}\right], \left[4, -\frac{10}{3}\right], \left[5, -\frac{10}{3}\right], \left[6, -\frac{13}{3}\right], \left[7, -\frac{16}{3}\right] \quad \textbf{(2.5.2.21)}$$

```
> for i from 0 to Mord do
     i,factor(getel(posFl,i,mnmaxB[i]));
  end;
```

$$0, -\frac{2^{1/3}\left(8\,q_1-3\right)}{n^{4/3}}$$

$$1, \frac{32}{9}\,\frac{a1\,m}{n^2}$$

$$2, -\frac{2}{9}\,\frac{2^{1/3}\,m^2\left(108\,p_3+216\,p_4+48\,q_1-65\right)}{n^{7/3}}$$

$$3,\ -\frac{32}{9}\ \frac{2^{1/3}\,m^3\,(9\,p_4-2)}{n^{7/3}}$$

$$4,\ -\frac{1}{9}\ \frac{2^{1/3}\,m^4\,(96\,p_3+549\,p_4-40)}{n^{10/3}}$$

$$5,\ -\frac{32}{3}\ \frac{2^{1/3}\,m^5\,p_4}{n^{10/3}}$$

$$6,\ -\frac{20}{3}\ \frac{2^{1/3}\,m^6\,p_4}{n^{13/3}}$$

$$7,\ -\frac{356}{45}\ \frac{2^{1/3}\,m^7\,p_4}{n^{16/3}} \qquad \textbf{(2.5.2.22)}$$

Again we derive the slopes of the convex hull and its corners;
note that if we choose q[1]=3/8 we eliminate the first term and decrease the slope.

TODO:
This will be useful in the proof of Lemma 5.3, in the same way as it was used in the proof of Lemma 4.4.

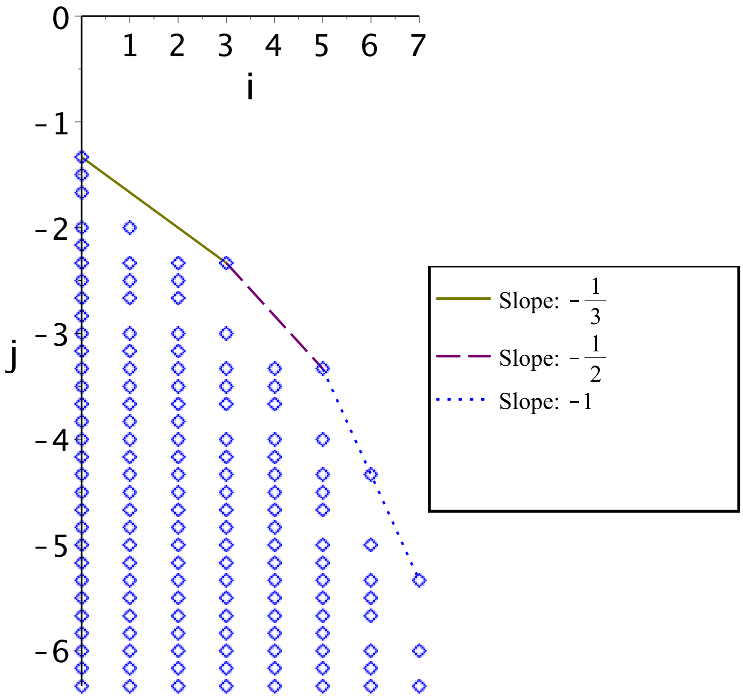
```
> ls,li := getslopes(mnmaxB,Mord);
```

$$ls,\ li := \left[-\frac{1}{3},\ -\frac{1}{2},\ -1\right],\ [0,\ 3,\ 5,\ 7] \qquad \textbf{(2.5.2.23)}$$

```
> colors := [brown,olive,purple,blue,olive,red,black]:
  styles := [spacedot,solid, dash, dot, dashdot, longdash,
  spacedash]:
```

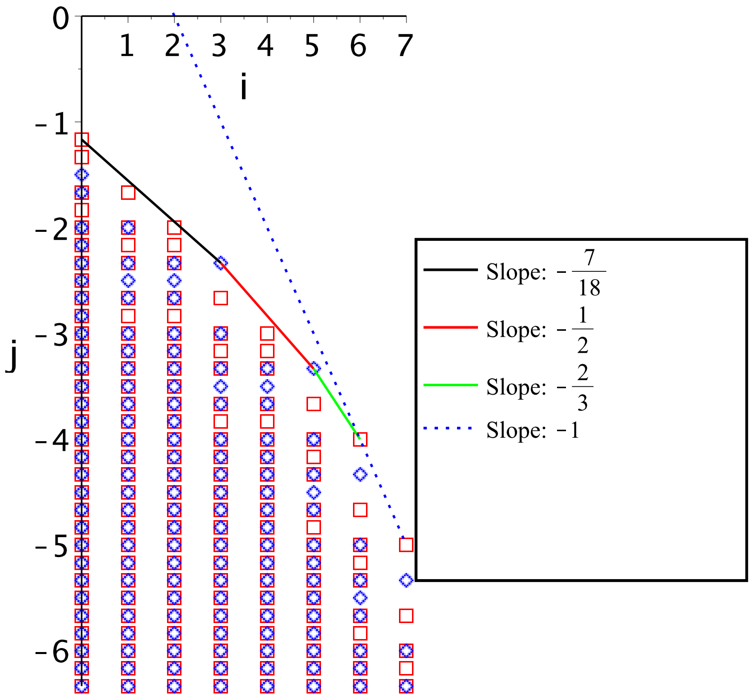
Draw the conveux hull in blue which still includes the term of order Theta(n^(-4/3))

```
> for i from 1 to nops(ls) do
     ls[i];li[i];
     tt[i] := plot((mnmaxB[li[i]]-ls[i]*li[i])+ls[i]*m,m=
  li[i]..li[i+1],color=colors[i mod nops(colors)+1],
  linestyle=styles[i mod nops(styles)+1],legend=[typeset
  ("Slope: ", ls[i])],legendstyle=[location=right]):
  end:
  Pconvblue := seq(tt[i],i=1..nops(ls)):
  #display(%);
> display(Pconvblue,P4b,myview,myoptionsUp,LegendSize);
```



We kill this term by setting q[1] = 3/8 and recompute the Newton polygons.
In the next picture the left-top blue point disappeared.
(note that the coefficients of lower order terms of posFk change as well, which is why we recompute them; however the picture will not change as the convex hull is not influenced by this change.)

```
> posFk2 := coeff(subs(q[1]=3/8,p[3]=0,p[2]=0,p[1]=0,p[0]=
  0,q[0]=0,posFfabcde),kappa):indets(%):
  newt4a2 := mynewt(posFk2,m,n):
  P4a2 := pointplot(newt4a2,labels=["m deg", "n deg"],
  symbolsize=15, symbol=box, color=red):
  posFl2 := coeff(subs(q[1]=3/8,p[3]=0,p[2]=0,p[1]=0,p[0]=
  0,q[0]=0,posFfabcde),lambda):indets(%):
  newt4b2 := mynewt(posFl2,m,n):
  P4b2 := pointplot(newt4b2,labels=["m deg", "n deg"],
  symbolsize=15, symbol=diamond, color=blue):
  display(P4a2,P4b2,P1dom1,P1dom2,P1dom2b,P1dom3a,myview,
  myoptionsUp,LegendSize);
```



Recompute blue with q[1]=1/4

```
> mnmaxB := getMaxNewt(Mord,newt4b2):
  seq([i,mnmaxB[i]],i=0..Mord);
```

$$\left[0, -\frac{3}{2}\right], [1, -2], \left[2, -\frac{7}{3}\right], \left[3, -\frac{7}{3}\right], \left[4, -\frac{10}{3}\right], \left[5, -\frac{10}{3}\right], \left[6, -\frac{13}{3}\right], \left[7, -\frac{16}{3}\right] \quad \textbf{(2.5.2.24)}$$

```
> for i from 0 to Mord do
      i,factor(getel(posFl2,i,mnmaxB[i]));
  end;
```

$$0, \frac{4\, 2^{1/3}}{n^{3/2}}$$

$$1, \frac{32}{9} \frac{a1\, m}{n^2}$$

$$2, -\frac{2}{9} \frac{2^{1/3}\, m^2 \left(-47 + 216\, p_4\right)}{n^{7/3}}$$

$$3, -\frac{32}{9} \frac{2^{1/3} m^3 (9 p_4 - 2)}{n^{7/3}}$$

$$4, -\frac{1}{9} \frac{2^{1/3} m^4 (-40 + 549 p_4)}{n^{10/3}}$$

$$5, -\frac{32}{3} \frac{2^{1/3} m^5 p_4}{n^{10/3}}$$

$$6, -\frac{20}{3} \frac{2^{1/3} m^6 p_4}{n^{13/3}}$$

$$7, -\frac{356}{45} \frac{2^{1/3} m^7 p_4}{n^{16/3}} \quad \textbf{(2.5.2.25)}$$

And we get new slopes, yet at the same m powers given in the second sequence;
here we need that the term m^3 is negative as it will dominate in the regime when Ai' is negative,
therefore we have to choose p[4]>2/9 (note that this also makes m^5 and m^7 multiplied by Ai' positive)

```
> ls,li := getslopes(mnmaxB,Mord);
```

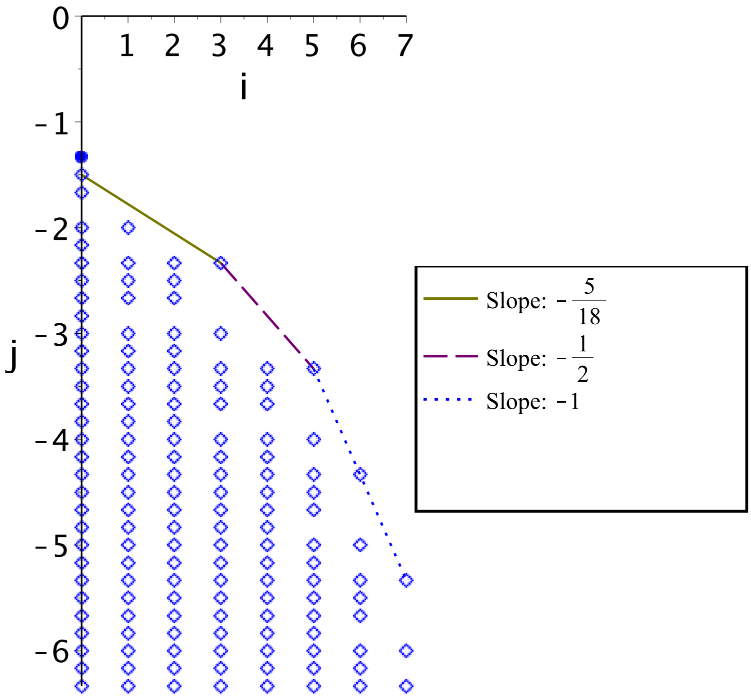
$$ls, li := \left[-\frac{5}{18}, -\frac{1}{2}, -1\right], [0, 3, 5, 7] \quad \textbf{(2.5.2.26)}$$

Draw the new conveux hull in blue

```
> for i from 1 to nops(ls) do
     ls[i];li[i];
     tt[i] := plot((mnmaxB[li[i]]-ls[i]*li[i])+ls[i]*m,m=
  li[i]..li[i+1],color=colors[i mod nops(colors)+1],
  linestyle=styles[i mod nops(styles)+1],legend=[typeset
  ("Slope: ", ls[i])],legendstyle=[location=right]):
  end:
  Pconvblue := seq(tt[i],i=1..nops(ls)):
  #display(%);
```

Plot the difference to before:
Only the solid circle on the top-left disappeared.

```
> P4bdiffshort := pointplot([0,-4/3],labels=["m deg", "n
  deg"], symbolsize=15, symbol=solidcircle, color=blue):
  display(Pconvblue,P4bdiffshort,P4b2,myview,myoptionsUp,
  LegendSize);
```



Lemma 8

We want to prove
for integers 0 <= j < k <= l <= 2*n,
k-j even,
and n large (here >= 10) that

```
> q(l,j,2*n)/(j+1) >= q(l,k,2*n)/(k+1);
```

$$\frac{q(l, k, 2\,n)}{k+1} \leq \frac{q(l, j, 2\,n)}{j+1} \quad \textbf{(3.1)}$$

It suffices to prove that the following is non-negative

```
> LLq := q(l,m-1,2*n)/m-q(l,m+1,2*n)/(m+2);
```

$$LLq := \frac{q(l, m-1, 2n)}{m} - \frac{q(l, m+1, 2n)}{m+2} \quad (3.2)$$

This is the recurrence holding for m non-negative

```
> qrec := q(l,m,2*n) = (l-m+1)/(l-m+2)*q(l+1,m-1,2*n)
                     + (l-m+2)/(l+m+2)*q(l+1,m+1,2*n)
                     + 1/(l-m+4)*q(l+2,m-2,2*n)
                     + 1/(l+m+2)*q(l+3,m-1,2*n);
```

$$qrec := q(l, m, 2n) = \frac{(l-m+1)\, q(l+1, m-1, 2n)}{l-m+2} + \frac{(l-m+2)\, q(l+1, m+1, 2n)}{l+m+2} + \frac{q(l+2, m-2, 2n)}{l-m+4} + \frac{q(l+3, m-1, 2n)}{l+m+2} \quad (3.3)$$

These relations are used in the induction step

```
> qup := subs(m=m+1,isolate(LLq,q(l,m-1,2*n)));
  qdown := subs(m=m-1,isolate(LLq,q(l,m+1,2*n)));
```

$$qup := q(l, m, 2n) = \frac{q(l, m+2, 2n)\,(m+1)}{m+3}$$

$$qdown := q(l, m, 2n) = \frac{q(l, m-2, 2n)\,(m+1)}{m-1} \quad (3.4)$$

Now, we perform an induction on l, with l=2*n being the base case.
We start by applying the recurrence to LLq.

```
> qe1 := collect(subs(subs(m=m-1,qrec),subs(m=m+1,qrec),LLq),
  q,factor);
```

$$qe1 := \frac{(-2\, l\, m^2 + 2\, m^3 + 2\, l^2 - m\, l - m^2 + 8\, l - 5\, m + 6)\, q(l+1, m, 2n)}{(l+m+1)\, m\, (l-m+1)\, (m+2)} + \frac{(l-m+2)\, q(l+1, m-2, 2n)}{(l-m+3)\, m} - \frac{(l-m+1)\, q(l+1, m+2, 2n)}{(l+m+3)\, (m+2)} + \frac{q(l+2, m-3, 2n)}{(l-m+5)\, m} - \frac{q(l+2, m-1, 2n)}{(l-m+3)\, (m+2)} + \frac{q(l+3, m-2, 2n)}{(l+m+1)\, m} - \frac{q(l+3, m, 2n)}{(l+m+3)\, (m+2)} \quad (3.5)$$

Now all l's are larger than m, so we can use the inductive hypothesis

Here we use the induction hypothesis and get the rational functions R1, R2, R3.
The idea is to change the parameter m such that all with the same l parameter have the same q.
(Note that if m=2, then the term q(l+2,m-3,2*n) is actually 0; but our qup-shift makes it 0)

```
> qe2 := collect(subs(
      subs(l=l+1,m=m-2,qup),
      subs(l=l+1,m=m+2,qdown),
      subs(l=l+2,m=m-3,qup),
      subs(l=l+3,m=m-2,qup),
  qe1), q,factor);
```

$$qe2 := \Big(2\,\big(2\, l^3 m^2 - 4\, l^2 m^3 + 2\, l\, m^4 + 4\, l^3 m + 3\, l^2 m^2 - 6\, l\, m^3 + 3\, m^4 + 3\, l^3 + 22\, l^2 m - 11\, l\, m^2 + 4\, m^3 + 19\, l^2 + 30\, m\, l - 16\, m^2 + 37\, l + 4\, m + 21\big)\, q(l+1, m, 2n)\Big) \Big/ \big((l+m+1)\, m\, (l-m+1)\, (m+2)\, (l-m+3)\, (m+1)\, (l+m+3)\big) - \frac{2\,(m^2 + 2\, l - 2\, m + 6)\, q(l+2, m-1, 2n)}{(l-m+5)\, m^2\, (l-m+3)\, (m+2)} - \frac{2\,(-m^2 + l + 3)\, q(l+3, m, 2n)}{(l+m+1)\, (m+1)\, m\, (l+m+3)\, (m+2)} \quad (3.6)$$

Next, we check if coefficient of q(l+1,m,2n) is non-negative
the denominator is positive, so we only deal with the numerator

```
> coeff(qe2,q(l+1,m,2*n));
  numer(%);
  simplify(subs(m=a,l=a+b,%));
```

$$\left(2\left(2l^3m^2-4l^2m^3+2lm^4+4l^3m+3l^2m^2-6lm^3+3m^4+3l^3+22l^2m-11lm^2+4m^3+19l^2+30ml-16m^2+37l+4m+21\right)\right)\Big/\left((l+m+1)\,m\,(l-m+1)\,(m+2)\,(l-m+3)\,(m+1)\,(l+m+3)\right)$$

$$4l^3m^2-8l^2m^3+4lm^4+8l^3m+6l^2m^2-12lm^3+6m^4+6l^3+44l^2m-22lm^2+8m^3+38l^2+60ml-32m^2+74l+8m+42$$

$$8a^4+4(b+3)^2a^3+\left(4b^3+30b^2+84b+66\right)a^2+\left(8b^3+62b^2+136b+82\right)a+6b^3+38b^2+74b+42 \quad \textbf{(3.7)}$$

Then, we use the following simple bounds, arising by taking just the first term on the right-hand side of qrec;
Note that we DO NOT have equality here but a >= sign;
however, for the subsitution process we use it like that.

```
> qrecUpBound := q(l,m,2*n) = coeff(rhs(qrec),q(l+1,m-1,2*n))*
  q(l+1,m-1,2*n);
```

$$qrecUpBound := q(l,m,2n) = \frac{(l-m+1)\,q(l+1,m-1,2n)}{l-m+2} \quad \textbf{(3.8)}$$

We will use this bound

```
> subs(l=l+1,qrecUpBound);
```

$$q(l+1,m,2n) = \frac{(l-m+2)\,q(l+2,m-1,2n)}{l-m+3} \quad \textbf{(3.9)}$$

Now the next 2 are the same

```
> qe3 := collect(subs(
     subs(l=l+1,qrecUpBound),
  qe2), q,factor);
```

$$qe3 := \left(2\left(2l^5m^3-8l^4m^4+12l^3m^5-8l^2m^6+2lm^7+4l^5m^2+8l^4m^3-50l^3m^4+62l^2m^5-26lm^6+2m^7+l^5m+45l^4m^2-43l^3m^3-57l^2m^4+70lm^5-16m^6-2l^5+20l^4m+177l^3m^2-256l^2m^3+97lm^4-8m^5-22l^4+124l^3m+277l^2m^2-375lm^3+140m^4-92l^3+334l^2m+119lm^2-136m^3-180l^2+403ml-38m^2-162l+174m-54\right)q(l+2,m-1,2n)\right)\Big/\left((l+m+1)\,m^2\,(l-m+1)\,(m+2)\,(l-m+3)^2\,(m+1)\,(l+m+3)\,(l-m+5)\right)-\frac{2\left(-m^2+l+3\right)q(l+3,m,2n)}{(l+m+1)\,(m+1)\,m\,(l+m+3)\,(m+2)} \quad \textbf{(3.10)}$$

Then, we check if coefficient of q(l+2,m-1,2n) is non-negative
the denominator is positive, so we only deal with the numerator

```
> coeff(qe3,q(l+2,m-1,2*n));
  numer(%);
  simplify(subs(m=a+2,l=a+b+2,%));
```

$$\left(2\left(2l^5m^3-8l^4m^4+12l^3m^5-8l^2m^6+2lm^7+4l^5m^2+8l^4m^3-50l^3m^4+62l^2m^5-26lm^6+2m^7+l^5m+45l^4m^2-43l^3m^3-57l^2m^4+70lm^5-16m^6-2l^5+20l^4m+177l^3m^2-256l^2m^3+97lm^4-8m^5-22l^4+124l^3m+277l^2m^2-375lm^3+140m^4-92l^3+334l^2m+119lm^2-136m^3-180l^2+403ml-38m^2-162l+174m-54\right)\right)\Big/\left((l+m+1)\,m^2\,(l-m+1)\,(m+2)\,(l-m+3)^2\,(m+1)\,(l+m+3)\,(l-m+5)\right)$$

$$4\,l^5\,m^3-16\,l^4\,m^4+24\,l^3\,m^5-16\,l^2\,m^6+4\,l\,m^7+8\,l^5\,m^2+16\,l^4\,m^3-100\,l^3\,m^4+124\,l^2\,m^5-52\,l\,m^6+4\,m^7+2\,l^5\,m+90\,l^4\,m^2-86\,l^3\,m^3-114\,l^2\,m^4+140\,l\,m^5-32\,m^6-4\,l^5+40\,l^4\,m+354\,l^3\,m^2-512\,l^2\,m^3+194\,l\,m^4-16\,m^5-44\,l^4+248\,l^3\,m+554\,l^2\,m^2-750\,l\,m^3+280\,m^4-184\,l^3+668\,l^2\,m+238\,l\,m^2-272\,m^3-360\,l^2+806\,m\,l-76\,m^2-324\,l+348\,m-108$$

$$(24\,b+56)\,a^5+(4\,b^4+44\,b^3+188\,b^2+612\,b+848)\,a^4+(4\,b^5+88\,b^4+646\,b^3+2254\,b^2+4862\,b+4994)\,a^3+(32\,b^5+532\,b^4+3294\,b^3+10046\,b^2+17426\,b+14462)\,a^2+(82\,b^5+1220\,b^4+6904\,b^3+19268\,b^2+29110\,b+20600)\,a+64\,b^5+908\,b^4+4912\,b^3+13016\,b^2+18064\,b+11420 \quad \textbf{(3.11)}$$

Finally, in the same way as before, we use the following bound, arising by taking just the second term on the right-hand side of qrec;
Note that we DO NOT have equality here but a >= sign;
however, for the subsitution process we use it like that.

```
> qrecUpBound2  := q(l,m,2*n) = coeff(rhs(qrec),q(l+1,m+1,2*n))
  *q(l+1,m+1,2*n);
```

$$qrecUpBound2 := q(l,m,2\,n)=\frac{(l-m+2)\,q(l+1,m+1,2\,n)}{l+m+2} \quad \textbf{(3.12)}$$

We will use this bound

```
> subs(l=l+2,m=m-1,qrecUpBound2);
```

$$q(l+2,m-1,2\,n)=\frac{(l-m+5)\,q(l+3,m,2\,n)}{l+m+3} \quad \textbf{(3.13)}$$

Now all are the same

```
> qe4  := collect(subs(
     subs(l=l+2,m=m-1,qrecUpBound2),
  qe3), q,factor);
```

$$qe4 := \left(2\,(2\,l^5\,m^3-8\,l^4\,m^4+12\,l^3\,m^5-8\,l^2\,m^6+2\,l\,m^7+4\,l^5\,m^2+9\,l^4\,m^3-52\,l^3\,m^4+62\,l^2\,m^5-24\,l\,m^6+m^7+47\,l^4\,m^2-33\,l^3\,m^3-75\,l^2\,m^4+73\,l\,m^5-12\,m^6-2\,l^5+7\,l^4\,m+199\,l^3\,m^2-222\,l^2\,m^3+45\,l\,m^4+m^5-22\,l^4+58\,l^3\,m+367\,l^2\,m^2-333\,l\,m^3+92\,m^4-92\,l^3+172\,l^2\,m+281\,l\,m^2-127\,m^3-180\,l^2+214\,m\,l+70\,m^2-162\,l+93\,m-54)\,q(l+3,m,2\,n)\right)\Big/\left((l+m+1)\,m^2\,(l-m+1)\,(m+2)\,(l-m+3)^2\,(m+1)\,(l+m+3)^2\right) \quad \textbf{(3.14)}$$

So we want to show that the factor of q(l+3,m,2*n) is positive;
note that l >= m >= 1:
therefore we set m=a+2 and l=a+b+2 in order to have a,b>=0 independently;
and we see that all terms have non-negative coefficients, which proves the claim.

```
> qe5  := simplify(subs(q(l+3,m,2*n)=1,m=a+1,l=a+b+1,qe4));
```

$$qe5 := \left((24\,b+56)\,a^5+(4\,b^4+48\,b^3+216\,b^2+552\,b+604)\,a^4+(4\,b^5+74\,b^4+502\,b^3+1658\,b^2+2942\,b+2324)\,a^3+(20\,b^5+292\,b^4+1632\,b^3+4496\,b^2+6452\,b+4116)\,a^2+(28\,b^5+372\,b^4+1890\,b^3+4674\,b^2+5874\,b+3258)\,a+8\,(b+5)\,(b+3)^2\left(b^2+\frac{9}{4}\,b+\frac{9}{4}\right)\right)\Big/\left((2\,a+b+3)\,(a+1)^2\,(b+1)\,(a \right. \quad \textbf{(3.15)}$$

$+ 3) (b + 3)^2 (a + 2) (2 a + b + 5)^2)$