Young tableaux with periodic walls: counting with the density method

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FPSAC 2021
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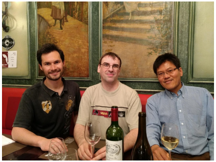
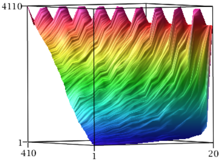
3	10	5	6	12	16	13	14
1	2	4	7	8	9	11	15





For details: see our FPSAC article

Young tableaux with local decreases

7	18	19	12	21	20	17
2	6	8	9	10	14	16
1	3	4	5	11	13	15

We consider Young tableaux in which some pairs of (horizontally or vertically) consecutive cells are allowed to have decreasing labels. Places where a decrease is allowed (but not compulsory) are drawn by a red edge, which we call a "wall".

Nice formulas for some specific tableaux of shape $n \times 2$:

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13	14
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Nice formulas for some specific tableaux of shape $n \times 2$:

- no walls:

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- no walls: $\frac{1}{n+1}\binom{2n}{n}$

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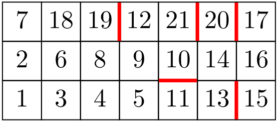
- horizontal walls everywhere:

Young tableaux with local decreases



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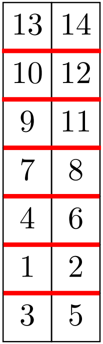

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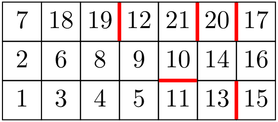

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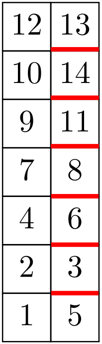

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- horizontal walls everywhere in 2nd col.:

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- vertical walls everywhere:

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- all k vertical walls:

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- horizontal walls everywhere: $\frac{(2n)!}{2^n}$
- horizontal walls everywhere in 2^{nd} col.: $\frac{(2n)!}{2^n n!} = (2n-1)!!$
- vertical walls everywhere: $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$
- all k vertical walls: $\frac{1}{n+1}\binom{2n}{n}\binom{n+1}{k}$ (We give 2 proofs)

Bijections with paths and trees

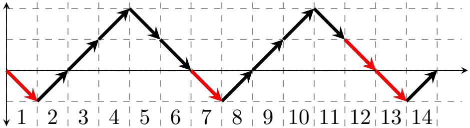
Theorem

The number of $n \times 2$ Young tableaux $\mathcal{Y}$ with k vertical walls is equal to

$$v_{n,k} = \frac{1}{n+1}\binom{2n}{n}\binom{n+1}{k}.$$

Proof #1: Bicolored down-steps in Dyck bridges + the Chung–Feller property

14	13
10	12
9	11
8	7
4	6
3	5
2	1

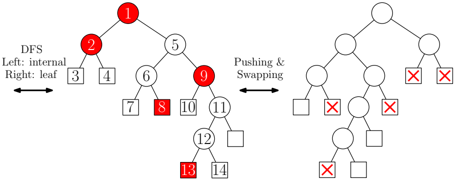


Proof #2: Leaf-marked binary trees

12	14
11	13
10	9
6	8
5	7
4	2
3	1

Marking & Sorting ⟷

12	14
11	13
9	10
6	8
5	7
2	4
1	3



Long walls with small holes: hook-length type formulas

Holes of size 1 on the border

13	14	16	17	19	20	21	25	27
11	2	10	12	15	18	6	23	26
4	1	8	5	7	9	3	22	24

λ_1 λ_2 λ_3 λ_4

Theorem [Gascom 2018]

The number of $n \times m$ Young tableaux of size mn with k walls from column 1 to $m-1$ at distance $0 < d_i := \sum_{j=1}^{i} \lambda_i < n$, $i = 1, \ldots, k$ with $h_i < h_{i+1}$ is equal to

$$\frac{(m-1)!}{(mn+m-1)_{m-1}} \left(\prod_{i=1}^{k+1} \prod_{j=1}^{m-2} \binom{\lambda_i + j}{j}^{-1} \right) \left(\prod_{i=1}^{k+1} \binom{md_i + m - 1}{\lambda_i, \ldots, \lambda_i} \right),$$

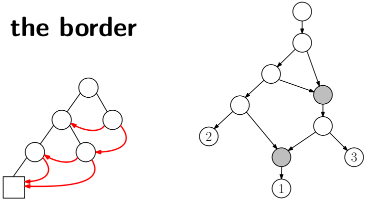
where the multinomial coefficients contain $m-1$ λ_i's.

Drawback: efficient formula but too ad-hoc. What about more complicated holes?

Larger holes lead to unusual asymptotics

The "simplest case" of **holes of size** 2 **on the border**

6	10	14	15	17	18
3	5	9	12	13	16
2	1	7	4	11	8



BAADBACFCBEDECDFEF

Theorem

The number f_n of such Young tableaux of size $n \times 3$ satisfies

$$f_n = \Theta\left(n!\ 12^n e^{a_1 (3n)^{1/3}} n^{-2/3}\right),$$

where $a_1 \approx -2.338$ is the largest root of the Airy function of the first kind.

- Bijections to phylogenetic networks, special words with n distinct letters, and related to compacted trees (special DAGs) [Fuchs–Yu–Zhang 21]
- General method to prove stretched exponentials in bivariate recurrences [Elvey Price–Fang–Wallner 21]. Here:

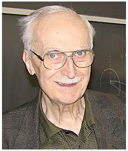
$$y_{n,k} = y_{n,k-1} + (2n + k - 1) y_{n-1,k} \qquad \text{and} \qquad f_n = y_{n,n}.$$

Generic approach:

The density method

* far origins in poset theory (volume of polytopes, log-concavity) [Stanley 1981]
* enumeration of linear extensions is #P-complete [Dyer Frieze 1988, Brightwell Winkler 1991]
* avatars in number theory [Zagier, Beukers Kolk Calabi 1993, Elkies 2003]
* applied to square Young tableaux [Barishnikov 2001]
  and variants of alternating permutations [Baryshnikov Romik 2010, Stanley 2010]
* generalized to further posets & random generation [Banderier Marchal Wallner 2016–2021]







Uniform random generation and enumeration

6	15	16
1	13	14
8	10	18
3	9	12
4	7	17
2	5	11

This example is "without loss of generality"
(i.e., our method works also
for non-periodic shapes).

How to generate/enumerate such tableaux? Brute-force is hopeless!
Solution = use our density method!

The density method will give thousands of coefficients in a few seconds.
The number of tableaux of size $2n \times 3$ is $f_n = (6n+1)! \int_0^1 p_n(z)dz$, with

$$p_{n+1}(z) = \int_0^z \frac{1}{24}(z-1)(x-z)(3x^3 - 7x^2z - xz^2 - z^3 - 2x^2 + 4xz + 4z^2)p_n(x)\,dx.$$

$\{f_n\}_{n\geq 0}$={1, 12, 8550, 39235950, 629738299350, 26095645151941500, 2323497950101372223250, 392833430654718548673344250, 115375222087417545717234273063750, 55038140590519890608190921051205837500, ...}.

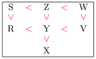
From (periodic) tableaux to tuples of reals and polytopes

6	15	16
1	13	14
8	10	18
3	9	12
4	7	17
2	5	11

7	16	17
2	14	15
9	11	19
4	10	13
5	8	18
3	6	12
	1	

.74	.96	.97
.25	.94	.95
.85	.91	.99
.42	.90	.93
.54	.82	.98
.35	.57	.92
	.06	

S	Z	W
R	Y	V
	X	



- The density method generates real numbers with the *same relative order*
- All possible values = a polytope $\mathcal{P}_n \subseteq [0,1]^{6n+1}$
- "Building blocks" of 7 cells for this periodic tableau

Uniformity via the "right" choice of densities

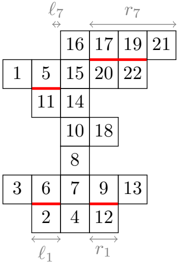
$$p_{n+1}(z) = \int_{0<x<z} \int_{x<y<z} \int_{0<r<y} \int_{r<s<z} \int_{z<w<1} \int_{y<v<w} p_n(v)\, dv\, dw\, ds\, dr\, dy\, dx$$

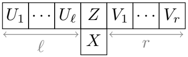
$$d(\boldsymbol{x}) = \frac{p_n(x_n)}{\int_0^1 p_n(t) dt} \prod_{k=0}^{n-1} \frac{p_k(x_k) 1_{\mathrm{block}_k}}{p_{k+1}(x_{k+1})} = \frac{p_0(x_0) 1_{\mathcal{P}_n}}{\int_0^1 p_n(t)\, dt} = \frac{1_{\mathcal{P}_n}}{\int_0^1 p_n(t)\, dt} = \frac{1_{\mathcal{P}_n}}{\mathrm{vol}(\mathcal{P}_\mathrm{n})}$$

$$\mathrm{Prob}(\boldsymbol{x} \in \mathcal{P}_n) = \mathrm{vol}(\mathcal{P}_n) = f_n/(6n+1)!$$

Jenga tableaux and the density method

Jenga! = Construct! in Swahili.





Given a shape $(\ell_i, r_i)_{i\in\mathbb{N}}$, what is the number f_n of tableaux with n lines?

Theorem

$$f_n = \Big(\sum_{i=1}^{n}(\ell_i + r_i + 1)\Big)! \int_0^1 p_n(x)\,dx$$

$$p_n(z) = \frac{z^{\ell_n}(1-z)^{r_n}}{\ell_n!\,r_n!}\int_0^z p_{n-1}(x)\,dx \qquad \textit{with} \qquad p_1(z) = \frac{z^{\ell_1}(1-z)^{r_1}}{\ell_1!\,r_1!}.$$

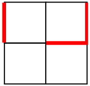
Proof: $p_n(z) = \int_{z<v_1<1}\cdots\int_{v_{r-1}<v_r<1}\int_{0<u_\ell<z}\cdots\int_{0<u_1<u_2}\int_{0<x<z} p_{n-1}(x)\,dx\,du_1\ldots du_\ell\,dv_r\ldots dv_1$

A classification of 2×2 periodic shapes

A periodic shape is the concatenation of n copies of a building block $\mathcal{B}$ of <cells:

$$\mathcal{Y} = \mathcal{B}^n.$$

A tableau $\mathcal{Y}$ with periodic walls is a periodic shape filled with all integers from $\{1, \ldots, |\mathcal{B}|n\}$ respecting the induced order constraints.

$\mathcal{B} =$ 

$\mathcal{B}^4 =$

3	10	5	6	12	16	13	14
1	2	4	7	8	9	11	15

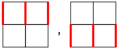
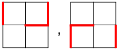
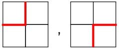
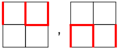
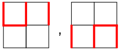
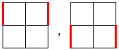
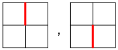
There are a priori $2^6 = 64$ shapes, but some are in bijection (e.g., turn by 180 degrees and reverse labels). It turns out that it leads to 32 different sequences.

We now characterize all 2×2 shapes according to the nature of the counting sequence/generating function, which is either

- "simple" hypergeometric
- hypergeometric,
- algebraic,
- D-algebraic and beyond.

Hypergeometric cases

= cases with uniquely determined minimum or maximum

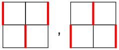
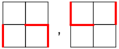
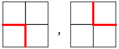
Class	Shape	Sequence	OEIS
H1		$\prod_{i=1}^{n}(4i-1)(4i-3)$	A101485
H2		$\prod_{i=1}^{n}(2i-1)(4i-1)$	A159605
H3		$2^{n+1}n!\prod_{i=1}^{n}(4i-3)$	$2^{n+1}\cdot$A084943
H4		$\binom{4n}{n}\prod_{i=1}^{n}(3i-1)$	$\binom{4n}{n}\cdot$A008544
H5		$\binom{4n}{n}\prod_{i=1}^{n}(3i-2)$	$\binom{4n}{n}\cdot$A007559
H6		$2^{n}n!\prod_{i=1}^{n}(4i-3)$	$n!\cdot$A084948
H7		$\prod_{i=1}^{n}(2i-1)(4i-1)$	A159605

Proofs:

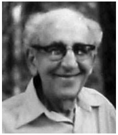
- Models H1–H5: variants of Jenga tableaux with $r_i = 0$ for all i
- Models H6–H7: recursively decompose with respect to the location of the unique minimum or maximum.

D-algebraic cases?

≈ cases with a zig-zag-like pattern

Class	Shape	GF	OEIS	Example
Z1		D-algebraic, and not D-finite: $\dfrac{\cos(t/\sqrt{2})^2+\cosh(t/\sqrt{2})^2}{2\cos(t/\sqrt{2})\cosh(t/\sqrt{2})}$	related to A211212	12 16 6 15 13 14 7 10 8 3 5 9 11 2 4 1
Z2		open problem!	—	3 5 8 9 11 13 14 15 2 10 4 7 1 16 6 12
Z3		open problem!	—	2 4 5 8 11 12 14 15 13 3 16 7 9 6 10 1

Proof for Z1: A permutation $(a_1,\dots,a_n)$ is an alternating permutation of type $(k_1,\dots,k_m)$ if
$a_1<\dots<a_{k_1}>a_{k_1+1}<\dots<a_{k_1+k_2}>a_{k_1+k_2+1}<\dots<a_n$.
Then, $k_i=1$ gives classical alternating permutations;
while $k_1=3$, $k_2=\dots=k_n=4$, and $k_{n+1}=1$ gives Z1.
A generalization of [Carlitz 73] then leads to



Leonard Carlitz
(1907-1999)
771 articles!

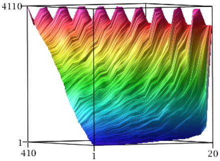
$$F(t)=\frac{E_{4,3}(t)E_{4,1}(t)}{E_{4,0}(t)}+E_{4,0}(t)\quad\text{where}\quad E_{k,r}(t)=\sum_{n\geq 0}(-1)^n\frac{t^{nk+r}}{(nk+r)!}.\qquad\square$$

Conclusion

- 3 ways to enumerate and generate Young tableaux with walls: hook-length type formulas, bijections, density method.
- Approach different from [Greene Nijenhuis Wilf 84]. They used the existence of a simple product formula (hook-length formula).
- Brute-force generation or P-partition formulas $\to$ exponential cost. Generation via our density method $\to$ $O(n^2)$ cost.
- A field to explore: examine more families of posets (e.g., permutations, Young tableaux, increasing trees, urn models in [Banderier Marchal Wallner 20]).
- Asymptotics? D-finite? D-algebraic? Links with other objects?

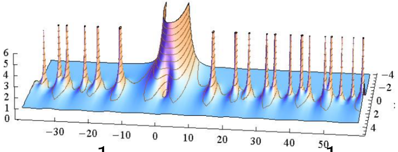
3	5	8	9	11	13	14	15
2	10	4	7	1	16	6	12

$$\Theta\left(n!\, C^n e^{a_1 n^\sigma} n^\alpha\right) \; ?$$



Bonus Slides

Values of the zeta function (after Zagier, Calabi, Elkies)



$$\zeta(s) = \sum_{k \geq 1} \frac{1}{k^s} \quad \rightarrow \quad \zeta(s)(1 - \frac{1}{2^s}) = \sum_{k \geq 0} \frac{1}{(2k+1)^s}$$

$$S(2) = \sum_{k \geq 0} \frac{1}{(2k+1)^2} = \sum_{k \geq 0} \int_0^1 \int_0^1 (xy)^{2k} = \int_0^1 \int_0^1 \frac{dxdy}{1-(xy)^2}$$

Change of variable $x = \frac{\sin u}{\cos v}$ and $y = \frac{\sin v}{\cos u}$.
The integration domain becomes the triangle $T = \{u > 0, v > 0, u + v < \pi/2\}$.

$$S(2) = \int_T dudv = \frac{\pi^2}{8}$$

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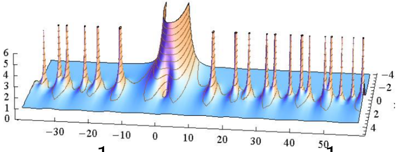

$$\zeta(s) = \sum_{k\geq 1} \frac{1}{k^s} \quad \to \quad \zeta(s)\left(1-\frac{1}{2^s}\right) = \sum_{k\geq 0} \frac{1}{(2k+1)^s}$$

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Calabi and Elkies generalisation:

n even: $\quad S(n) = \text{vol(polytope of dimension } n) = \left(\frac{\pi}{2}\right)^n \frac{A(n)}{n!}$

$A(n) = \#$ alternating permutations of length n.

Exponential cost formulas via P-partitions

E.g. for the zigzag shape Z3, Christian Krattenthaler obtained via Stanley's P-partition theory a nice formula (but with exponential cost):

$$Z3(n) = \frac{(4n)!}{2^n} \sum_{\epsilon \in \{0,1\}^{n-1}} (-1)^{|\epsilon|} \left(\frac{1}{f_1(\epsilon)} - \frac{1}{f_2(\epsilon)} \right)$$

where $|\epsilon| := \epsilon_1 + \cdots + \epsilon_{n-1}$ and

$$f_1(\epsilon) := \prod_{i=1}^{n-1} (2i + 1 + 2(\epsilon_1 + \cdots + \epsilon_i))(2i + 2 + 2(\epsilon_1 + \cdots + \epsilon_i)),$$

$$f_2(\epsilon) := 3 \prod_{i=1}^{n-1} (2i + 2 + 2(\epsilon_1 + \cdots + \epsilon_i))(2i + 3 + 2(\epsilon_1 + \cdots + \epsilon_i)).$$

Open problem: to infer from it asymptotics, (non?) D-finiteness, etc.

Advantage of our density method: polynomial cost via the integrals of densities.