More Models of Walks Avoiding a Quadrant

CanaDAM 2021 Online 05/2021

Michael Wallner

joint work with Mireille Bousquet-Mélou

https://dmg.tuwien.ac.at/mwallner

Institute of Discrete Mathematics and Geometry, TU Wien, Austria

(Austrian Science Fund (FWF): J 4162 and P 34142)

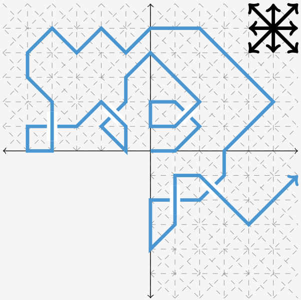
More Models of Walks Avoiding a Quadrant, LIPIcs, AofA 2020

I have an open PhD position to offer in my project "Stretched exponentials and beyond". Feel free to contact me if you are interested!

The problem

Question

How many walks of length n starting from $(0,0)$ avoid the quadrant?



- We fix the starting point $(0,0)$,
- a step set $\mathcal{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$ of small steps, and
- the three-quadrant cone $\mathcal{C} = \{(i,j) : i \geq 0 \text{ or } j \geq 0\}$.

Real-life applications

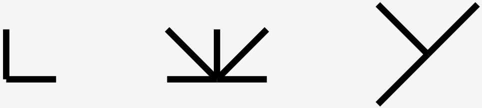
More seriously ...

it is a model for many discrete objects in

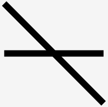
- combinatorics, statistical physics
- probability theory, queueing theory
- ...

How many interesting models are there?

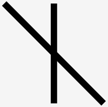
- $\mathcal{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\} \quad \Rightarrow \quad 2^8 = 256$ models
- However, some are equivalent
  - to a model of walks in the full or half-space ($\Rightarrow$ algebraic)



  - to another model in the collection (diagonal symmetry)

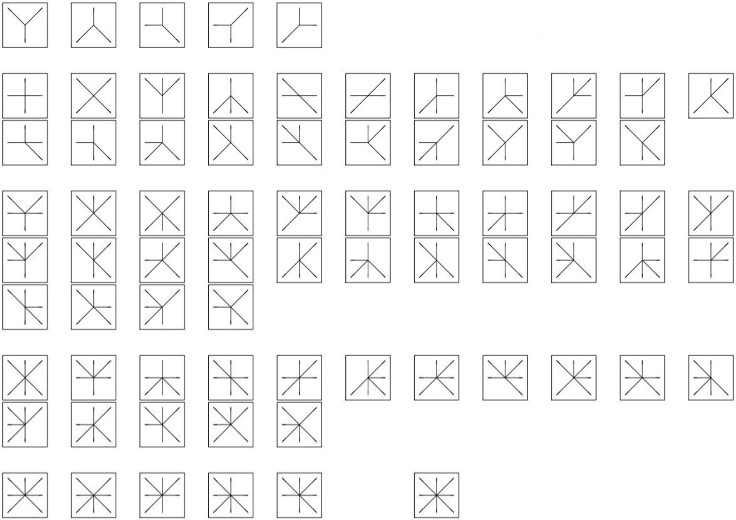


$\cong$



- We are left with 74 interesting models (79 in the quarter plane)

The 74 interesting models in the three-quadrant cone



Interesting questions

- Closed form/asymptotics for the number $c(n)$ of walks of length n?
- Closed form/asymptotics for the number $c_{i,j}(n)$ of walks ending at (i,j)?
- The generating functions and their nature?

$$C(t) = \sum_{n\geq 0} c(n)t^n, \qquad C(x,y;t) = \sum_{(i,j)\in\mathcal{C}} \sum_{n\geq 0} c_{i,j}(n)t^n x^i y^j$$

  - Can we express these series?
  - Are they rational/algebraic/D-finite?

A hierarchy of formal power series

The formal power series $C(t)$ is

- rational if it can be written as
$$C(t) = \frac{P(t)}{Q(t)},$$
where $P(t)$ and $Q(t)$ are polynomials in t.
- algebraic (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation
$$P(t, C(t)) = 0.$$
- D-finite if it satisfies a (non-trivial) linear differential equation with polynomial coefficients:
$$p_k(t)C^{(k)}(t) + \cdots + p_0(t)C(t) = 0.$$

Why is it important to be D-finite?

- Nice and effective closure properties (sum, product, differentiation, ...)
- Fast algorithms to compute coefficients
- Asymptotics of coefficients

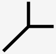
Solved cases

D-finite $C(x, y; t)$

1  [Bousquet-Mélou 16]

2  [Bousquet-Mélou 16]

3  [Raschel, Trotignon 19]

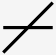
4  [Raschel, Trotignon 19]

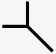
5  [Raschel, Trotignon 19]

6  King [This talk!]

D-finite excursions $\sum_{n\geq 0} c_{0,0}(n)t^n$

7  [Budd 20]

8  [Budd 20]

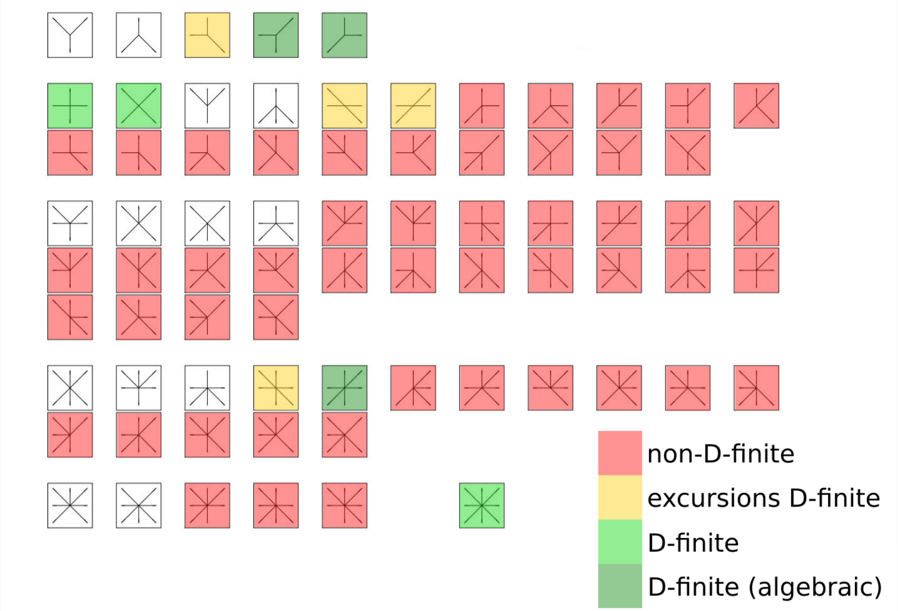
9  [Elvey-Price 20]

10  [Elvey-Price 20]

Non-D-finite

- 51 models [Mustapha 19]

The taxonomy so far



The group of the walk for the king

- From now on we use $\bar{x} := \frac{1}{x}$ and $\bar{y} := \frac{1}{y}$
- The step polynomial encodes the possible steps

$$S(x,y) = x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y}.$$

- $S(x,y)$ is left unchanged by the rational transformations

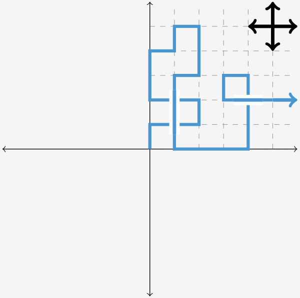
$$\Phi : (x,y) \mapsto (\bar{x},y) \qquad \text{and} \qquad \Psi : (x,y) \mapsto (x,\bar{y}).$$

- They are involutions and generate a finite dihedral group G:

$$\begin{array}{ccccc} & & (\bar{x},y) & & \\ & \overset{\Phi}{\nearrow} & & \overset{\Psi}{\searrow} & \\ (x,y) & & & & (\bar{x},\bar{y}) \\ & \underset{\Psi}{\searrow} & & \underset{\Phi}{\nearrow} & \\ & & (x,\bar{y}) & & \end{array}$$

- The group can be defined for any model with small steps!

The quarter plane



- Quarter plane
  $$\mathcal{Q} = \{(i,j) : i \geq 0 \text{ and } j \geq 0\}.$$
- Generating function
  $$Q(x,y;t) = \sum_{i,j\geq 0} \sum_{n \geq 0} q_{i,j}(n) t^n.$$

Theorem [Bousquet-Mélou, Mishna 10], [Bostan, Kauers 10], [Kurkova, Raschel 12], [Mishna, Rechnitzer 07], [Melczer, Mishna 13], [and more!]

The series $Q(x,y;t)$ is D-finite if and only if G is finite.

An algebraicity phenomenon for the king 

Theorem

The generating function $C(x,y;t) \equiv C(x,y)$, *of king walks starting from* $(0,0)$ *that are confined to* $\mathcal{C}$, *satisfies*

$$C(x,y) = A(x,y) + \frac{1}{3}\left(Q(x,y) - \bar{x}^2 Q(\bar{x},y) - \bar{y}^2 Q(x,\bar{y})\right),$$

where $A(x,y)$ *is algebraic of degree* 216 *over* $\mathbb{Q}(x,y,t)$.

This series satisfies

$$K(x,y)A(x,y) = \frac{2+\bar{x}^2+\bar{y}^2}{3} - t\bar{y}(x+1+\bar{x})A_-(\bar{x}) - t\bar{x}(y+1+\bar{y})A_-(\bar{y}) - t\bar{x}\bar{y}A_{0,0},$$

where $A_-(x) \in \mathbb{Q}[x][[t]]$ *is algebraic of degree* 72 *over* $\mathbb{Q}(x,t)$ *and* $A_{0,0} \in \mathbb{Q}[[t]]$ *is algebraic of degree* 24 *over* $\mathbb{Q}(t)$.

Such a phenomenon already proved for  and  in [Bousquet-Mélou 16].
Proof: Follow [Bousquet-Mélou 16] + guess-and-check + neat algebraic extensions (see next).

Asymptotics

Corollary

The number $c_{0,0}(n)$ of n-step king walks confined to $\mathcal{C}$ and ending at the origin, and the number $c(n)$ of walks of $\mathcal{C}$ ending anywhere satisfy for $n \to \infty$:

$$c_{0,0}(n) \sim \left(\frac{2^{29}K}{3^7}\right)^{1/3} \frac{\Gamma(2/3)}{\pi} \frac{8^n}{n^{5/3}},$$

$$c(n) \sim \left(\frac{2^{32}K}{3^7}\right)^{1/6} \frac{1}{\Gamma(2/3)} \frac{8^n}{n^{1/3}},$$

where K is the unique real root of

$$101^6 K^3 - 601275603\, K^2 + 92811\, K - 1.$$

- Refines results of [Denisov, Wachtel 15] and [Mustapha 19] by the precise multiplicative constant
- Lower order terms are easily computable

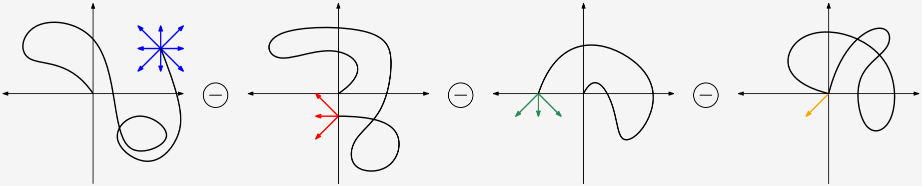
A functional equation

Step by step construction ($S(x, y) = x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y}$):

$$C(x, y) = 1 + tS(x, y)C(x, y) - t\bar{x}(\bar{y} + 1 + y)C_-(\bar{y}) - t\bar{y}(\bar{x} + 1 + x)C_-(\bar{x}) - t\bar{x}\bar{y}C_{0,0}$$

with

$$C_-(x) = \sum_{\substack{i>0 \\ n\geq 0}} c_{-i,0}(n)x^i t^n \in \bar{x} \qquad \text{and} \qquad C_{0,0} = \sum_{n\geq 0} c_{0,0}(n)x^i t^n \in \bar{x}.$$



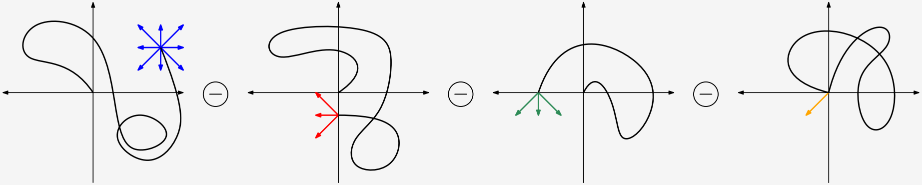
A functional equation

Step by step construction ($S(x,y) = x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y}$):

$$C(x,y) = 1 + tS(x,y)C(x,y) - t\bar{x}(\bar{y}+1+y)C_-(\bar{y}) - t\bar{y}(\bar{x}+1+x)C_-(\bar{x}) - t\bar{x}\bar{y}C_{0,0}$$

with

$$C_-(x) = \sum_{\substack{i>0 \\ n\geq 0}} c_{-i,0}(n)x^i t^n \in \bar{x} \qquad \text{and} \qquad C_{0,0} = \sum_{n\geq 0} c_{0,0}(n)x^i t^n \in \bar{x}.$$



The kernel equation for $C(x,y)$

$$K(x,y)xyC(x,y) = xy - t(1+y+y^2)C_-(\bar{y}) - t(1+x+x^2)C_-(\bar{x}) - tC_{0,0}$$

$$K(x,y) := 1 - tS(x,y)$$

An important observation

The kernel equation for $C(x, y)$

$$K(x,y)xyC(x,y) = xy - t(1+y+y^2)C_-(\bar{y}) - t(1+x+x^2)C_-(\bar{x}) - tC_{0,0}$$

The kernel equation of $Q(x, y)$ is very similar

$$K(x,y)xyQ(x,y) = xy - t(1+y+y^2)Q(0,y) - t(1+x+x^2)Q(x,0) + tQ(0,0)$$

An important observation

The kernel equation for $C(x,y)$

$$K(x,y)xyC(x,y) = xy - t(1+y+y^2)C_-(\bar{y}) - t(1+x+x^2)C_-(\bar{x}) - tC_{0,0}$$

The kernel equation of $Q(x,y)$ is very similar

$$K(x,y)xyQ(x,y) = xy - t(1+y+y^2)Q(0,y) - t(1+x+x^2)Q(x,0) + tQ(0,0)$$

Hence, they have the same orbit sum:

$$xyQ(x,y) - \bar{x}yQ(\bar{x},y) + \bar{x}\bar{y}Q(\bar{x},\bar{y}) - x\bar{y}Q(x,\bar{y}) =$$

$$xyC(x,y) - \bar{x}yC(\bar{x},y) + \bar{x}\bar{y}C(\bar{x},\bar{y}) - x\bar{y}C(x,\bar{y}) = \frac{(x-\bar{x})(y-\bar{y})}{K(x,y)}.$$

(As well as $-\bar{x}^2Q(\bar{x},y)$, $-\bar{y}^2Q(x,\bar{y})$, and $\bar{x}^2\bar{y}^2Q(\bar{x},\bar{y})$.)

Idea: create a zero orbit sum (hence an algebraic GF?!)

Hence, they have the same orbit sum:

$$xyQ(x,y) - \bar{x}yQ(\bar{x},y) + \bar{x}\bar{y}Q(\bar{x},\bar{y}) - x\bar{y}Q(x,\bar{y}) =$$

$$xyC(x,y) - \bar{x}yC(\bar{x},y) + \bar{x}\bar{y}C(\bar{x},\bar{y}) - x\bar{y}C(x,\bar{y}) = \frac{(x-\bar{x})(y-\bar{y})}{K(x,y)}.$$

(As well as $-\bar{x}^2Q(\bar{x},y)$, $-\bar{y}^2Q(x,\bar{y})$, and $\bar{x}^2\bar{y}^2Q(\bar{x},\bar{y})$.)

Idea: create a zero orbit sum (hence an algebraic GF?!)

Hence, they have the same orbit sum:

$$xyQ(x,y) - \bar{x}yQ(\bar{x},y) + \bar{x}\bar{y}Q(\bar{x},\bar{y}) - x\bar{y}Q(x,\bar{y}) =$$

$$xyC(x,y) - \bar{x}yC(\bar{x},y) + \bar{x}\bar{y}C(\bar{x},\bar{y}) - x\bar{y}C(x,\bar{y}) = \frac{(x-\bar{x})(y-\bar{y})}{K(x,y)}.$$

(As well as $-\bar{x}^2Q(\bar{x},y)$, $-\bar{y}^2Q(x,\bar{y})$, and $\bar{x}^2\bar{y}^2Q(\bar{x},\bar{y})$.)

We introduce the formal power series

$$A(x,y) := C(x,y) - \frac{1}{3}\left(Q(x,y) - \bar{x}^2Q(\bar{x},y) - \bar{y}^2Q(x,\bar{y})\right).$$

A (lattice path) functional equation for $A(x,y)$ and orbit sum 0

$$K(x,y)A(x,y) = \frac{2+\bar{x}^2+\bar{y}^2}{3} - t\bar{y}(x+1+\bar{x})A_-(\bar{x}) - t\bar{x}(y+1+\bar{y})A_-(\bar{y}) - t\bar{x}\bar{y}A_{0,0},$$

and

$$xyA(x,y) - \bar{x}yA(\bar{x},y) + \bar{x}\bar{y}A(\bar{x},\bar{y}) - x\bar{y}A(x,\bar{y}) = 0.$$

Continue with $A(x,y)$

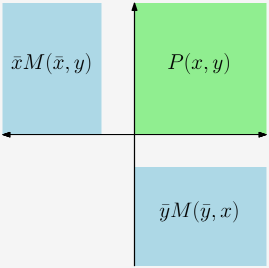
In order to characterize $C(x,y)$ it suffices to solve for $A(x,y)$:

$$K(x,y)A(x,y) = \frac{2+\bar{x}^2+\bar{y}^2}{3} - t\bar{y}(x+1+\bar{x})A_-(\bar{x}) - t\bar{x}(y+1+\bar{y})A_-(\bar{y}) - t\bar{x}\bar{y}A_{0,0}.$$

We want to cancel the kernel, BUT $A(x,y)$ contains negative powers of x and y. Hence, we split it into 3 parts:

$$A(x,y) = P(x,y) + \bar{x}M(\bar{x},y) + \bar{y}M(\bar{y},x),$$

where now $P(x,y), M(x,y) \in \mathbb{Q}[x,y][[t]]$.



A quadrant-like problem for $M(x,y)$

...we get

$$P(x,y) = \bar{x}\left(M(x,y) - M(0,y)\right) + \bar{y}\left(M(y,x) - M(0,x)\right),$$

and (after some work)

$$K(x,y)(2M(x,y) - M(0,y)) = \frac{2x}{3} - 2t\bar{y}(x+1+\bar{x})M(x,0) + t\bar{y}(y+1+\bar{y})M(y,0)$$
$$+ t(x-\bar{x})(y+1+\bar{y})M(0,y) - t\left(1+\bar{y}^2 - 2\bar{x}\bar{y}\right)M(0,0) - t\bar{y}M_x(0,0).$$

A quadrant-like problem for $M(x, y)$

...we get

$$P(x,y) = \bar{x}\,(M(x,y) - M(0,y)) + \bar{y}\,(M(y,x) - M(0,x)),$$

and (after some work)

$$K(x,y)(2M(x,y) - M(0,y)) = \frac{2x}{3} - 2t\bar{y}(x+1+\bar{x})M(x,0) + t\bar{y}(y+1+\bar{y})M(y,0)$$
$$+ t(x-\bar{x})(y+1+\bar{y})M(0,y) - t\left(1+\bar{y}^2 - 2\bar{x}\bar{y}\right)M(0,0) - t\bar{y}M_x(0,0).$$

Now it is legitimate to cancel the kernel. After more work we arrive at an equation for $M(0,x)$ with one catalytic variable only:

$$\mathrm{Pol}(M(0,x), B_1, B_2, B_3, B_4, t, x) = 0,$$

where $B_i \in \mathbb{Q}(t)$ are half-known power series; e.g., $B_4 = M(0,0)$.

- This is a (big !) polynomial equation with one catalytic variable x, in theory solvable using [Bousquet-Mélou, Jehanne 06].
- Unfortunately, the polynomial system was too big for our computers.
- Hence, we used a *guess-and-check* approach.

Guessing

We guessed polynomial equations using $c_{i,j}(n)$ for $0 \leq n \leq 2000$:

GF	Deg. GF	Deg. t	# terms
B_1	12	26	229
B_2	24	60	477
B_3	24	12	323
B_4	24	32	823

Hence, these equations define algebraic power series. In order to prove that they are the ones involved in

$$\mathrm{Pol}(M(0,x), B_1, B_2, B_3, B_4, t, x) = 0,$$

we needed to investigate their algebraic relations.

The algebraic structure of the B_i's

$\mathbb{Q}(t)$

$\downarrow$ ④

1. Let $u = t + t^2 + \mathcal{O}(t^3)$ be the only series satisfying the
$$(1-3u)^3(1+u)t^2 + (1+18u^2-27u^4)t - u = 0.$$
$\mathbb{Q}(t,u)$

$\downarrow$ ③

2. Let $v = t + 3t^2 + \mathcal{O}(t^3)$ be the only series satisfying
$$(1+3v-v^3)u - v(v^2+v+1) = 0.$$
$\mathbb{Q}(t,v)$

$\downarrow$ ②

3. Define
$$w = \sqrt{1+4v-4v^3-4v^4} = 1 + 2t + 4t^2 + \mathcal{O}(t^3).$$
$\mathbb{Q}(t,w)$

We get
$$B_1 \in \mathbb{Q}(t,v) \qquad \text{and} \qquad B_2, B_3, B_4 \in \mathbb{Q}(t,w).$$
This gives
$$B_4 = M(0,0) = C_{-1,0} = \frac{1}{2t}\left(\frac{w(1+2v)}{1+4v-2v^3} - 1\right)$$
$$= t + 2t^2 + 17t^3 + 80t^4 + 536t^5 + \mathcal{O}(t^6).$$

The final result

The precise knowledge of v and w allows us to prove that the guesses are correct and finishes the proof.

Theorem

The generating function $C(x, y; t) \equiv C(x, y)$, of walks starting from $(0, 0)$ that are confined to $\mathcal{C}$, satisfies

$$C(x,y) = A(x,y) + \frac{1}{3}\left(Q(x,y) - \bar{x}^2 Q(\bar{x},y) - \bar{y}^2 Q(x,\bar{y})\right),$$

where $A(x, y)$ is algebraic of degree 216 over $\mathbb{Q}(x, y, t)$.

This series satisfies

$$\begin{aligned} K(x,y)A(x,y) = \frac{2+\bar{x}^2+\bar{y}^2}{3} &- t\bar{y}(x+1+\bar{x})A_-(\bar{x}) \\ &- t\bar{x}(y+1+\bar{y})A_-(\bar{y}) - t\bar{x}\bar{y}A_{0,0}, \end{aligned}$$

where $A_-(x) \in \mathbb{Q}[x][[t]]$ is algebraic of degree 72 over $\mathbb{Q}(x, t)$ and $A_{0,0} \in \mathbb{Q}[[t]]$ is algebraic of degree 24 over $\mathbb{Q}(t)$.

More models

- For each of the following 7 models we can define $A(x, y)$ with orbit sum 0

- First 3 models are now solved
- Methods of this presentation applicable
- For last 3 models: guessed equations of degree 24 for $A_{-1,0}$ (resp. $A_{-2,0}$)
  (For the first 3: degree 4, 8, 24, respectively)

More models

- For each of the following 7 models we can define $A(x, y)$ with orbit sum 0

- First 3 models are now solved
- Methods of this presentation applicable
- For last 3 models: guessed equations of degree 24 for $A_{-1,0}$ (resp. $A_{-2,0}$) (For the first 3: degree 4, 8, 24, respectively)

Thank you!