Compacted binary trees and minimal automata admit stretched exponentials

CanaDAM 2021 Online 05/2021

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joint work with Andrew Elvey Price and Wenjie Fang

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(Austrian Science Fund (FWF): J 4162 and P 34142)

Compacted binary trees admit a stretched exponential, JCTA, Vol. 177(105306), Jan. 2021; ArXiv:1908.11181

Asymptotics of minimal deterministic finite automata recognizing a finite binary language, LIPIcs, AofA 2020

I have an open PhD position to offer in my project "Stretched exponentials and beyond". Feel free to contact me if you are interested!

Motivation: Efficiently store redundant information

Example

Consider the labeled tree necessary to store the arithmetic expression

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(* (- (* x x) (* y y)) (+ (* x x) (* y y)))
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which represents $(x^2 - y^2)(x^2 + y^2)$.

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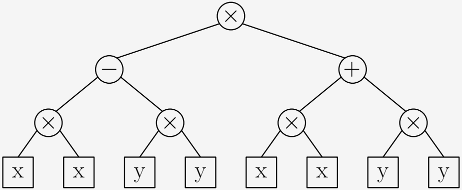

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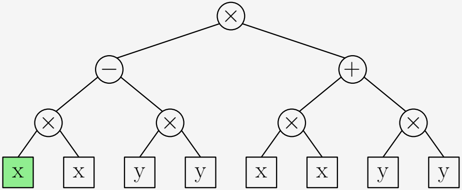

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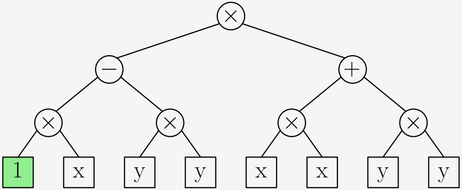

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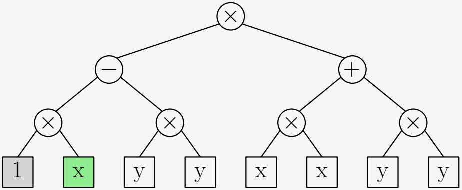

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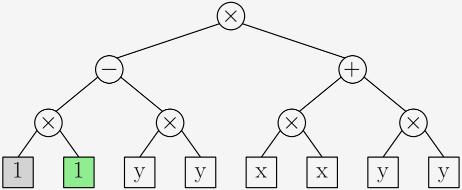

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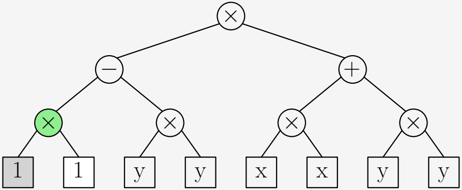

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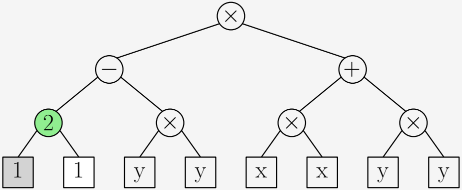

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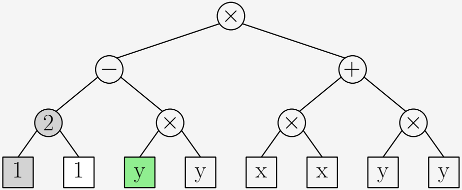

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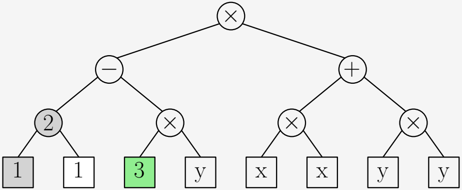

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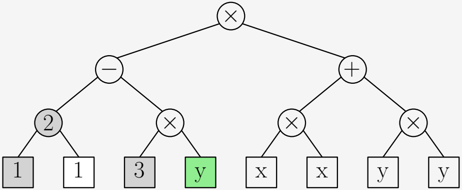

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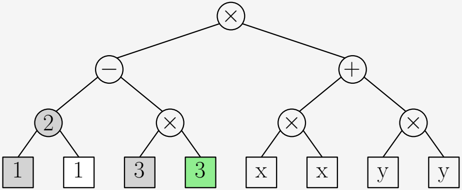

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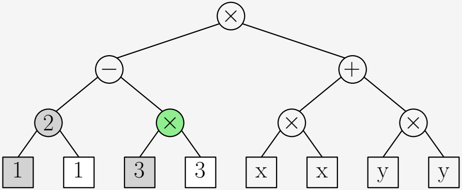

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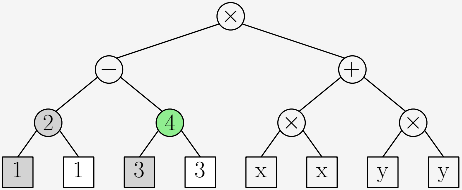

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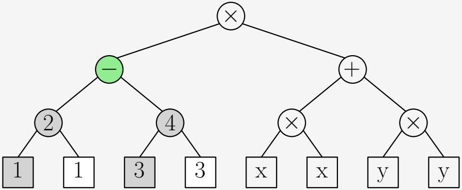

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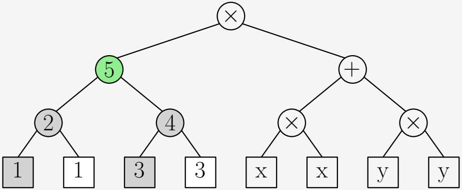

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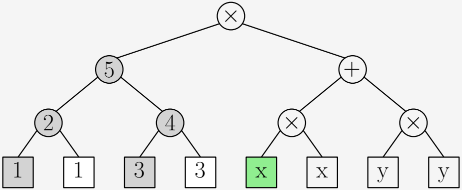

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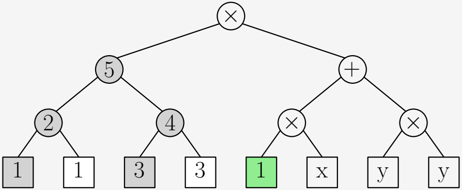

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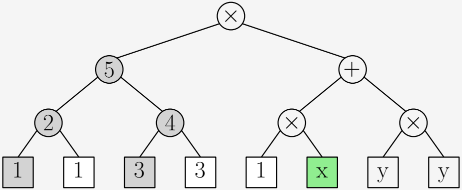

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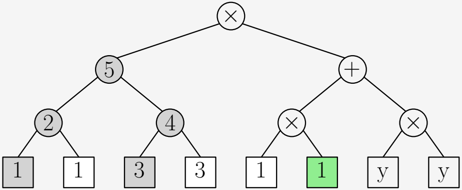

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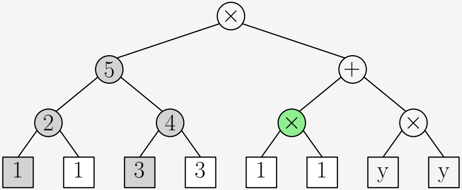

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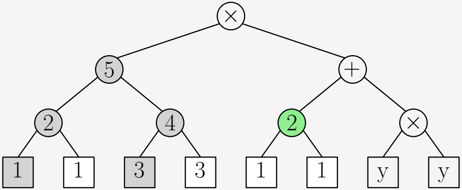

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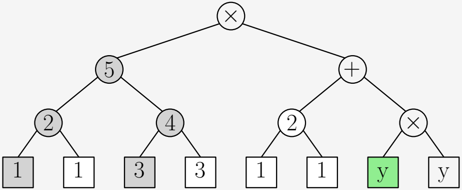

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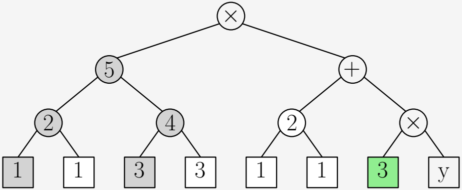

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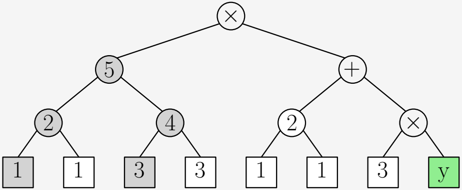

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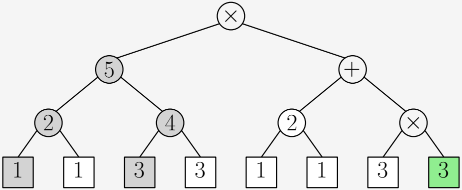

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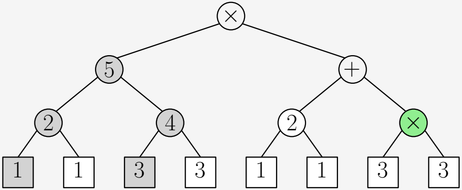

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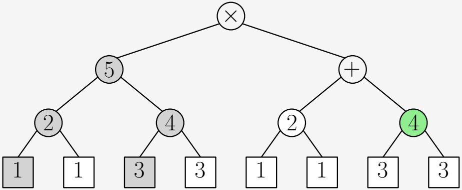

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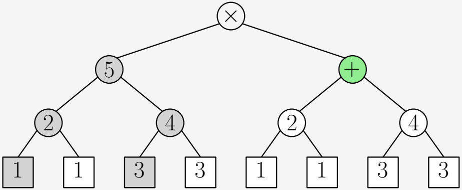

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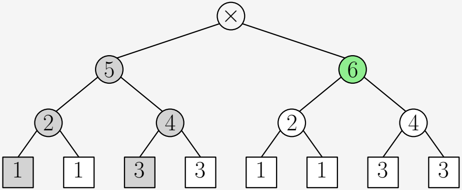

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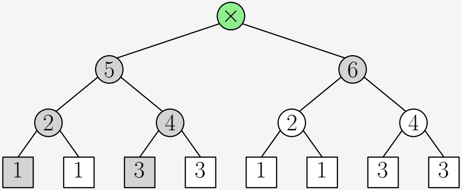

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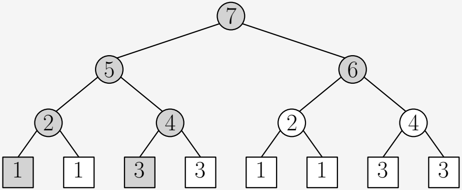

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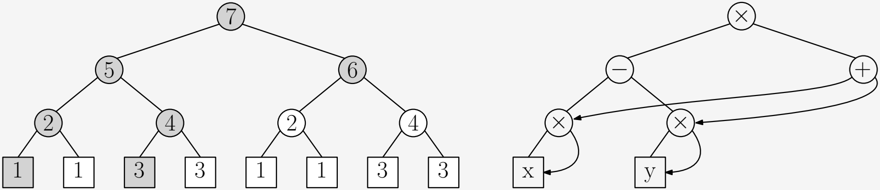

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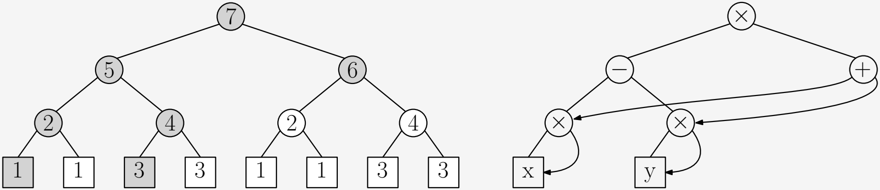

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Definition

Compacted tree is the directed acyclic graph computed by this procedure.

Compacted trees

- Efficient algorithm to compute compacted tree: expected time $\mathcal{O}(n)$
- Analyzed by [Flajolet, Sipala, Steyaert 1990]: A tree of size n has a compacted form of expected size

$$C\frac{n}{\sqrt{\log n}},$$

where C is explicit related to the type of trees and the statistical model.

- Applications:
  - **XML-Compression** [Bousquet-Mélou, Lohrey, Maneth, Noeth 2015]
  - **Compilers** [Aho, Sethi, Ullman 1986]
  - **LISP** [Goto 1974]
  - **Data storage** [Meinel, Theobald 1998], [Knuth 1968], etc.

Reverse question

How many compacted trees of (compacted) size n exist?

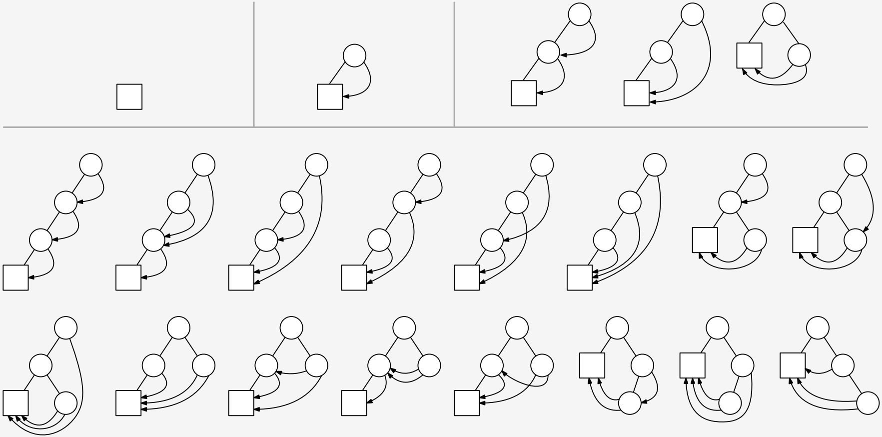
Compacted (unlabeled binary) trees

- Size: number of internal nodes
- c_n: number of compacted trees of size n

  $(c_n)_{n \geq 0} = (1, 1, 3, 15, 111, 1119, 14487, \dots)$
- Important: Subtrees are unique!

Simple bounds

$$n! \leq c_n \leq \frac{1}{n+1}\binom{2n}{n} n!$$



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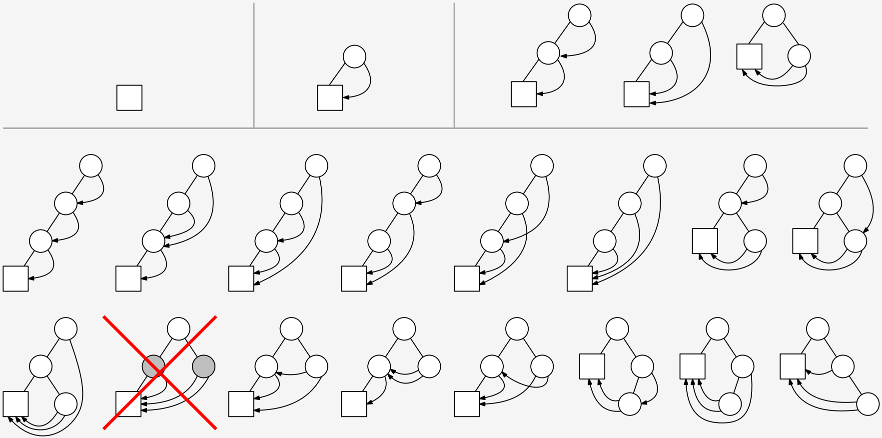
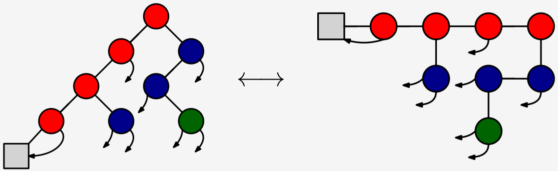


Bounded right height (Previous work)

The **right height** of a binary tree is the maximal number of right children on any path from the root to a leaf (not going through pointers).



Theorem [Genitrini, Gittenberger, Kauers, W 2020]

The number $c_{k,n}$ of compacted trees with right height at most k is for $n \to \infty$ asymptotically equivalent to

$$c_{k,n} \sim \kappa_k n! \left(4 \cos\left(\frac{\pi}{k+3}\right)^2\right)^n n^{-\frac{k}{2} - \frac{1}{k+3} - \left(\frac{1}{4} - \frac{1}{k+3}\right)\cos\left(\frac{\pi}{k+3}\right)^{-2}},$$

where $\kappa_k \in \mathbb{R} \setminus \{0\}$ is independent of n.

Main result compacted trees

A stretched exponential μ^{n^σ} appears!

Theorem [Elvey Price, Fang, W 2021]

The number of compacted binary trees satisfies for $n \to \infty$

$$c_n = \Theta\left(n!\, 4^n e^{3a_1 n^{1/3}} n^{3/4}\right),$$

with $a_1 \approx -2.338$: largest root of the Airy function $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt$.

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Conjecture

Experimentally we find

$$c_n \sim \gamma_c n! 4^n e^{3a_1 n^{1/3}} n^{3/4},$$

where

$$\gamma_c \approx 173.12670485.$$

Other appearances of stretched exponentials

Known exactly:

- Integer partitions:

$$\sim (4\sqrt{3})^{-1} e^{\pi(2n/3)^{1/2}} n^{-1}$$

- Pushed Dyck paths [Beaton, McKay 2014], [Guttmann 2015]:

$$\sim C_1 4^n e^{-3\left(\frac{\pi \log 2}{2}\right)^{2/3} n^{1/3}} n^{-5/6}$$

- Cogrowth sequence of a lamplighter group variant of $\mathbb{Z}_2 \wr \mathbb{Z}$ [Revelle 2003]:

$$\sim C_2 \mu^n e^{-3(\pi \log(2)/2)^{2/3} n^{1/3}} n^{1/6}$$

- Phylogenetic tree-child networks [Fuchs, Yu, Zhang 2020]:

$$\Theta\left(n^{2n} (12e^{-2})^n e^{a_1 (3n)^{1/3}} n^{-2/3} \right)$$

Conjectured:

- Permutations avoiding 1324 [Conway, Guttmann, Zinn-Justin 2018]:

$$\approx \mu^n e^{-cn^{1/2}}$$

- Pushed self avoiding walks [Beaton, Guttmann, Jensen, Lawler 2015]:

$$\approx \mu^n e^{-cn^{3/7}}$$

- and recently more and more appear in group theory, queuing theory, ...

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Deterministic finite automata (DFA)

DFA on alphabet $\{a, b\}$

Graph with

- two outgoing edges from each node (state), labelled a and b
- An initial state q_0
- A set F of *final states* (coloured green).

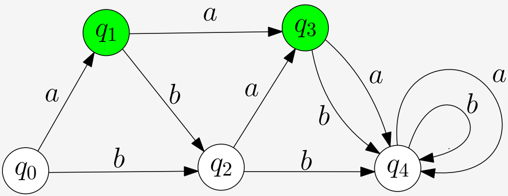

Figure: DFA

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DFA on alphabet $\{a, b\}$

Graph with

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Properties

- **Language:** the set of accepted words
- **Minimal:** no DFA with fewer states accepts the same language
- **Acyclic:** no cycles (except loops at unique sink)

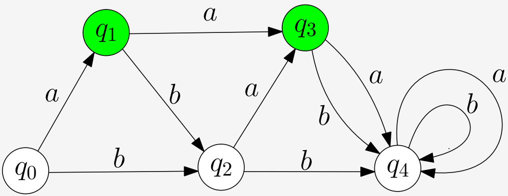
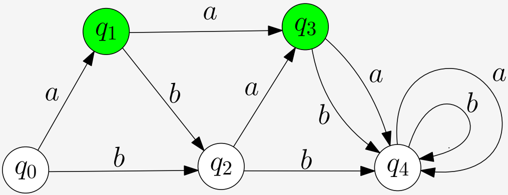


Figure: DFA, which is the minimal DFA recognizing the language $\{a, aa, ba, aba\}$.

Counting minimal acyclic DFAs

- Studied by Domaratzki, Kisman, Shallit, and Liskovets 2002–2006
- **Open problem:** Asymptotics
- Best bounds were out by an exponential factor



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Main result minimal DFAs

A stretched exponential μ^{n^σ} appears again!

Theorem [Elvey Price, Fang, W 2020]

The number m_n of minimal DFAs with $n+1$ states recognizing a finite binary language satisfies for $n \to \infty$

$$m_n = \Theta\left(n!\, 8^n e^{3a_1 n^{1/3}} n^{7/8}\right),$$

with $a_1 \approx -2.338$: largest root of the Airy function $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt$.

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Experimentally we find

$$m_n \sim \gamma n! 8^n e^{3a_1 n^{1/3}} n^{7/8},$$

where

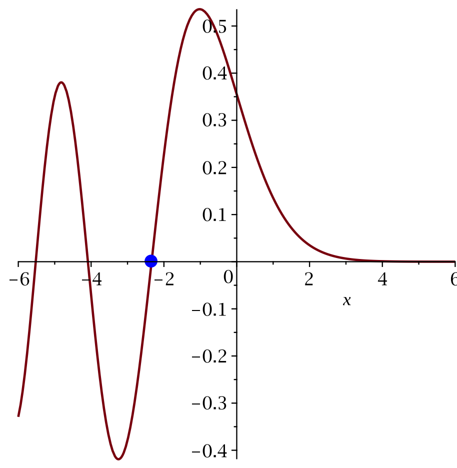
$$\gamma \approx 76.438160702.$$

What is the Airy function?

Properties

- $\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt$
- Largest root $a_1 \approx -2.338$
- $\lim_{x\to\infty} \mathrm{Ai}(x) = 0$
- Also defined by $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$

- [Banderier, Flajolet, Schaeffer, Soria 2001]: Random Maps
- [Flajolet, Louchard 2001]: Brownian excursion area



How to prove this?

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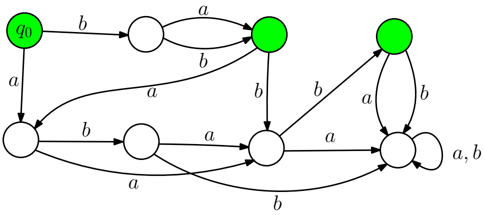
1. Bijection to decorated Dyck paths
2. Two-parameter recurrence relation for decorated Dyck paths
3. Heuristic analysis of recurrence
4. Inductive proof of asymptotically tight bounds using heuristics

Bijection to decorated paths



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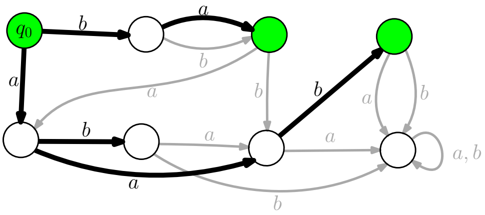

- Highlight spanning tree given by depth first search (ignoring the sink)
- I.e., black path to each vertex is first in lexicographic order
- Colour other edges red
- Draw as a binary tree with *a* edges pointing left and *b* edges pointing right

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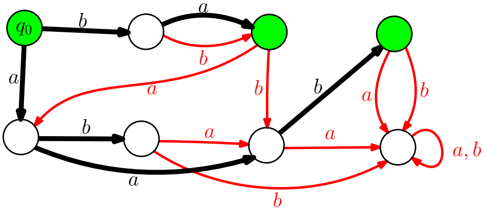

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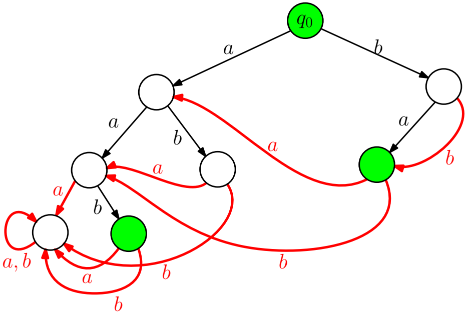

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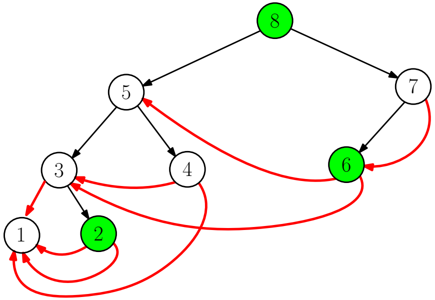
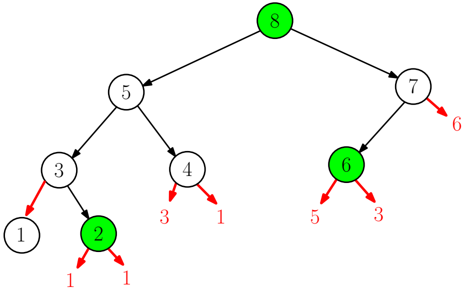


- Label nodes in post-order. By construction red edges point from a larger number to a smaller number
- → Label pointers

Bijection to decorated paths



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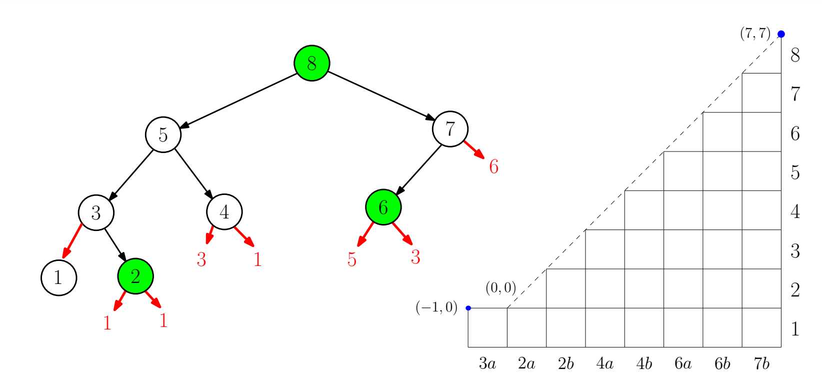

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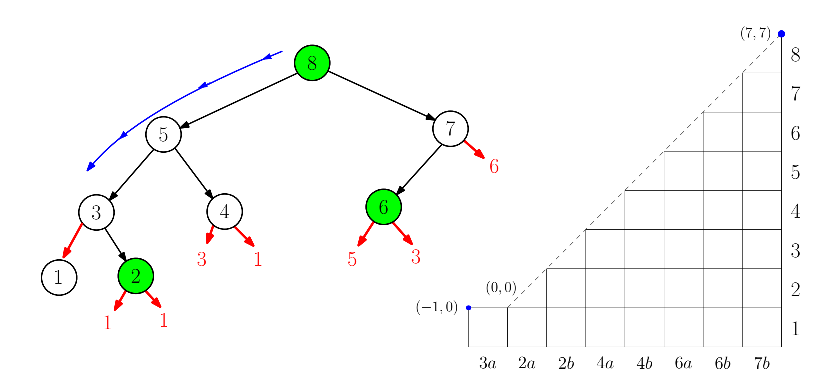

When the tree traversal...

- goes up: add up step with color matching the corresponding node.
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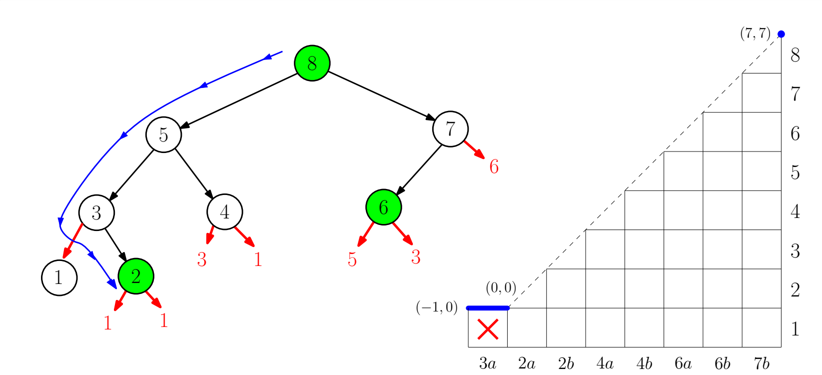

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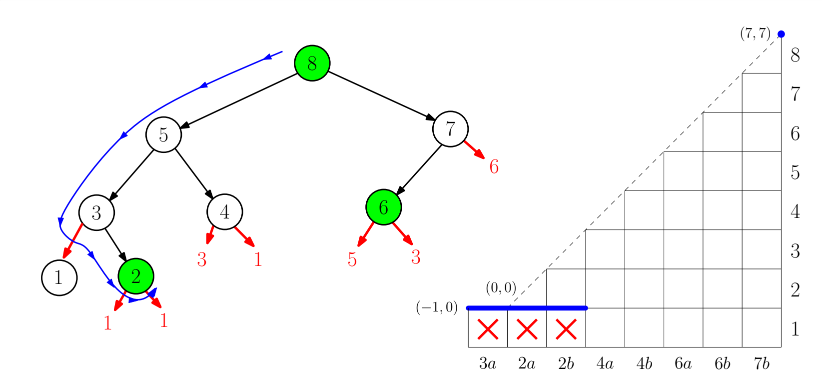

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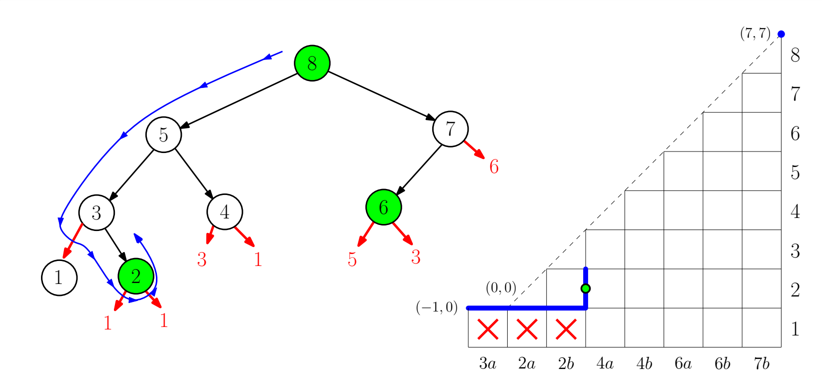

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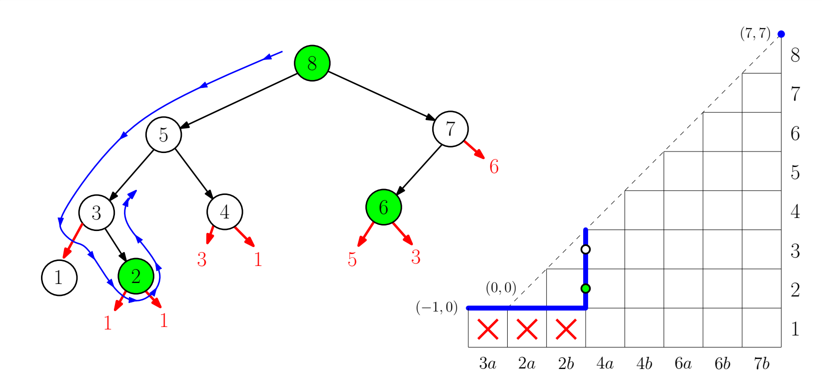

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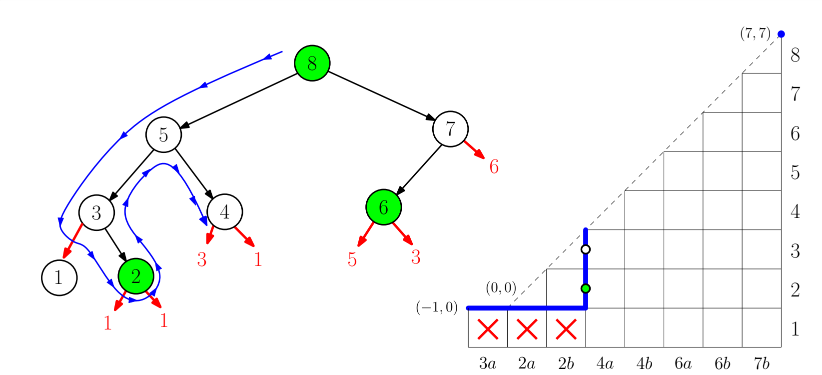

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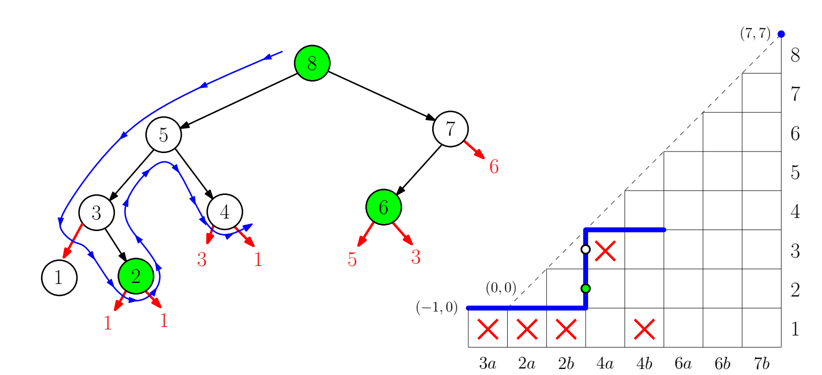

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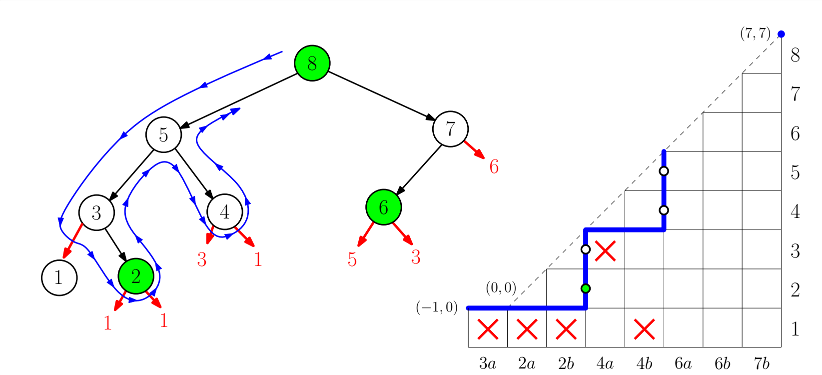

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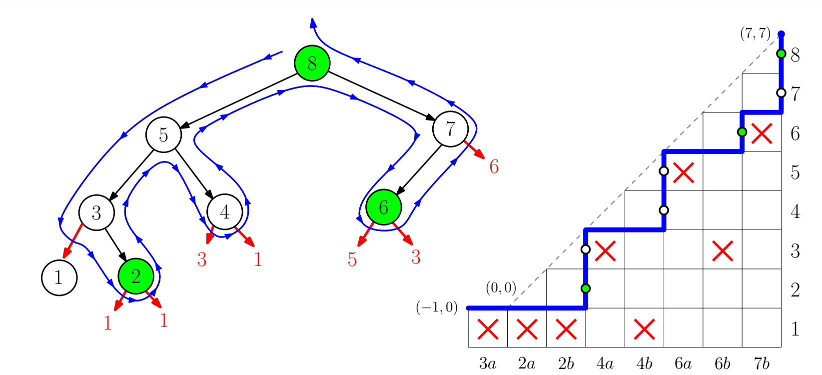

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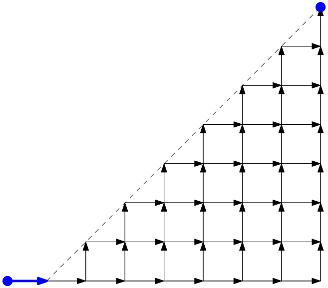
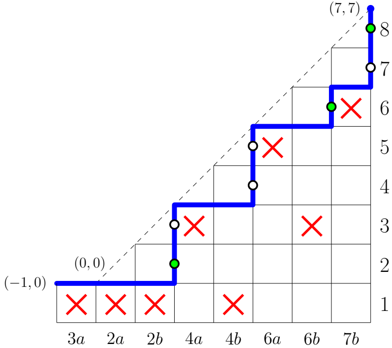
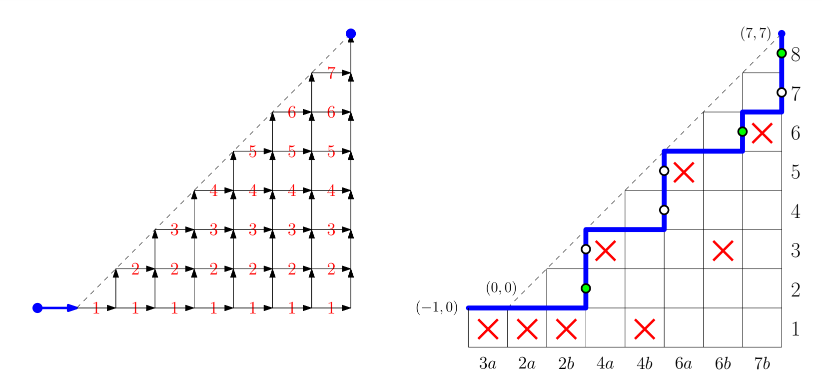




- Path starts at $(-1, 0)$ and ends at (n, n)
- Path stays below diagonal (after first step)
- One box is marked below each horizontal step
- Each vertical step is colored white or green

Decorated paths



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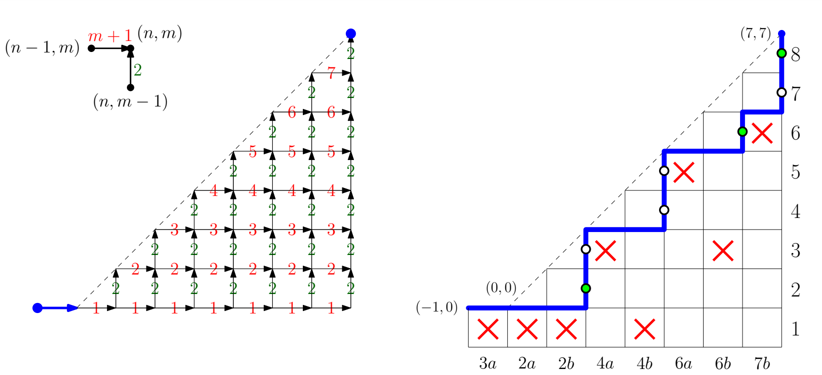

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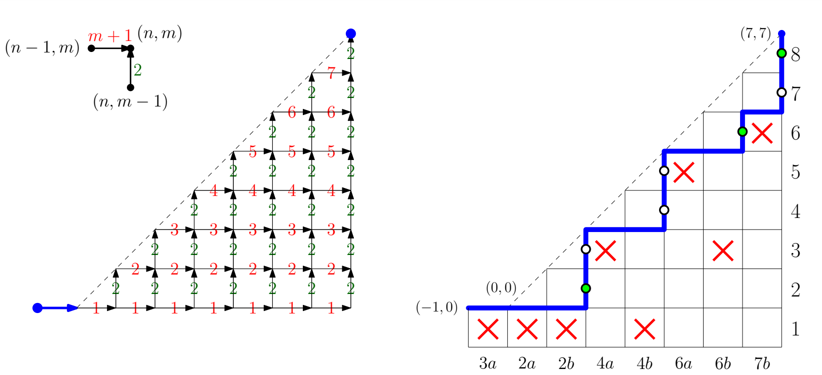
By the bijection: The number of these paths is the number d_n of acyclic DFAs with $n+1$ nodes.

Decorated paths



Recurrence: Denote by $a_{n,m}$ the number of paths ending at (n, m).

$$a_{n,m} = 2a_{n,m-1} + (m+1)a_{n-1,m}, \qquad \text{for } n \geq m$$

$$a_{-1,0} = 1.$$

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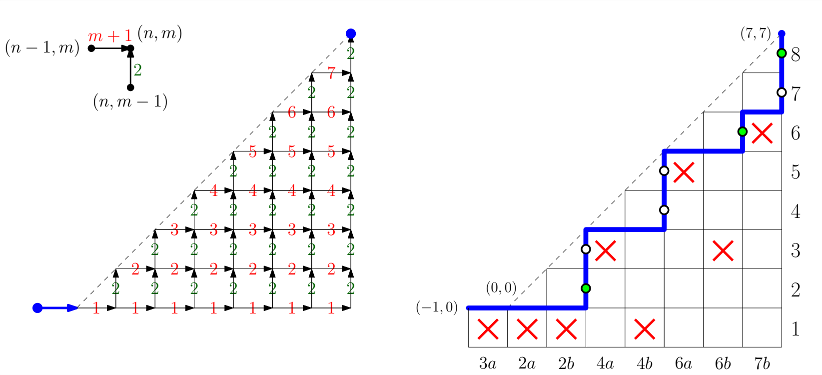

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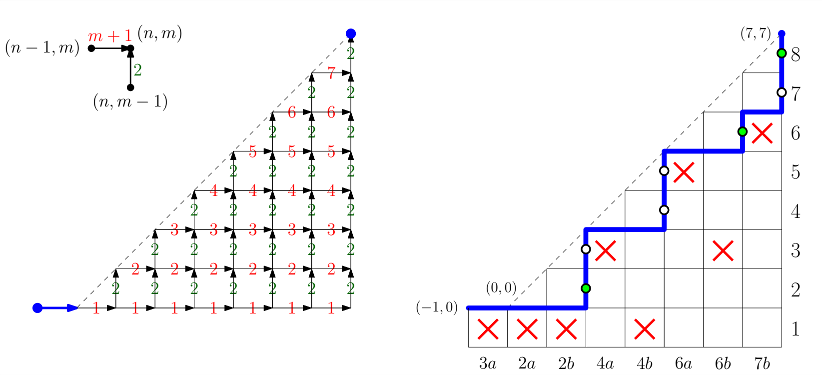
What about minimality?

Recurrence for minimal DFAs



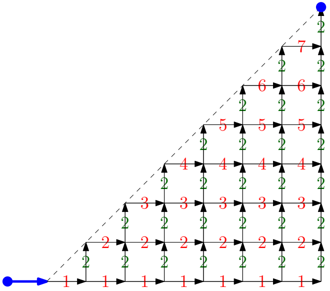
Recurrence: Denote by $b_{n,m}$ the number of paths ending at (n, m).

$$b_{n,m} = 2b_{n,m-1} + (m+1)b_{n-1,m} - mb_{n-2,m-1}, \qquad \text{for } n \geq m,$$

$$b_{-1,0} = 1.$$

Now: $m_n = b_{n,n}$ is the number of minimal acyclic DFAs with $n+1$ nodes.

Transforming the recurrence for minimal DFAs



Transformation:

$$e_{n+m,n-m} = \frac{1}{n!2^m} b_{n,m}.$$

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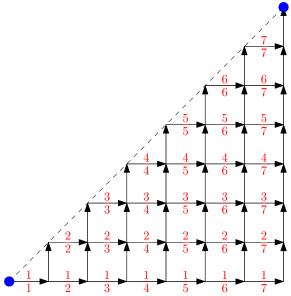

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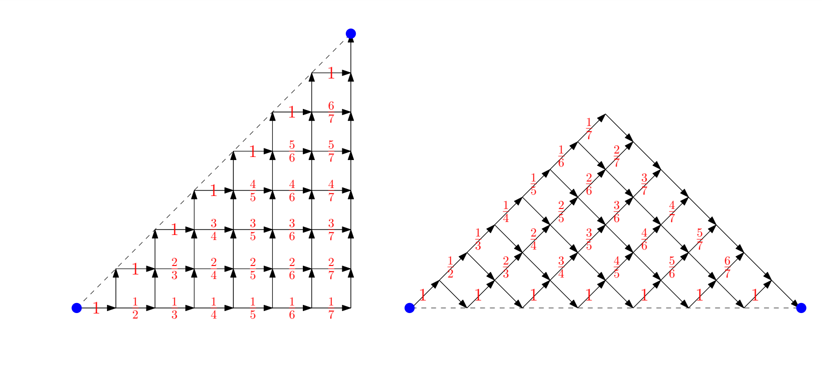

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New recurrence:

$$e_{n,m} = \left(1 - \frac{2(m+1)}{n+m}\right) e_{n-1,m-1} + e_{n-1,m+1} - \frac{n-m}{(n+m)(n+m-2)} e_{n-3,m-1}.$$

Now: $m_n = n! 2^n e_{2n,0}$.

Heuristics

We want to understand $e_{n,m}$ for large (fixed) n.

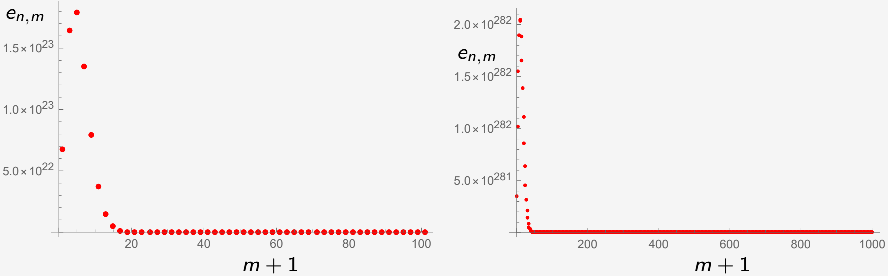

Figure: Plots of $e_{n,m}$ against $m+1$. **Left:** $n = 100$, **Right:** $n = 1000$.

- Let's zoom in to the left (small m) where interesting things are happening.
- It seems to be converging to something...

Guess: $e_{n,m} \approx h(n) f\left(\frac{m+1}{g(n)}\right)$. Moreover, we guess $g(n) = \sqrt[3]{n}$.

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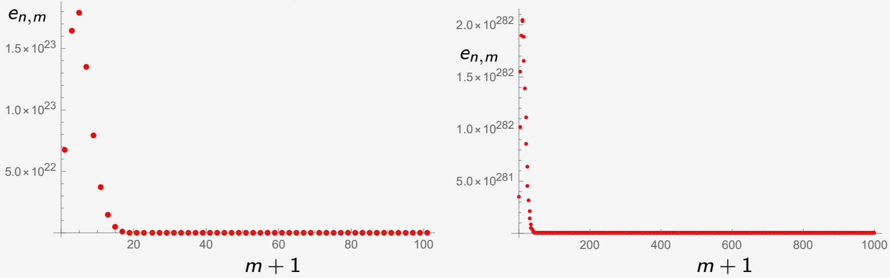

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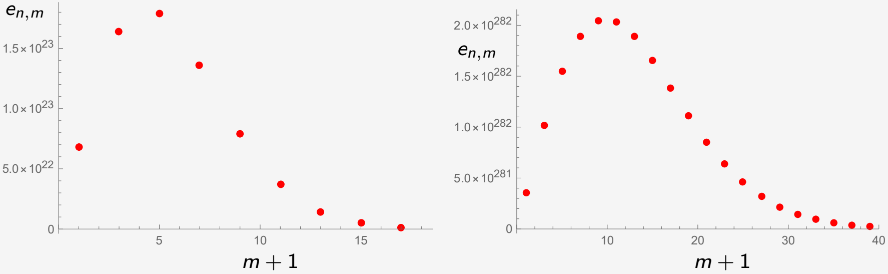

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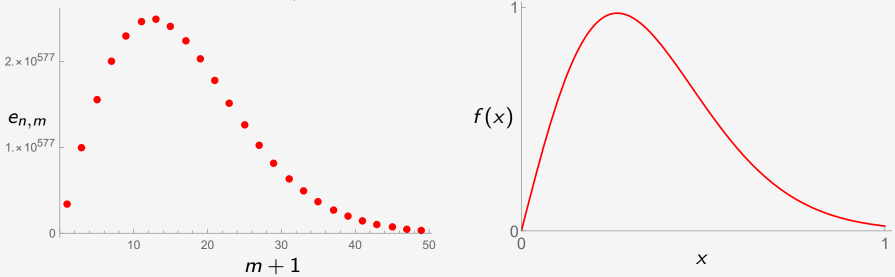

Figure: **Left:** Plot of $e_{n,m}$ against $m+1$ for $n = 2000$. **Right:** Limiting function $f(x)$.

- Let's zoom in to the left (small m) where interesting things are happening.
- It seems to be converging to something...

Guess: $e_{n,m} \approx h(n) f\left(\frac{m+1}{g(n)}\right)$. Moreover, we guess $g(n) = \sqrt[3]{n}$.

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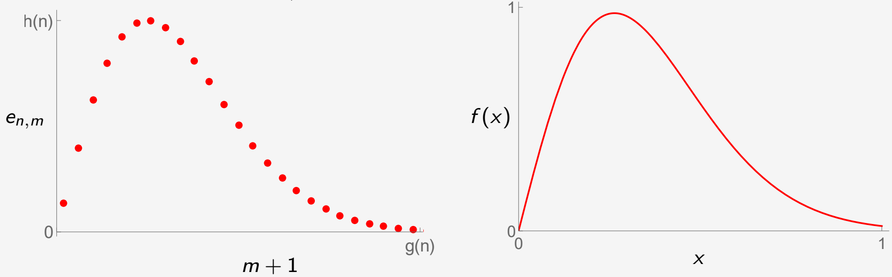

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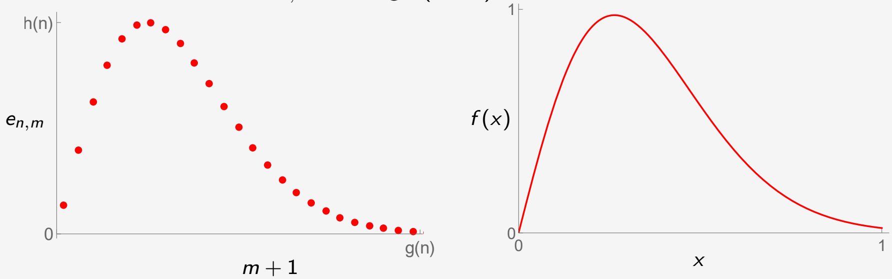

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Heuristic analysis of weighted paths

Recurrence

$$e_{n,m} = \left(1 - \frac{2(m+1)}{n+m}\right) e_{n-1,m-1} + e_{n-1,m+1} - \frac{n-m}{(n+m)(n+m-2)} e_{n-3,m-1}.$$

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- **Ansatz (a):** $e_{n,m} \approx h(n) f\left(\frac{m+1}{\sqrt[3]{n}}\right)$.
- Substitute into recurrence and set $m = \kappa\sqrt[3]{n} - 1$:

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$$f''(\kappa) = (2\kappa + c) f(\kappa) \qquad \Rightarrow \qquad f(\kappa) = \mathrm{Ai}(2^{-2/3}(2\kappa + c))$$

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  Where c is constant and Ai is the Airy function.
- Boundary condition $e_{n,-1} = 0$. Then $f(0) = 0$ implies $c = 2^{2/3} a_1$, where $a_1 \approx -2.338$ satisfies $\mathrm{Ai}(a_1) = 0$.

Refined heuristic analysis

1. **Ansatz of order 1:**

$$e_{n,m} \approx h(n) f\left(\frac{m+1}{\sqrt[3]{n}}\right),$$
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yields estimates $c = 2^{2/3} a_1$ such that

$$h(n) \approx 2^n e^{3a_1(n/2)^{1/3}} \quad \text{and} \quad f(\kappa) = \mathrm{Ai}(2^{1/3}\kappa + a_1).$$

2. **Ansatz of order 2:**

$$e_{n,m} \approx h(n)\left(f_0\left(\frac{m+1}{\sqrt[3]{n}}\right) + n^{-1/3} f_1\left(\frac{m+1}{\sqrt[3]{n}}\right)\right),$$
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yields estimates $d = 29/12$ such that

$$h(n) \sim const \cdot 2^n e^{3a_1(n/2)^{1/3}} n^{29/24} \quad \text{and} \quad f_0(\kappa) = \mathrm{Ai}(2^{1/3}\kappa + a_1).$$

This way we conjecture the asymptotic form for acyclic minimal DFAs:

$$m_n = 2^n n! e_{2n,0} = \Theta\left(n! 8^n e^{3a_1 n^{1/3}} n^{7/8}\right).$$

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Theorem

The number m_n of minimal DFAs recognizing a finite binary language and the number c_n (r_n) of compacted (relaxed) binary trees satisfy for $n \to \infty$

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with $a_1 \approx -2.338$: largest root of the Airy function $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt$.

Many future research directions:

- Multiplicative constant? Does it exist?
- Characterizing 2-parameter recurrences admitting stretched exponentials.
- Limit shapes: expected height? longest word? etc.
- Further applications to biology and queuing theory.

Open PhD position in my project "Stretched exponentials and beyond"!

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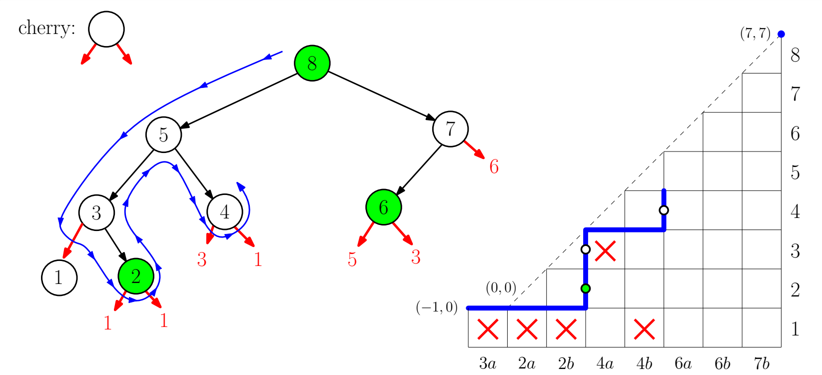
Backup

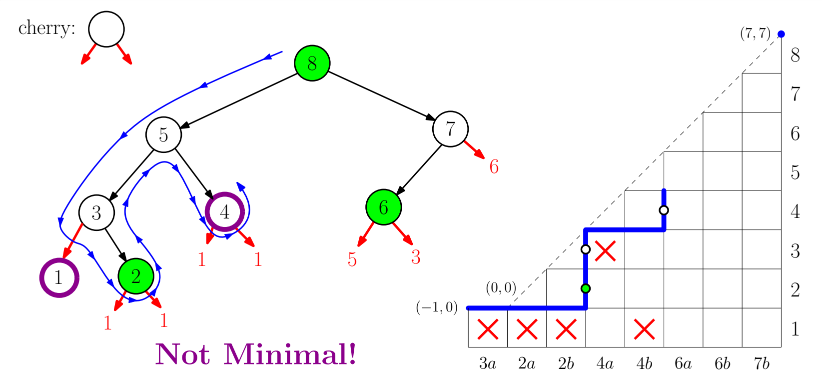
Minimal acyclic DFAs



For the DFA to be minimal, no state can be equivalent to a previous state:

- only possible if the new node is a cherry.
- If cherry is labeled m, then $m-1$ choices (of pointer labels and state color) must be avoided.
- Cherry corresponds to $\rightarrow\rightarrow\uparrow$ in path.

Minimal acyclic DFAs



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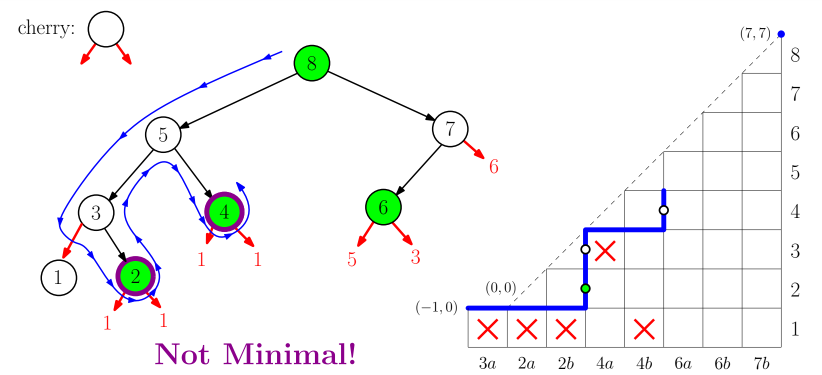

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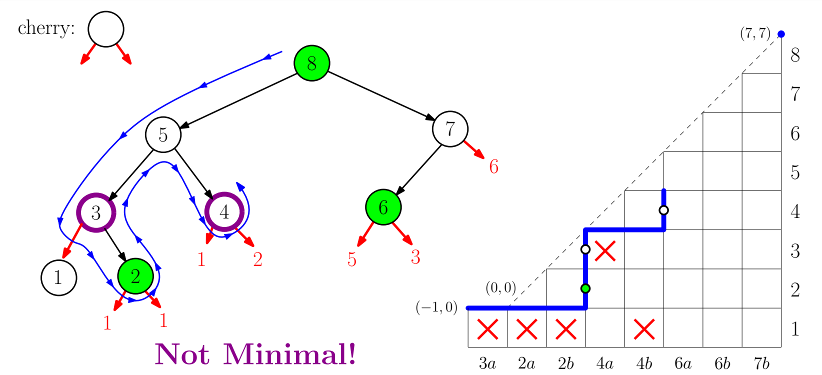

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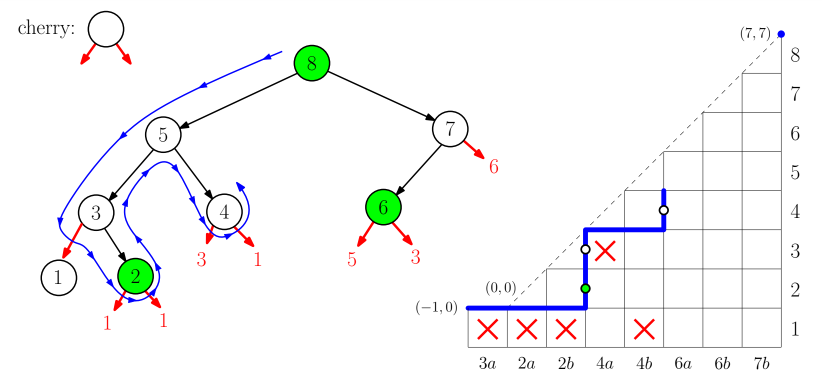

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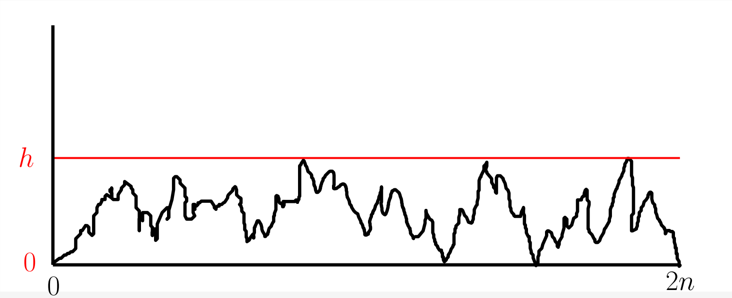
Dyck paths of length $2n$ where paths of height h get weight 2^{-h}



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Number of paths $\approx 4^n e^{-c_1 n^{1-2\alpha}}$, **Weight** $= 2^{-n^{\alpha}} = e^{-\log(2) n^{\alpha}}$.

Weighted number of paths is $\approx 4^n e^{-c_1 n^{1-2\alpha} - \log(2) n^{\alpha}}$.

Maximum occurs when $\alpha = 1/3$ and is equal to $4^n e^{-cn^{1/3}}$.

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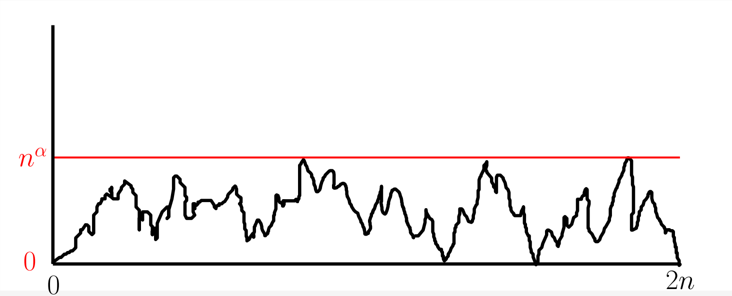

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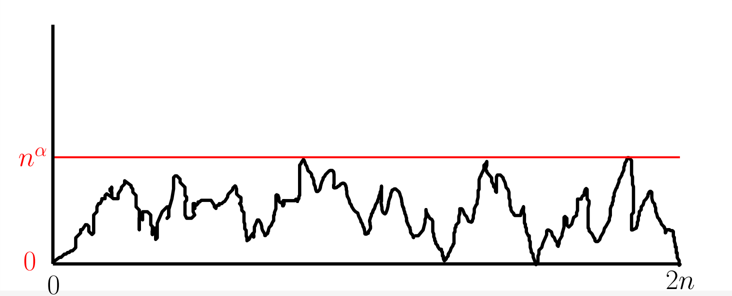

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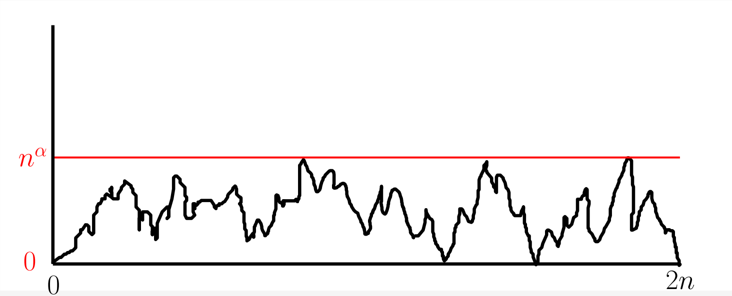

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Proof method

Recall:

$$e_{n,m} = \left(1 - \frac{2(m+1)}{n+m}\right) e_{n-1,m-1} + e_{n-1,m+1} - \frac{n-m}{(n+m)(n+m-2)} e_{n-3,m-1}$$

Number of minimal acyclic DFAs is $m_n = 2^n n! e_{2n,0}$.

Method:

Find sequences $X_{n,m}$ and $Y_{n,m}$ with the same asymptotic form, such that

$$X_{n,m} \leq e_{n,m} \leq Y_{n,m},$$

for all m and all n large enough.

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How to find them?

1. Use heuristics
2. Fiddle until $X_{n,m}$ and $Y_{n,m}$ satisfy the recurrence of $e_{n,m}$ with the equalities replaced by inequalities:

   $$= \quad \longrightarrow \quad \leq \text{ and } \geq$$

3. Prove $X_{n,m} \leq e_{n,m} \leq Y_{n,m}$ by induction.

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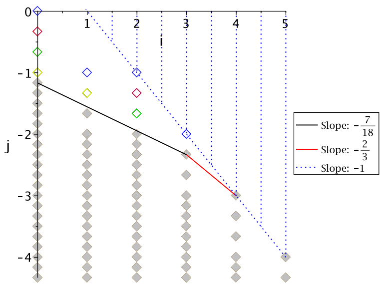
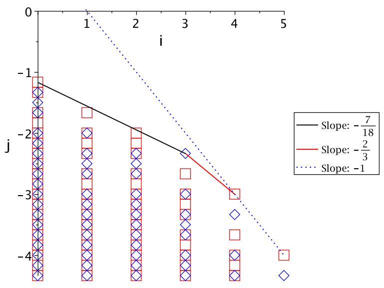
Unfortunately very technical (and not suited for the end of a talk ;))

Technicalities for compacted trees and minimal DFAs

Lots of technicalities:

- Before induction, we have to remove the negative term from the recurrence, but we have to do so precisely for asymptotics to stay the same.
- We only prove bounds for small *m*; we prove that large *m* terms don't matter
- The lower bound is negative for very large *m*, so we have to be careful with induction
- We only prove the bounds for sufficiently large *n*, but this only makes a difference to the constant term. Proof involves colorful Newton polygons:





Relaxed problem (relaxed compacted trees)

Recurrence for relaxed compacted trees

$$d_{n,m} = \frac{n-m+2}{n+m} d_{n-1,m-1} + d_{n-1,m+1}.$$

Lemma (lower bound)

For all $n, m \geq 0$ let

$$\tilde{X}_{n,m} := \left(1 - \frac{2m^2}{3n} + \frac{m}{2n}\right) \mathrm{Ai}\left(a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right) \qquad \text{and}$$

$$\tilde{s}_n := 2 + \frac{2^{2/3} a_1}{n^{2/3}} + \frac{8}{3n} - \frac{1}{n^{7/6}}.$$

Then, for any $\varepsilon > 0$, there exists an $\tilde{n}_0$ such that

$$\tilde{X}_{n,m} \tilde{s}_n \leq \frac{n-m+2}{n+m} \tilde{X}_{n-1,m-1} + \tilde{X}_{n-1,m+1},$$

for all $n \geq \tilde{n}_0$ and for all $0 \leq m < n^{1-\varepsilon}$.

Lower bound – Expansion

1. Transform to $P_{n,m} \geq 0$ for

$$P_{n,m} := -\tilde{X}_{n,m}\tilde{s}_n + \frac{n-m+2}{n+m}\tilde{X}_{n-1,m-1} + \tilde{X}_{n-1,m+1}.$$

where $(\sigma_i, \tau_j \in \mathbb{R})$

$$\tilde{s}_n := \sigma_0 + \frac{\sigma_1}{n^{1/3}} + \frac{\sigma_2}{n^{2/3}} + \frac{\sigma_3}{n} + \frac{\sigma_4}{n^{7/6}},$$

$$\tilde{X}_{n,m} := \left(1 + \frac{\tau_2 m^2 + \tau_1 m}{n}\right) \mathrm{Ai}\left(a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right).$$

2. Expand $\mathrm{Ai}(z)$ in a neighborhood of

$$\alpha = a_1 + \frac{2^{1/3}m}{n^{1/3}},$$

using $\mathrm{Ai}''(z) = z\mathrm{Ai}(z)$. Then

$$P_{n,m} = p_{n,m}\mathrm{Ai}(\alpha) + p'_{n,m}\mathrm{Ai}'(\alpha),$$

where $p_{n,m}$ and $p'_{n,m}$ are power series in $n^{-1/6}$ whose coefficients are polynomials in m.

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$$\tilde{X}_{n,m} := \left(1 + \frac{\tau_2 m^2 + \tau_1 m}{n}\right) \mathrm{Ai}\left(a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right).$$

2. Expand $\mathrm{Ai}(z)$ in a neighborhood of

$$\alpha = a_1 + \frac{2^{1/3}m}{n^{1/3}},$$

using $\mathrm{Ai}''(z) = z\mathrm{Ai}(z)$. Then

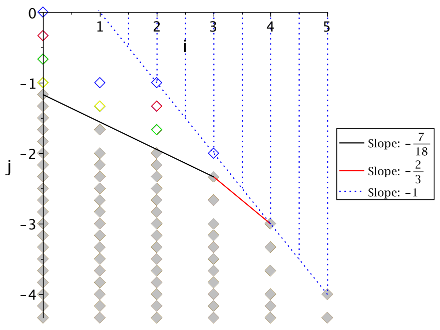
$$P_{n,m} = p_{n,m}\mathrm{Ai}(\alpha) + p'_{n,m}\mathrm{Ai}'(\alpha),$$

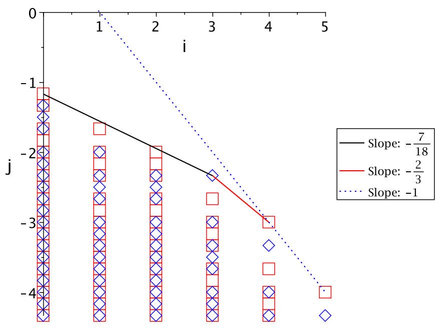
where $p_{n,m}$ and $p'_{n,m}$ are power series in $n^{-1/6}$ whose coefficients are polynomials in m.

Lower bound – Polygon

We get

$$P_{n,m} = \mathrm{Ai}(\alpha)\left(-\frac{\sigma_4}{n^{7/6}} - \frac{2^{5/3}a_1 m}{3n^{5/3}} - \frac{41m^2}{9n^2} - \frac{2^{8/3}a_1 m^3}{3n^{8/3}} - \frac{34m^4}{9n^3} - \frac{62m^5}{135n^4} + \dots\right) +$$
$$\mathrm{Ai}'(\alpha)\left(\frac{2^{1/3}(2\tau_1 - 1)}{n^{4/3}} + \frac{2^{1/3}}{n^{3/2}} - \frac{8a_1 m}{9n^2} + \frac{2^{1/3}(24\tau_1 - 31)m^2}{9n^{7/3}} - \frac{2^{13/3}m^3}{9n^{7/3}} + \dots\right.$$



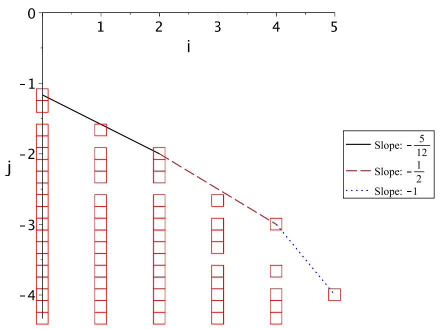


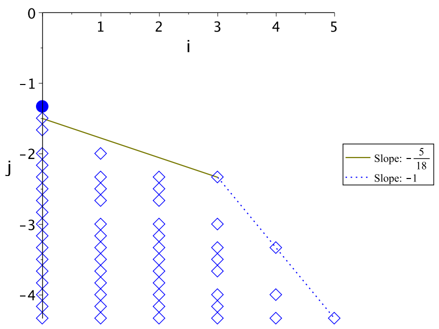
Lower bound – Case analysis

3. Treat $p_{n,m}$ and $p'_{n,m}$ separately and prove that all dominating terms (corners of convex hull) are positive.

$$p_{n,m} = \sum \tilde{a}_{i,j} m^i n^j$$

$$p'_{n,m} = \sum \tilde{a}'_{i,j} m^i n^j$$





non-zero coefficients

Relaxed trees: Proof idea – lower bound

Main idea

Suppose $(X_{n,m})_{n \geq m \geq 0}$ and $(s_n)_{n \geq 1}$ satisfy

$$X_{n,m} s_n \leq \frac{n-m+2}{n+m} X_{n-1,m-1} + X_{n-1,m+1}, \qquad (1)$$

for all sufficiently large n and all integers $m \in [0, n]$.

Define $(h_n)_{n \geq 0}$ by $h_0 = 1$ and $h_n = s_n h_{n-1}$; then prove that

$$X_{n,m} h_n \leq b_0 d_{n,m}$$

for some constant b_0 by induction:

Relaxed trees: Proof idea – lower bound

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$$X_{n,m} h_n \overset{(1)}{\leq} \frac{n-m+2}{n+m} X_{n-1,m-1} h_{n-1} + X_{n-1,m+1} h_{n-1}$$

$$\overset{\text{(Induction)}}{\leq} \frac{n-m+2}{n+m} b_0 d_{n-1,m-1} + b_0 d_{n-1,m+1}$$

$$\overset{\text{Rec. } d_{n,m}}{=} b_0 d_{n,m}.$$

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